Chapter C

Vectors and Relative Motion

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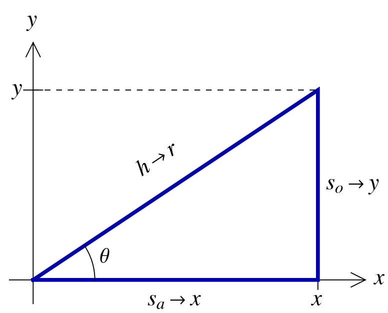
C.1 - Vector Algebra

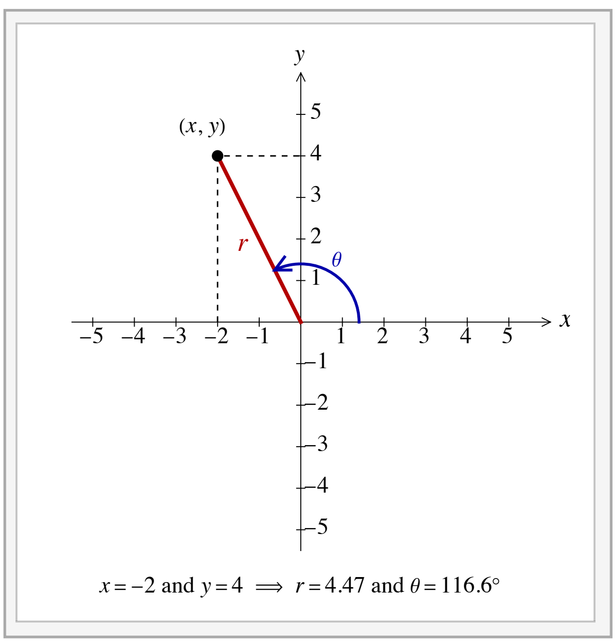
Polar Coordinates

(x, y) are the Cartesian (or rectangular) coordinates of some point on a plane. r and θ are the polar coordinates; $r = \sqrt{x^2 + y^2}$ is the distance from the origin to the point and θ is the angle measured counterclockwise from the positive x axis to the point. The student first sees the definition of the trig functions in terms of a right triangle with s_a the side adjacent to the angle θ and s_o opposite θ. By placing the right triangle on the Cartesian plane we can generalize these definitions to general angle.



$$\sin\theta = \frac{s_o}{h} = \frac{y}{r}, \quad \cos\theta = \frac{s_a}{h} = \frac{x}{r} \quad \text{and} \quad \tan\theta = \frac{s_o}{s_a} = \frac{y}{x}.$$

When written this way we can now define the trig functions for general angles, not just acute angles. Using the above definitions it is a straightforward matter to find the formulas for converting between polar and rectangular coordinates.



Interactive Figure

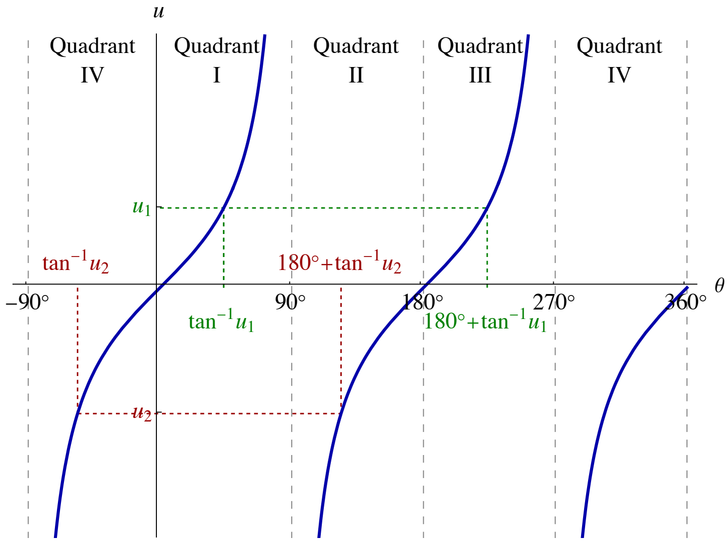
$$\boldsymbol{r \text{ and } \theta \implies x \text{ and } y}$$

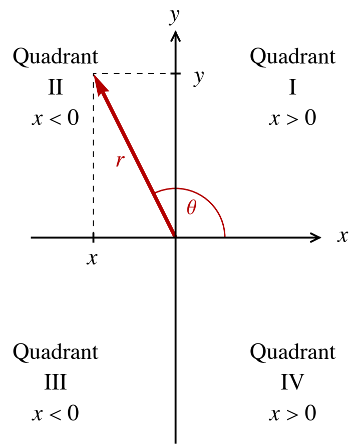
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

$$\boldsymbol{x \text{ and } y \implies r \text{ and } \theta}$$

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & \text{when } x > 0 \\ 180° + \tan^{-1}\left(\frac{y}{x}\right) & \text{when } x < 0 \end{cases}$$

The subtlety in solving for the angle follows from the identity: $\tan\theta = \tan(180° + \theta)$. The range of the $\tan^{-1}$ (also written arctan) function is $-90° < \theta < 90°$, which corresponds to the quadrants I and IV; this is equivalent to the condition that $x > 0$. The case of quadrants II and III, when $x < 0$, requires shifting the result of the inverse tangent by $180°$.





Example C.1 - Polar Coordinates

Consider the point $(x, y) = (-2, 4)$ in the Cartesian plane. Find the polar coordinates r and θ of this point.

Solution

This is a straightforward application of the formulas above, where $x = -2$ and $y = 4$.

$$r = \sqrt{x^2 + y^2} = \sqrt{20} = 4.47$$

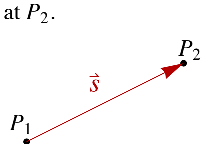
$$\theta = 180° + \tan^{-1}\left(\frac{y}{x}\right) = 116.6°$$

Vector Basics

A vector is a quantity with both a magnitude and a direction. A scalar has only a magnitude; it is just a real number. The magnitude of a vector is a non-negative (positive or zero) scalar. Velocity is a vector quantity and speed is its magnitude. Acceleration, force and momentum are also vectors. Time, temperature, mass and pressure are examples of scalars.

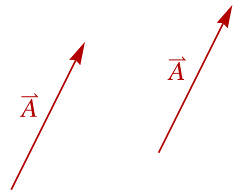
We will write a vector variable by a symbol with an "$\rightharpoonup$" over it. The magnitude of a vector is given by the symbol without the arrow or by applying the "|| ||" brackets to the vector. If $\vec{A}$ is some vector then $A = \|\vec{A}\|$ is its magnitude.

We can represent vectors by arrows. Suppose someone walks from a starting point P_1 to a stopping point P_2. A displacement vector $\vec{s}$ (or $\Delta\vec{r}$) may be viewed as an arrow with its tail at P_1 and its tip at P_2.



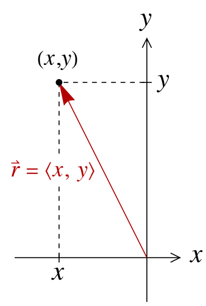
A position vector $\vec{r}$ is a displacement vector with its tail at the origin. For more general vectors we represent them by arrows pointing in the direction of the vectors and the length of the arrow is proportional to the magnitude of the vector. For instance, if one velocity vector has a magnitude of 60 mi/hr and another has 30 mi/hr then the arrow representing the first should have twice the length of the second.

A vector has no fixed position. If a vector arrow is moved keeping its length and direction fixed then it still is the same vector.



Component Definition - Position and Displacement Vectors

A position vector is a way to label a position in the Cartesian plane; it has its tail at the origin and its head at the position it labels. We will use an angled bracket notation for vectors. The position vector that labels the point (x, y) will be written as $\vec{r} = \langle x, y \rangle$.

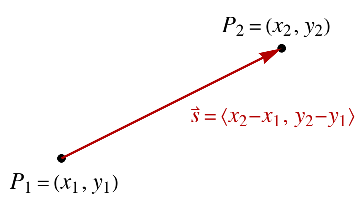


We will define the magnitude and direction of a vector so that the polar coordinates, r and θ, are the magnitude and direction of the vector.

In the Cartesian plane we will denote vector from (x_1, y_1) to (x_2, y_2) by:

$$\vec{s} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

This is called a displacement vector. A position vector is clearly a displacement vector with its tail at the origin.

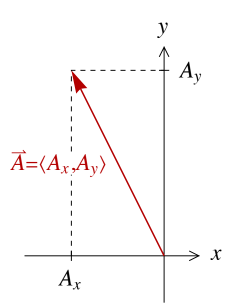


Component Definition - General Vectors

We will write a 2-dimensional vector $\vec{A}$ as a pair of real numbers A_x and A_y called components. (A 3D vector is a triple.) We will use the "angled-bracket" notation for vectors.

$$\vec{A} = \langle A_x, A_y \rangle$$

The component A_x has the interpretation as the amount the vector $\vec{A}$ is in the x-direction and A_y the y-direction.



Unit Vectors and Notation

A unit vector is a vector of magnitude one. We denote unit vectors with a " ^ " over its top. For any vector $\vec{A}$ we can simply find the unit vector in its direction $\hat{A}$ by

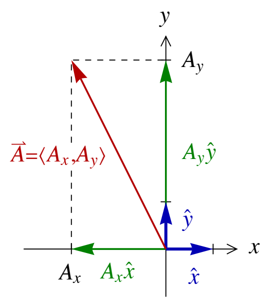
$$\hat{A} = \frac{\vec{A}}{||\vec{A}||}.$$

Basis unit vectors are unit vectors along the coordinate axes. We will use $\hat{x}$ and $\hat{y}$ for the unit vectors in the x and y directions. The more traditional notation for these unit vectors is to write them as $\hat{i}$ and $\hat{j}$.

$$\hat{x} = \langle 1, 0 \rangle \text{ and } \hat{y} = \langle 0, 1 \rangle$$

Any vector can then be written in terms of these basis unit vectors

$$\vec{A} = \langle A_x, A_y \rangle = A_x\,\hat{x} + A_y\,\hat{y}$$



Magnitude and Direction Angle

We define A_x and A_y as the components of the vector $\vec{A}$. A_x is the part of $\vec{A}$ in the x direction and similarly A_y is the y part. The Cartesian coordinates x and y are the components of a position vector $\vec{r}$. For a two dimensional vector we can represent the direction with an angle, measured as in the polar coordinates. To convert between the magnitude and direction angle and the components of a two dimensional vector we have analogous expressions to the ones for polar coordinates.

$$\boldsymbol{A \text{ and } \theta \implies A_x \text{ and } A_y}$$

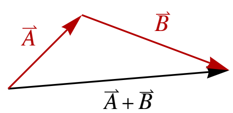
$$A_x = A\cos\theta \text{ and } A_y = A\sin\theta \tag{C.1}$$

$$\boldsymbol{A_x \text{ and } A_y \implies A \text{ and } \theta}$$

$$A = \sqrt{A_x{}^2 + A_y{}^2} \text{ and } \theta = \begin{cases} \tan^{-1}\left(\frac{A_y}{A_x}\right) \text{ when } A_x > 0 \\ 180\,° + \tan^{-1}\left(\frac{A_y}{A_x}\right) \text{ when } A_x < 0 \end{cases} \tag{C.2}$$

Vector Addition

Suppose a displacement vector $\vec{s}_1$ corresponds to someone walking from P_1 to P_2. Suppose that then the person walks from P_2 to P_3; call this displacement $\vec{s}_2$. The net displacement is the vector from P_1 to P_3; this is what we will define as the sum of the two displacements $\vec{s}_1 + \vec{s}_2$. To generalize this to any vectors, we will define the sum of general vectors $\vec{A}$ and $\vec{B}$. Draw the vectors as shown, with the tail of $\vec{B}$ at the tip of $\vec{A}$. The sum the vectors $\vec{A} + \vec{B}$ is the vector drawn from the tail of $\vec{A}$ to the tip of $\vec{B}$.



With our component definition vector addition takes the very simple form:

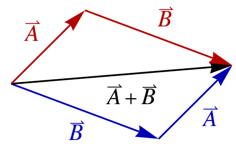
$$\vec{A} + \vec{B} = \langle A_x, A_y \rangle + \langle B_x, B_y \rangle = \langle A_x + B_x, A_y + B_y \rangle. \tag{3.3}$$

Commutative Property

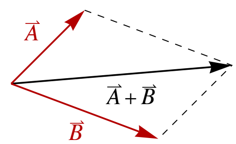
An algebraic operation is commutative when changing the order of the items doesn't affect the result. For vector addition this takes the form.

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

The commutative property of the addition of reals implies this for vectors.



Because of the commutative property there are two more ways of adding vectors we can consider. In addition to placing the tail of $\vec{B}$ at the tip of $\vec{A}$, we can place the tail of $\vec{A}$ at the tip of $\vec{B}$. Also there is the parallelogram rule: Draw the two vector together tail to tail and complete the parallelogram; the sum it the vector from the common tail of the vectors to the opposite corner.

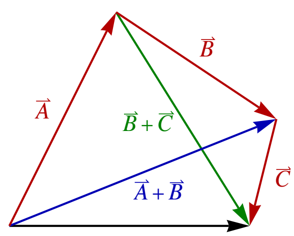


Associative Property

From the definition of vector addition it is clear it satisfies.

$$\left(\vec{A}+\vec{B}\right)+\vec{C}=\vec{A}+\left(\vec{B}+\vec{C}\right)$$

This property is called associativity. For an associative operation it is not necessary to use brackets, since the order of the operation is unimportant.



$(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$

Identity

The identity vector $\vec{0}$ is the vector that leaves any other vector $\vec{A}$ unchanged under addition.

$$\vec{A}+\vec{0}=\vec{A}$$

It is clear that the zero vector has zeros as components.

$$\vec{0}=\langle 0,0\rangle$$

The magnitude of the identity vector is 0, so we can write $0=\|\vec{0}\|$. Note that the direction of $\vec{0}$ is undefined; in fact, it is the only vector with an undefined direction.

Additive Inverse.

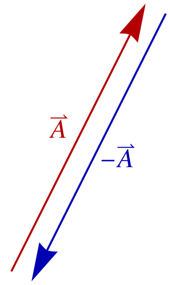
For any vector $\vec{A}$ there is an additive inverse vector $-\vec{A}$ with the property:

$$\vec{A}+\left(-\vec{A}\right)=\vec{0}$$

Clearly, this has the value

$$-\vec{A}=\langle -A_x,-A_y\rangle$$

and has the same magnitude and is in the opposite direction.



Vector Subtraction

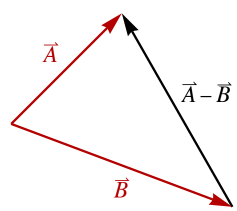
We define vector subtraction by adding the inverse.

$$\vec{A} - \vec{B} = \vec{A} + \left(-\vec{B}\right)$$

In terms of components we have

$$\vec{A} - \vec{B} = \langle A_x,\, A_y\rangle - \langle B_x,\, B_y\rangle = \langle A_x - B_x,\, A_y - B_y\rangle.$$

The simplest way to view the vector $\vec{A} - \vec{B}$ is as the vector that when added to $\vec{B}$ gives $\vec{A}$; if the vectors are drawn tail to tail then it is the vector from the tip of $\vec{B}$ to the tip of $\vec{A}$



Scalar Multiplication

If $\vec{A}$ is a vector and c is a scalar then we can define their product $c\,\vec{A}$ as a vector.

$$c\,\vec{A} = \langle c\,A_x,\, c\,A_y\rangle \tag{C.4}$$

It is clear that its magnitude is given by

$$\left\|c\,\vec{A}\right\| = |c|\;\|\vec{A}\|,$$

where |c| is the absolute value of the scalar. The direction of $c\,\vec{A}$ is the same as $\vec{A}$ when $c > 0$ and opposite to $\vec{A}$ when $c < 0$. When $c = 0$ we get $0\,\vec{A} = \vec{0}$. Note also that $1\,\vec{A} = \vec{A}$ and $(-1)\,\vec{A} = -\vec{A}$.

The scalar multiplication operation has the associative and distributive properties.

Associative Property

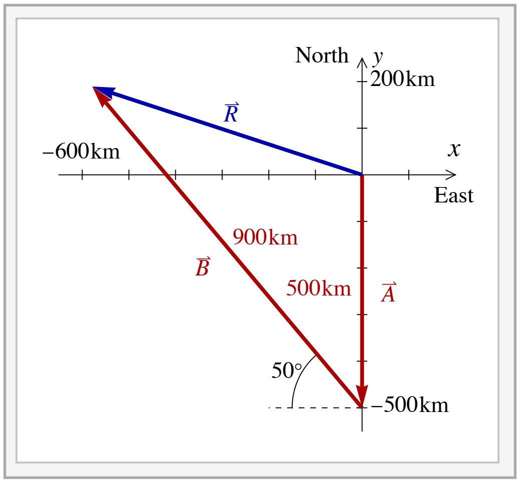
$$(c\,d)\,\vec{A} = c\left(d\,\vec{A}\right)$$

Distributive Properties

$$(c + d)\,\vec{A} = c\,\vec{A} + d\,\vec{A} \;\text{ and }\; c\left(\vec{A} + \vec{B}\right) = c\,\vec{A} + c\,\vec{B}$$

Example C.2 - Vector Addition

A plane flies 500 km to the south and then 900 km in the direction 50° north of west. What is the plane's net displacement? What are the magnitude and direction angle of the plane's net displacement?



Interactive Figure

Solution

Here we are adding two vectors, starting with magnitude and direction information. To find the components using equation (C.1) we must identify the direction angles which are measured counterclockwise from the positive x direction, which we take to be East. Vector $\vec{A}$ is to the South so it has direction angle $-90\,°$ and it has magnitude 500 km. Although we could use (C.1) find A_x and A_y it is easier to see that since it is purely in the negative y direction:

$$\vec{A} = \langle A_x, A_y \rangle = \langle 0, -500 \text{ km} \rangle \, .$$

The direction angle for vector $\vec{B}$ is $\theta_B = 180\,° - 50\,° = 130\,°$. (See the interactive diagram above.) The magnitude of $\vec{B}$ is $B = 900$ km so using (C.1) we get

$$\vec{B} = \langle B_x, B_y \rangle = \langle B \cos \theta_B, B \sin \theta_B \rangle = \langle -578.51, 689.44 \rangle \text{ km} \, .$$

The net displacement is sum of these two vectors, the resultant vector $\vec{R} = \vec{A} + \vec{B}$.

$$\begin{aligned} \vec{R} &= \vec{A} + \vec{B} = \langle R_x, R_y \rangle \\ &= \langle 0, -500 \rangle \text{ km} + \langle -578.51, 689.44 \rangle \text{ km} \\ &= \langle -578.51, 189.44 \rangle \text{ km} \end{aligned}$$

Using (C.2) we can find the Magnitude and direction angle of the net displacement.

$$R = \sqrt{R_x^2 + R_y^2} = 609 \text{ km} \qquad \text{(Magnitude)}$$

$$\theta = 180\,° + \tan^{-1} \frac{R_y}{R_x} = 161.9\,° \qquad \text{(Direction Angle)}$$

The net displacement is just $\vec{R}$.

$$\vec{R} = \langle -579, 189 \rangle \text{ km} \qquad \text{(Net Displacement)}$$

C.2 - Relative Motion

Suppose you are moving in a car with a velocity of $\vec{v}_{\text{car}}$; this is your velocity relative to the road, so we will write this as $\vec{v}_{C,R}$, where C stands for car and R for the road. Now imagine there is a truck that, relative to you (in the car) is moving at $\vec{v}_{T,C}$, that is: the velocity of the truck (labelled T) relative to the car. What is the truck's velocity relative to the road? The simple vector formula is: $\vec{v}_{T,R} = \vec{v}_{T,C} + \vec{v}_{C,R}$; that is, the velocity of the truck relative to the road is the vector sum of the velocity of the truck to the car and the velocity of the car to the road. With $\vec{v}_{C,R}$ being the velocity of the car relative to the road, then what is the velocity of the road, relative to the car; well, it has to be opposite: $\vec{v}_{C,R} = -\vec{v}_{R,C}$; this means, for instance, that if a car is moving to the north relative to the road, the road moves to the south relative to the car at the same speed.

For this type of problem we are comparing the relative velocity of three things, which we will generically label and A, B and C. The general expression is

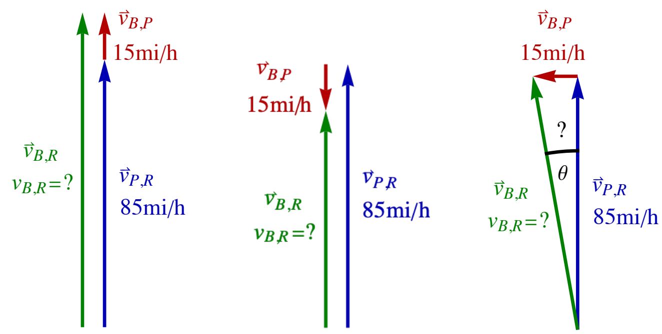
$$\vec{v}_{A,C} = \vec{v}_{A,B} + \vec{v}_{B,C}$$

To solve a relative motion problem you have to identify and label the three things you are comparing. It is also important to note that generally:

$$\vec{v}_{B,A} = -\vec{v}_{A,B}$$

Example C.3 - Bubba

Bubba drives his pickup at 85 mi/h while throwing his beer bottle out the window at 15 mi/h, relative to the pickup.



(a) (b) (c) and (d)

(a) What is the speed of the bottle, relative to the road, if the bottle is thrown forward?

Solution

The first thing we need to do is identify the three things we are comparing. Label them as P for the pickup, R for the road and B for the bottle. The speed of the pickup relative to the road $v_{P,R}$ is (the magnitude of $\vec{v}_{P,R}$) is 85 mi/h. The speed of the bottle relative to the pickup $v_{B,P}$ is (the magnitude of $\vec{v}_{B,P}$) is 15 mi/h. We want the velocity of the bottle relative to the road $\vec{v}_{P,R}$.

$$\vec{v}_{B,R} = \vec{v}_{B,P} + \vec{v}_{P,R}$$

Take the fixed frame K to be the frame of the road and K' to be the truck's frame. We are studying the motion of the bottle with respect to these two frames. For a one dimensional problem remember that real numbers are one-dimensional vectors where the sign gives the direction. We then have the preceding expression without the vector arrows.

$$v_{B,R} = v_{B,P} + v_{P,R} = 15 \text{ mi/h} + 85 \text{ mi/h} = 100 \text{ mi/h}$$

(b) What is the speed of the bottle, relative to the road, if the bottle is thrown backward?

Solution

Now $v_{B,P}$ is in the opposite direction, so we will make it negative.

$$v_{B,R} = v_{B,P} + v_{P,R} = (-15 \text{ mi/h}) + 85 \text{ mi/h} = 70 \text{ mi/h}$$

(c) What is the speed of the bottle, relative to the road, if the bottle is thrown directly to Bubba's left?

Solution

This is now two-dimensional. $\vec{v}_{P,R}$ and $\vec{v}_{B,P}$ are perpendicular and form two sides of a right triangle. $\vec{v}_{B,R}$ is the hypotenuse; to find its magnitude use the Pythagorean theorem.

$$v_{B,R} = \sqrt{(85 \text{ mi/h})^2 + (15 \text{ mi/h})^2} = 86.3 \text{ mi/h}$$

(d) Relative to the road, what is the angle the bottle makes from the truck's direction?

Solution

From the right triangle we my find θ using trig.

$$\theta = \arctan \frac{15 \text{ mi/h}}{85 \text{ mi/h}} = 10.0\,°$$

Example C.4 - A Plane in a Cross-wind

A plane flies to a city 800 mi to the east. In still air, the trip takes two hours. Suppose there is a strong 100 mi/h cross-wind blowing to the south. We will assume that the speed of the plane with respect to the air is fixed.

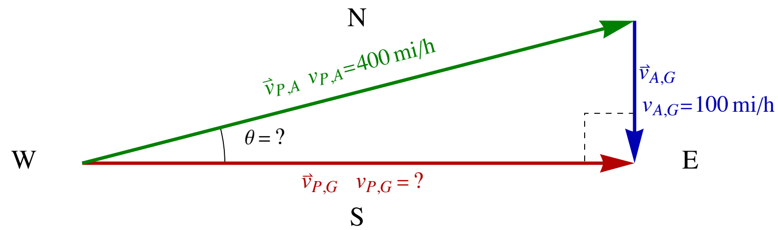
(a) At what angle, north of east, must the plane aim to have a trajectory toward the east?

Solution

In this problem the three things whose motion we are relating are the ground G, the plane P and the air A. We know that the direction of the plane relative to the ground $\vec{v}_{P,G}$ is to the east. The wind velocity is then $\vec{v}_{A,G}$; choosing east as the x-direction and north as the y-direction we have

$$\vec{v}_{A,G} = \langle 0, -100 \rangle \frac{\text{mi}}{\text{h}} = -100 \frac{\text{mi}}{\text{h}} \hat{y}$$

For $\vec{v}_{P,A}$ (the plane relative to the air) we can find its magnitude. 800 mi in two hours implies the speed of the plane relative to the air of $v_{P,A} = 400\,\text{mi/h}$. As for the direction of $\vec{v}_{P,A}$, it is directed at the unknown angle θ north of east. The velocity of the plane relative to the ground $\vec{v}_{P,G}$ is directed to the east with an unknown magnitude.



Given the right triangle above, we can solve for the angle θ.

$$\theta = \sin^{-1} \frac{100}{400} = 14.5\,°$$

(b) How long does the trip take.

Solution

We know that the distance is $d = 800$ mi and $d = v\,t$, where $v = v_{P,G}$ is the speed of the plane relative to the ground. To find that we use the Pythagorean theorem.

$$v_{P,G}^2 + v_{A,G}^2 = v_{P,A}^2 \implies v_{P,G} = \sqrt{v_{P,A}^2 - v_{A,G}^2} = \sqrt{(400\,\text{mi/h})^2 - (100\,\text{mi/h})^2} = 387.30\,\text{mi/h}$$

From this we can solve for the time.

$$d = v\,t = v_{P,G}\,t \implies t = \frac{d}{v_{P,G}} = 2.07\,\text{h}$$