Chapter B

One-Dimesnsional Kinematics

Blinn College - Physics 1401 - Terry Honan

Kinematics is the study of motion. This chapter will introduce the basic definitions of kinematics. The definitions of the velocity and acceleration will require the introduction of the basic notions of calculus, most specifically the derivative. We will also consider in detail the simple special cases of motion with constant velocity and constant acceleration. Free fall will be discussed as an example of motion with constant acceleration.

B.1 - The General Problem

By one dimensional motion we mean motion constrained to a line. As examples, consider a car driving on a straight road or the vertical motion of an elevator. The problem of motion in two or three dimensions will be discussed in the next chapter.

Position as a Function of Time

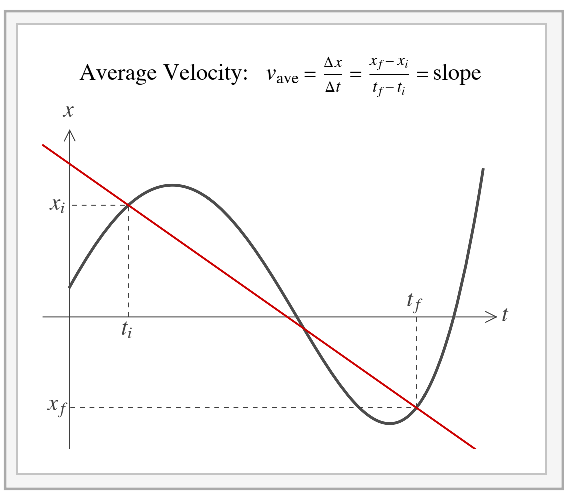
To mathematically define position we need to attach a real number line (the x-axis) to the line of motion. To do this there are two arbitrary choices; we must choose the $x = 0$ position and then we must choose the positive direction.

Any 1D motion can be represented graphically. Time is the independent variable, so it will be the horizontal axis. We will then consider graphs of x as a function of time, where x is the vertical axis.

> **Units:** The SI unit for distance is: m = meter

Velocity and the Derivative

Average Velocity



Interactive Figure

If a car drives 130 mi in 2 hours, we can calculate a velocity of 65 mi/hr. This is not necessarily what the speedometer would read; the speedometer reads the magnitude of the instantaneous velocity. In this case 65 mi/hr is what we call the average velocity.

We will define the average velocity by

$$v_{\text{ave}} = \frac{\Delta x}{\Delta t}$$

where Δ (Delta) generally will represent the final value minus the initial value

$$\Delta x = x_f - x_i \text{ and } \Delta t = t_f - t_i.$$

Note that the average velocity corresponds to two times t_i and t_f, and x_i and x_f are the positions at the two times. In a graph of x vs. t the average velocity has the interpretation as the slope of the secant line between the two points (t_i, x_i) and (t_f, x_f).

> **Units:** The SI unit for velocity is: m/s

The average velocity is not the distance per unit time, it is the displacement per unit time. The distinction is important. If you move along a line from an initial position to a different position a distance d away and then back to the initial position, your displacement is zero since $\Delta x = x_f - x_i = 0$ but the total distance travelled is $2d$. We may define the average speed as the total distance divided by the total time.

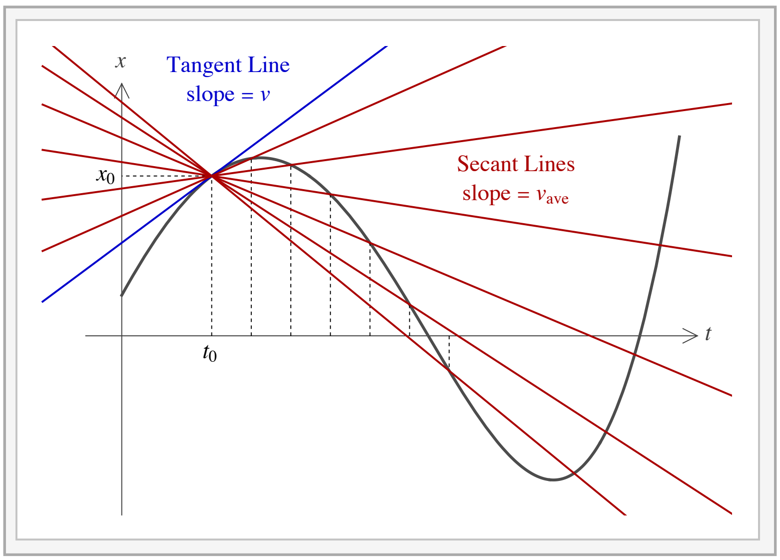
$$\text{Average Speed} = \frac{\text{total distance}}{\text{total time}}$$

Instantaneous Velocity

The instantaneous velocity refers to a single time t. Take the position at t to be x. We can then consider a later time $t + \Delta t$, where the position is $x + \Delta x$. The average velocity between these two times is $\Delta x / \Delta t$. To get the instantaneous velocity we let Δt become small; we do this by taking the limit as $\Delta t \to 0$. This gives the derivative of calculus; instantaneous velocity is the time rate of change (or time derivative) of position.

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

In a calculus class, or a calculus-based physics class, such derivatives are used for calculations; in this class we will *only* use expressions like this to define things and never in calculations.



Interactive Figure

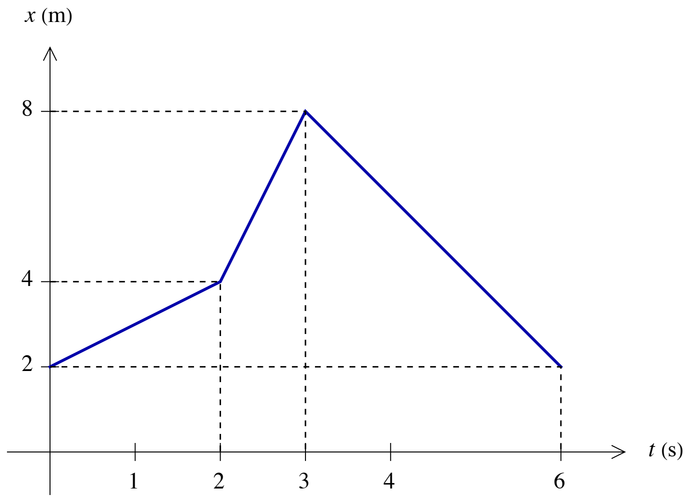
The graphical interpretation of the instantaneous velocity is simple. The average velocity is the slope of the secant lines. If we consider the secant lines corresponding to different Δt values then as $\Delta t \to 0$, these secant lines approach what we call the tangent line. The velocity at t is then the slope of that tangent line. When we refer to the slope of a graph at some time, we mean the slope of the line tangent to the graph at that time.

The direction of motion in one dimension is given by the sign of the velocity. The speed is the velocity without its direction and is thus the absolute value of the velocity.

$$\text{speed} = |v|$$

Example B.1 - Graphical Analysis of x vs. t

Consider the following graph of x vs. t showing three segments with different constant velocities.



(a) What is the average velocity between 0 s and 3 s?

Solution

The coordinates on the graph allow us to read off what is needed.

$$v_{\text{ave}} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} = \frac{x(3\text{ s}) - x(1\text{ s})}{3\text{ s} - 0\text{ s}} = \frac{8\text{ m} - 2\text{ m}}{3\text{ s} - 0\text{ s}} = 2\,\frac{\text{m}}{\text{s}}$$

(b) What is the average velocity between 2 s and 6 s?

Solution

$$v_{\text{ave}} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} = \frac{x(6\text{ s}) - x(2\text{ s})}{6\text{ s} - 2\text{ s}} = \frac{2\text{ m} - 4\text{ m}}{6\text{ s} - 2\text{ s}} = -\frac{1}{2}\,\frac{\text{m}}{\text{s}}$$

Note that when the net displacement is negative the average velocity is negative.

(c) What is the average velocity between 0 s and 6 s?

Solution

$$v_{\text{ave}} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} = \frac{x(6\text{ s}) - x(0\text{ s})}{6\text{ s} - 0\text{ s}} = \frac{2\text{ m} - 2\text{ m}}{6\text{ s} - 4\text{ s}} = 0$$

(d) What is the average speed between 0 s and 6 s?

Solution

Speed is the magnitude but of the velocity but the average speed is not the magnitude of the acceleration. Average velocity depend on the displacement Δx and the average speed depends on the total distance traveled.

$$\text{Average Speed} = \frac{\text{total distance}}{\text{total time}}$$

When you stop where you started your displacement is zero but the distance traveled is not. Between 0 s and 3 s the motion is in the positive direction and the distance is 8 m − 2 m = 6 m. Between 3 s and 6 s the displacement is −6 m but the distance traveled is 6 m

$$\text{Average Speed} = \frac{6\text{ m} + 6\text{ m}}{6\text{ s}} = 2\,\frac{\text{m}}{\text{s}}$$

(e) What is the instantaneous velocity at 1 s?

Solution

The instantaneous velocity at some time, or just velocity, is the slope of the x vs. t graph at that time. Since the graph is a straight segment at that time we can just find the slope. Note that from the graph, the velocity at 1 s is the average velocity between 0 s and 2 s.

$$v(1\text{ s}) = v_{\text{ave}} = \frac{x(2\text{ s}) - x(0\text{ s})}{2\text{ s} - 0\text{ s}} = \frac{4\text{ m} - 2\text{ m}}{2\text{ s}} = 1\ \frac{\text{m}}{\text{s}}$$

(f) What is the instantaneous velocity at 4 s?

Solution

Using the same logic at 4 s, the velocity then is the average velocity between 3 s and 6 s.

$$v(4\text{ s}) = v_{\text{ave}} = \frac{x(6\text{ s}) - x(3\text{ s})}{6\text{ s} - 3\text{ s}} = \frac{2\text{ m} - 8\text{ m}}{3\text{ s}} = -2\ \frac{\text{m}}{\text{s}}$$

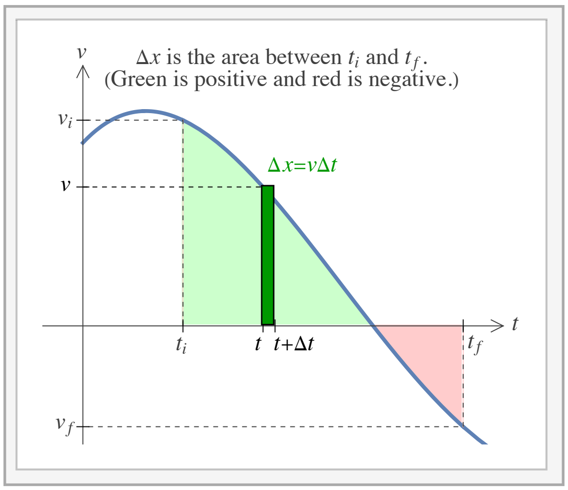
(g) What is the speed at 4 s?

Solution

The speed is the magnitude of the velocity and that means in one dimension that it is the absolute value of the velocity.

$$|v| = |v(4\text{ s})| = 2\ \frac{\text{m}}{\text{s}}$$

Displacement and Velocity vs. Time Graphs



In the graph of v vs. t above, let us first concentrate on the small time interval Δt. From the definition of velocity, we know that for a small Δt something will move the small displacement $\Delta x = v\,\Delta t$. In the graph above, that distance Δx is just the area of the rectangle between t and $t + \Delta t$. If we want the total distance traveled between t_i and t_f, it will then be the sum of all the small areas between t_i and t_f. When we talk about area this way it is implied that when the function is below the axis, meaning that the velocity is negative, the contribution to the area $\Delta x = v\,\Delta t$ is negative.

$$\Delta x = \sum v\,\Delta t\ = \text{Area under } v \text{ vs. } t\ \ (\Delta t \text{ is small})$$

Acceleration

Acceleration is to velocity as the velocity is to the position. Velocity is the time rate of change of position, so acceleration is the time rate of change of the velocity.

Average Acceleration

Since the average velocity is related to the position by $v_{\text{ave}} = \Delta x / \Delta t$ we can similarly write the average acceleration in terms of the velocity by

$$a_{\text{ave}} = \frac{\Delta v}{\Delta t}.$$

We can think of average acceleration graphically as the slope of the secant lines of a v vs. t graph.

Units: The SI unit for acceleration is: $\mathrm{m/s^2}$

Example B.2 - Graphical Analysis of *x* vs. *t* (continued)

(h) What is the average acceleration between 1 s and 4 s?

Solution

In parts **(e)** and **(f)** we saw that the instantaneous velocities at 1 s and 4 s were $v(1\,\mathrm{s}) = 1\,\frac{\mathrm{m}}{\mathrm{s}}$ and $v(4\,\mathrm{s}) = -2\,\frac{\mathrm{m}}{\mathrm{s}}$. From these values we can find the average acceleration.

$$a_{\text{ave}} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i} = \frac{v(4\,\mathrm{s}) - v(1\,\mathrm{s})}{4\,\mathrm{s} - 1\,\mathrm{s}} = \frac{-2\,\mathrm{m/s} - 1\,\mathrm{m/s}}{3\,\mathrm{s}} = -1\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

Instantaneous Acceleration

The instantaneous acceleration (or just acceleration) is the time rate of change (or time derivative) of the velocity.

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

B.2 - Constant Velocity and Acceleration

Now that we have considered the general problem of one dimensional kinematics we can now consider special cases, first constant velocity, then constant acceleration. An important case of constant acceleration is free fall.

Constant Velocity

If velocity is a constant then the acceleration is zero, since the slope of a constant function is zero. Since velocity is the slope of the x vs. t graph, the graph must be a straight line. $y = m x + b$, where as usual m is the slope and b is the y-intercept. For x vs. t this becomes: $x(t) = v\,t + x_0$where the x-intercept is the initial position x_0; this is the position at $t = 0$, $x_0 = x(0)$. Rearranging the terms we get:

$$x(t) = x_0 + v\,t$$

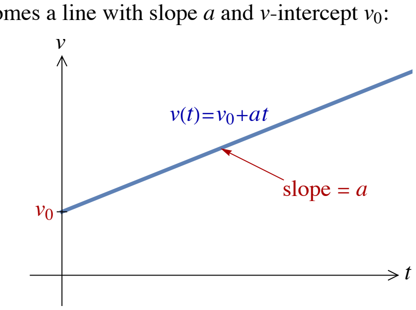
For this discussion, we are choosing the convention that the initial time is zero. Also, we will label the final time and position as just t and x: $t_i = 0, t_f = t, x_i = x_0$ and $x_f = x$. This gives: $\Delta x = x_f - x_i = x - x_0$.

$$x = x_0 + v\,t \text{ or } \Delta x = v\,t.$$

This is a simple expression; for constant velocity, the distance is the product of the velocity and time.

Constant Acceleration

If the acceleration is a constant then to get the velocity we repeat the preceding procedure for going from a constant velocity to the position. Acceleration is the slope of the v vs. t graph. As before, define the initial velocity v_0 to be the velocity at $t = 0$, $v_0 = v(0)$ and the final time and velocity become t and v. The v vs. t graph becomes a line with slope a and v-intercept v_0:

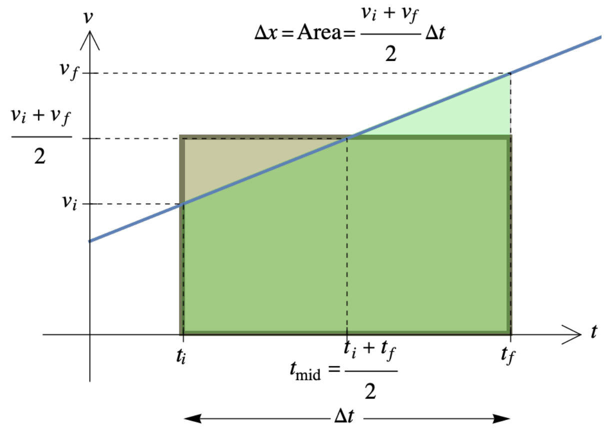


velocity vs. time for constant acceleration

$$v(t) = v_0 + a\,t$$

We will now derive a set of four equations for motion with constant acceleration, where the expression above is our first.

Before finding the second expression, recall that for a graph of v vs. t, the area under the curve is the displacement, Δx. Since the graph is a straight line the area under the curve is the area of a trapezoid. If we consider some interval from t_i to t_f then we get the following graph.



The area of a trapezoid with base b and heights h_i and h_f is the base times the average of the heights: Area = $b\,\frac{h_i+h_f}{2}$. In this case, it becomes: $\Delta x = \frac{1}{2}\left(v_i + v_f\right)\cdot\Delta t$. To understand this formula for area compare the areas of the trapezoid and the rectangle whose height is the average of the trapezoid's heights, as shown above. Applying $v_{\text{ave}} = \Delta x / \Delta t$ so this we see that, for motion with constant acceleration only, the average velocity at the midpoint of an interval is the average of the initial and final velocities.

$$v_{\text{ave}} = \frac{\Delta x}{\Delta t} = \frac{1}{2}\left(v_i + v_f\right)$$

To find the second equation for constant acceleration, let us rewrite $\Delta x = \frac{1}{2}\left(v_i + v_f\right)\cdot\Delta t$ using the convention: $t_i = 0$, $t_f = t$, $v_i = v_0$ and $v_f = v$. This gives:

$$\Delta x = \frac{1}{2}\,(v_0 + v)\,t\,.$$

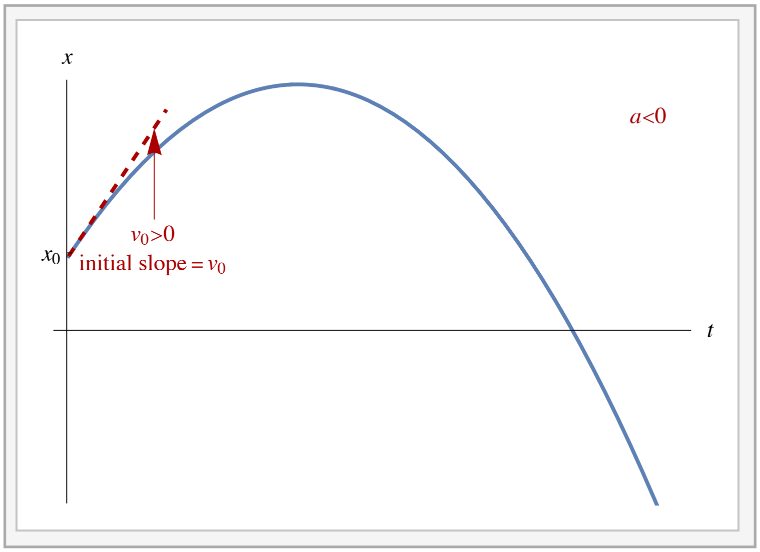
Using this as the second expression and $v = v_0 + a\,t$ as the first, we can find the third expression by eliminating v and the fourth equation by eliminating t. Eliminating v gives

$$\Delta x = \frac{1}{2}\,(v_0 + v)\,t = \frac{1}{2}\left(v_0 + (v_0 + a\,t)\right)t = v_0\,t + \frac{1}{2}\,a\,t^2$$

To write x as a function of t we use $\Delta x = x - x_0$ to get out third equation.

$$x(t) = x_0 + v_0\,t + \frac{1}{2}\,a\,t^2.$$

Note that for constant acceleration, the graph of x vs. t is a parabola. (Recall that a parabola in y vs. x has the form $y = a\,x^2 + b\,x + c$.)



The graph of x vs. t for constant acceleration is a parabola.

To get the fourth equation eliminate t from from the first two equations. Solve for t with the first equation

$$v = v_0 + a\,t \implies t = \frac{v - v_0}{a}$$

and then plug into the second.

$$\Delta x = \frac{1}{2}\,(v_0 + v)\,t = \frac{1}{2}\,(v_0 + v)\,\frac{v - v_0}{a} \implies 2\,a\,\Delta x = v^2 - v_0^2$$

where $(A + B)\,(A - B) = A^2 - B^2$ was used.

With this we have derived a set of four equations for kinematics with constant acceleration. These relate the variables t, Δx, v_0, v and a. These will be useful for a large class of problems in this chapter.

Constant Acceleration Equations

$$v = v_0 + a\,t$$
$$\Delta x = \frac{1}{2}\,(v_0 + v)\,t$$
$$\Delta x = v_0\,t + \frac{1}{2}\,a\,t^2$$
$$v^2 = v_0^2 + 2\,a\,\Delta x$$

Example B.3 - Decelerating Car

A car brakes uniformly to a stop from 30.0 m/s while moving 150. m.

(a) What is the car’s acceleration while braking?

Solution

The first step is to establish what we are given and what we are looking for in terms of the variables in our constant acceleration equations. Here we are given the initial velocity, the final velocity, the displacement and we are looking for the acceleration.

$$v_0 = 30.0\text{ m/s}\,,\quad v = 0\,,\quad \Delta x = 150.\text{ m and } a = ?$$

Often the best way to approach the constant acceleration equations is by identifying the equations in terms of the variable it does not include. Here we are not given time or looking for it so that leaves us to the fourth equation, the one that doesn’t involve t.

$$v^2 = v_0^2 + 2\,a\,\Delta x \implies a = -\frac{v_0^2}{2\,\Delta x} = -3.00\text{ m/s}^2$$

(b) How long does it take for the car to stop?

Solution

Since we have already solved for a we can use any equation involving time to get t. Here will solve part (b) without reference to part (a); this leads us to the second equation, the one that does not involve a.

$$\Delta x = \frac{1}{2}(v_0 + v)\,t \implies t = \frac{2\,\Delta x}{v_0} = 10.0\text{ s}$$

Free Fall

Free fall is one dimensional motion under the influence of only gravity. Assuming that only gravity acts implies that we are ignoring any friction effects. We will choose the convention that up is the positive direction. Also, we will take y as the position variable; this will be consistent with our later usage where y is typically taken as the upward vertical variable. Galileo discovered that the acceleration of all bodies in the presence of gravity (ignoring air resistance) is the same. The value of the downward acceleration is

$$g = 9.80\,\frac{\text{m}}{\text{s}^2} = 32.0\,\frac{\text{ft}}{\text{s}^2}\,.$$

Since up is the positive y direction and the acceleration is downward we take the acceleration to be:

$$a = -g$$

Using this value of a and replacing x with y takes the constant acceleration equations to the free fall expressions.

Free Fall Equations

$$v = v_0 - g\,t$$
$$\Delta y = \frac{1}{2}(v_0 + v)\,t$$
$$\Delta y = v_0\,t - \frac{1}{2}\,g\,t^2$$
$$v^2 = v_0^2 - 2\,g\,\Delta y$$

Example B.4 - A Dropped Ball

A ball is dropped from a height 6.00 m. (We will assume there is no air resistance for free-fall problems.)

(a) What is its time of fall?

Solution

We will first label the relevant variable and constants.

$$h = 6.00\text{ m and } g = 9.80\text{ m}/\text{s}^2$$

When solving a problem in terms of symbols, one must write the answer in terms of the symbols given and in terms of physical constants. Here what is given is h and the relevant constant is g. h is related to Δy and, since the net motion is downward, $\Delta y < 0$. Since the ball is dropped we conclude its initial velocity is zero. We are looking for t.

$$\Delta y = -h\,,\ v_0 = 0 \text{ and } t = ?$$

(Another way to understand $\Delta y = -h$ is to set $y_0 = h$, $y = 0$ and use $\Delta y = y - y_0$.) The variable that we do not know or need is the final velocity v, so we are led to the third equation.

$$\Delta y = v_0\,t - \frac{1}{2}\,g\,t^2 \implies -h = 0 - \frac{1}{2}\,g\,t^2$$

Note that generally for a dropped object, the time of fall is related to the height by

$$h = \frac{1}{2}\,g\,t^2.$$

Solving for t and taking the positive square root we get our answer, first with symbols and then with the numbers.

$$t = \sqrt{\frac{2\,h}{g}} == \sqrt{\frac{2\times 6.00\text{ m}}{9.80\text{ m}/\text{s}^2}} = 1.11\text{ s}$$

(b) What is the ball's velocity when (just before) it hits the ground?

Solution

Now we are looking for v. To find this without reference to the result of part (a) we should use the fourth equation.

$$v^2 = v_0^2 - 2\,g\,\Delta y$$

We still have $\Delta y = -h$ and $v_0 = 0$, so we get:

$$v^2 = 2\,g\,h\,.$$

The motion is downward so we want the negative square root when solving for v.

$$v = -\sqrt{2\,g\,h} = -\sqrt{2 \times 9.80\ \mathrm{m/s^2} \times 6.00\ \mathrm{m}} = -10.8\ \mathrm{m/s}$$

If the problem asked for the speed instead of velocity then you would take the positive square root.

Example B.5 - pward Initial Velocity

A ball is thrown straight upward from the ground at $30.0\ \mathrm{m/s}$.

(a) What is the maximum height reached by the ball?

Solution

We are given the initial velocity. How do we mathematically describe the ball's highest point? At the very top of its vertical path the ball is instantaneously at rest, $v = 0$. The maximum height y_{max} is Δy when $v = 0$.

$$v_0 = 30.0\ \mathrm{m/s}\,,\ \ v = 0\,,\ \ y_{max} = \Delta y = ?$$

Since time is neither given or desired we are led to the fourth equation.

$$v^2 = v_0^2 - 2\,g\,\Delta y \implies y_{max} = \Delta y = \frac{v_0^2}{2\,g} = 45.9\ \mathrm{m}$$

(b) How long does it take for the ball to return to the ground?

Solution

When the ball returns to where it began $\Delta y = 0$. Note that Δy is not the distance traveled; it is the net displacement. It is zero because $\Delta y = y - y_0$ and $y = y_0$. Since we are solving for time and we still have $v_0 = 30\ \mathrm{m/s}$ we should use the third equation.

$$\Delta y = v_0\,t - \frac{1}{2}\,g\,t^2 \implies 0 = t\left(v_0 - \frac{1}{2}\,g\,t\right) \implies t = 0 \ \text{ and } \ v_0 - \frac{1}{2}\,g\,t = 0$$

It is trivially true that $\Delta y = 0$ when $t = 0$; we want the other solution, which becomes:

$$t = \frac{2\,v_0}{g} = 6.12\ \mathrm{s}$$

(c) Solve for t_{up}, the time for the ball to move to its highest point. Also show that when $\Delta y = 0$, as in the previous part, $t = 2\,t_{up}$.

Solution

Since $v = 0$ at the top, we have

$$v = v_0 - g\,t \implies 0 = v_0 - g\,t_{up} \implies t_{up} = \frac{v_0}{g} = 3.06\ \mathrm{s}.$$

It follows that $t = 2\,t_{up}$. This is generally the case for free-fall problems with $\Delta y = 0$.

(d) When does the ball pass a 35 m high window?

Solution

We now want to find t when $\Delta y = 35.0\ \mathrm{m}$. We still have $v_0 = 30.0\ \mathrm{m/s}$, so the third equation is needed.

$$\Delta y = v_0\,t - \frac{1}{2}\,g\,t^2 \implies 35.0\ \mathrm{m} = (30.0\ \mathrm{m/s})\,t - \frac{1}{2}\left(9.80\ \mathrm{m/s^2}\right)t^2$$

Solving for t gives two solutions. Both solutions are required here; it passes the window twice, moving upward and then moving downward.

$$t = 1.57\text{ s} \text{ and } t = 4.55\text{ s}$$

(The above can be solved using the quadratic formula $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, with $a = \frac{1}{2}(9.80\text{ m/s}^2)$, $b = -30.0\text{ m/s}$ and $c = 35.0\text{ m}$.)

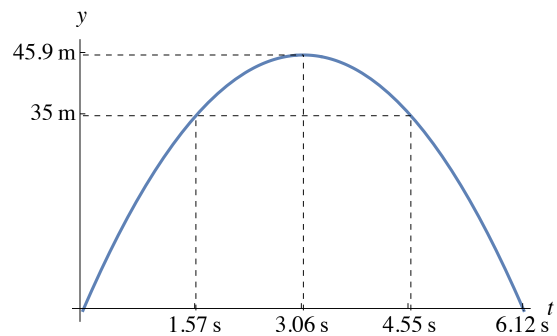


Figure - *y vs. t* graph showing the values calculated in this problem