Chapter D

Two Dimensional Kinematics

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D.1 - Kinematical Variables in 2D

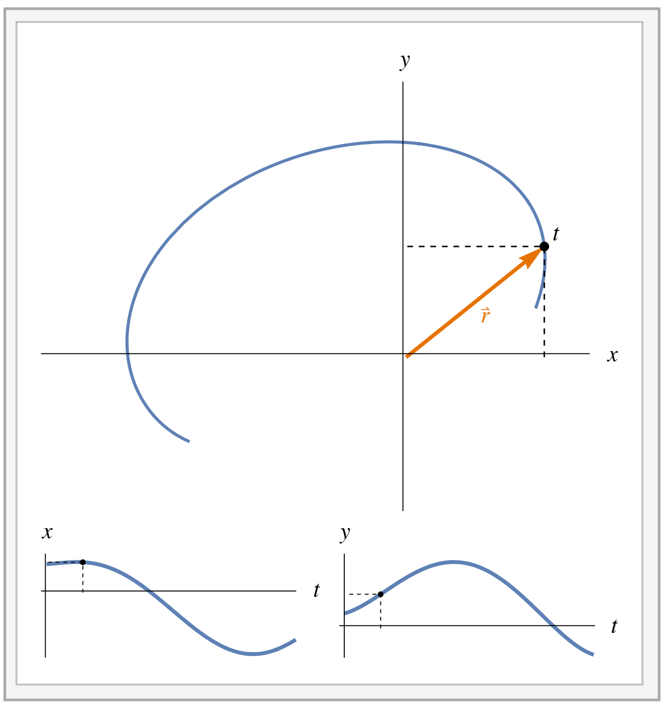
The General Problem

Position as a Function of Time

There are two equivalent ways of describing position in two dimensions. One is by labeling the coordinates (x, y). The other is by giving the position vector $\vec{r}$ of the position. The two are related; the coordinates are the components of the position vector. To label position as a function of time we can consider x and y as separate functions of time or as $\vec{r}$ as a function of time.

$$x(t) \text{ and } y(t) \iff \vec{r}(t) = \langle x(t),\, y(t)\rangle$$

The actual path followed by the body is called the trajectory. It is represented by a plot of a path in the xy plane.



Interactive Figure - Two dimensional motion may be visualized as vectors or as two separate cases of one dimensional motion, each representing a component of the vectors.

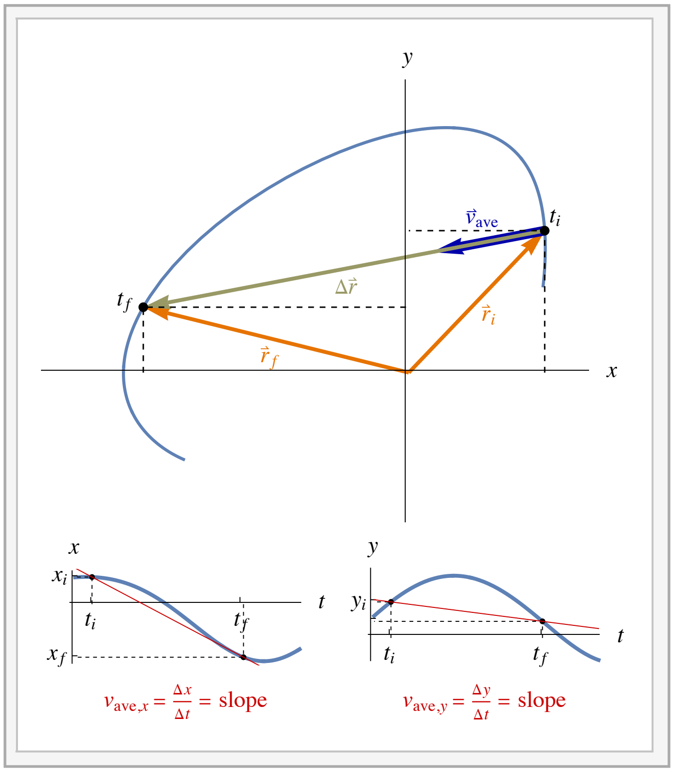
Average Velocity

The average velocity, as we saw in the one dimensional case, refers to two times. At t_i the position vector is $\vec{r}_i = \vec{r}(t_i)$ and at t_f it is $\vec{r}_f = \vec{r}(t_f)$. The displacement is the difference of these two positions

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i = \langle x_f - x_i,\, y_f - y_i\rangle.$$

The average velocity vector is then defined as

$$\vec{v}_{\text{ave}} = \frac{\Delta\vec{r}}{\Delta t} = \left\langle \frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t} \right\rangle = \langle v_{\text{ave},x}, v_{\text{ave},y} \rangle. \tag{D.1}$$



Interactive Figure - The average velocity can be found from $\Delta\vec{r}$ using $\vec{v}_{\text{ave}} = \Delta\vec{r}/\Delta t$.

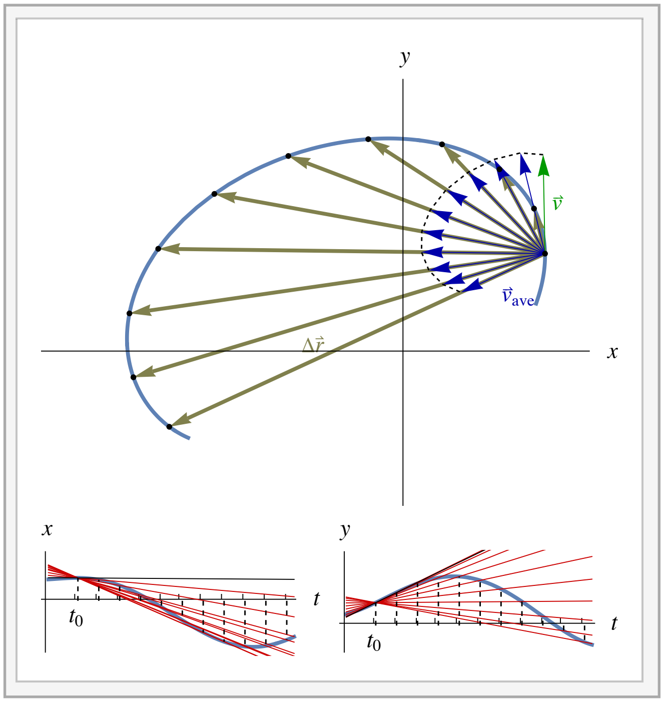
Instantaneous Velocity

The instantaneous velocity is defined as the limit of the average velocity as Δt approaches zero; it is the time derivative of the position vector.

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \left\langle \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}, \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \right\rangle = \langle v_x, v_y \rangle. \tag{D.2}$$

The magnitude of the velocity is called the speed.

$$v = \|\vec{v}\| = \sqrt{v_x^2 + v_y^2} = \text{speed} \tag{D.3}$$

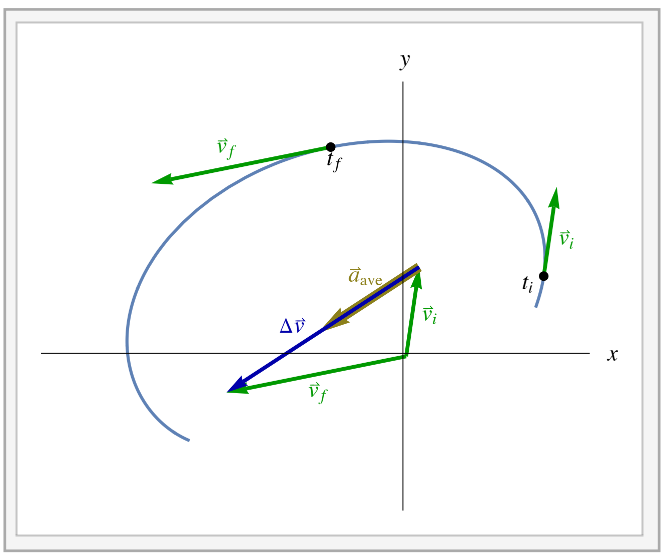


Interactive Figure - The instantaneous velocity is the limit of the average velocity as $\Delta t \to 0$.

Average Acceleration

As with the one dimensional case acceleration is to velocity as the velocity is to position.

$$\vec{a}_{\text{ave}} = \frac{\Delta \vec{v}}{\Delta t} = \left\langle \frac{\Delta v_x}{\Delta t}, \frac{\Delta v_y}{\Delta t} \right\rangle = \langle a_{\text{ave},x}, a_{\text{ave},y} \rangle. \tag{D.4}$$

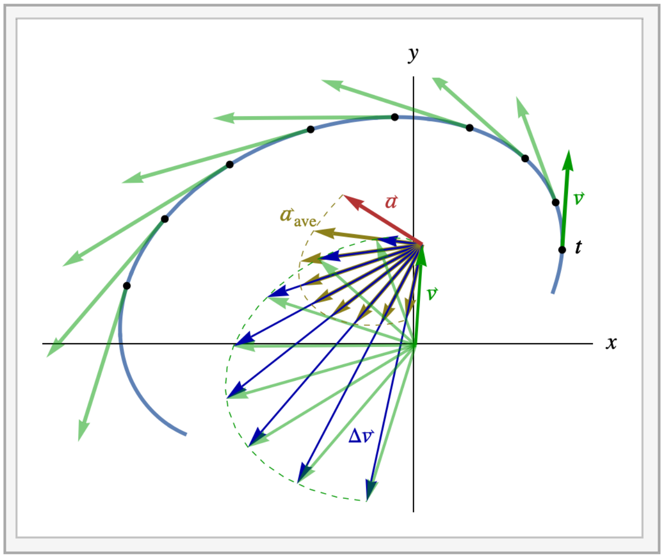


Interactive Figure - To find $\Delta\vec{v}$, you must redraw the velocities with a common tail. The average acceleration can then be found $\vec{a}_{\text{ave}} = \Delta\vec{v} / \Delta t$.

Instantaneous Acceleration

The instantaneous acceleration is similarly defined as a limit of the average acceleration or simply as the time derivative of the velocity.

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \left\langle \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t}, \lim_{\Delta t \to 0} \frac{\Delta v_y}{\Delta t} \right\rangle = \langle a_x, a_y \rangle. \tag{D.5}$$



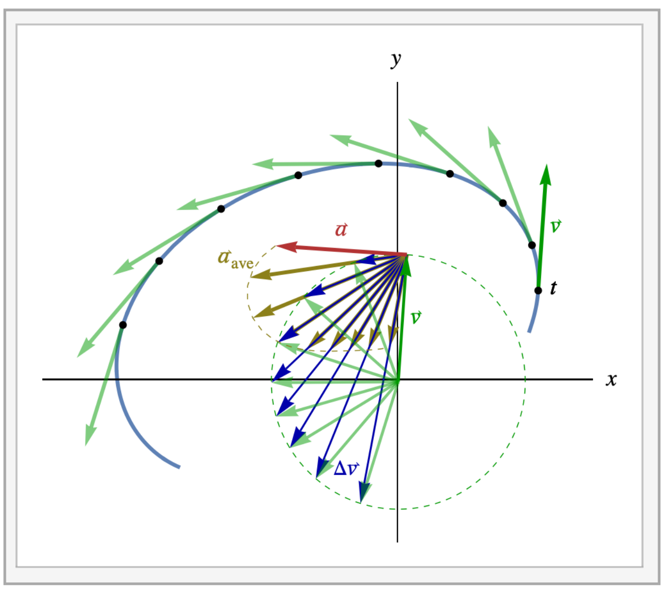
Interactive Figure - The instantaneous acceleration is the limit of the average acceleration as $\Delta t \to 0$.

Acceleration with Constant Speed

The acceleration is the time derivative of the velocity. Constant velocity implies zero acceleration. Speed is the magnitude of the acceleration so there can be an acceleration if the speed is constant, because the direction can change. If the speed is constant then the acceleration must be perpendicular to the direction of motion, which is the direction of the velocity.

$$\text{constant speed} \quad \Longleftrightarrow \quad \vec{a} \perp \vec{v}$$

If a car is driving at a constant speed and turning left then the acceleration is toward the left.



Interactive Figure - For motion with constant speed, the acceleration is perpendicular to the velocity. Note that when the velocity vectors are redrawn with a common tail, their tips are on a circle with the speed as its radius. As $\Delta t \to 0$, the average acceleration approaches the instantaneous acceleration which becomes perpendicular to the velocity.

Parallel and Perpendicular Components of Acceleration

If the speed is changing then there is a forward or backward component of the acceleration, where an increasing speed gives a forward component and a decreasing speed gives a backward component. If the speed and direction are changing then you have both components parallel and perpendicular to the direction of motion. Suppose a car is braking while turning left. There are then two components of the acceleration, the parallel component points backward and the perpendicular component points to the right.

D.2 - Constant Velocity and Acceleration

The cases of constant velocity and acceleration follows the one dimensional case. Since we are now dealing with vectors the initial position and initial velocity, introduced previously, become vector quantities.

$$\text{const}\ \vec{v} \quad \Longrightarrow \quad \vec{a} = \vec{0} \ \text{ and } \ \vec{r}(t) = \vec{r}_0 + \vec{v}\, t$$

$$\text{const}\ \vec{a} \quad \Longrightarrow \quad \vec{v}(t) = \vec{v}_0 + \vec{a}\, t \ \text{ and } \ \vec{r}(t) = \vec{r}_0 + \vec{v}_0\, t + \frac{1}{2}\, \vec{a}\, t^2$$

For 2D motion with constant acceleration we can modify the four constant acceleration equations from Chapter 2 into two sets of four equations.

$$\begin{array}{cc}
\boldsymbol{x}\ \textbf{Equations} & \boldsymbol{y}\ \textbf{Equations} \\
v_x = v_{0x} + a_x\, t & v_y = v_{0y} + a_y\, t \\
\Delta x = \frac{1}{2}\,(v_x + v_{0x})\, t & \Delta y = \frac{1}{2}\,(v_y + v_{0y})\, t \\
\Delta x = v_{0x}\, t\ +\ \frac{1}{2}\, a_x\, t^2 & \Delta y = v_{0y}\, t\ +\ \frac{1}{2}\, a_y\, t^2 \\
v_x^2 = v_{0x}^2 + 2\, a_x\, \Delta x & v_y^2 = v_{0y}^2 + 2\, a_y\, \Delta y.
\end{array}$$

Projectile Motion

An important case of motion with constant acceleration is that of projectile motion. Projectile motion is two dimensional motion of body under the influence of only gravity. If gravity is all that acts means that we ignore air resistance. Spin effects like a curving baseball or a slicing golf ball are associated with air resistance and will also be ignored.

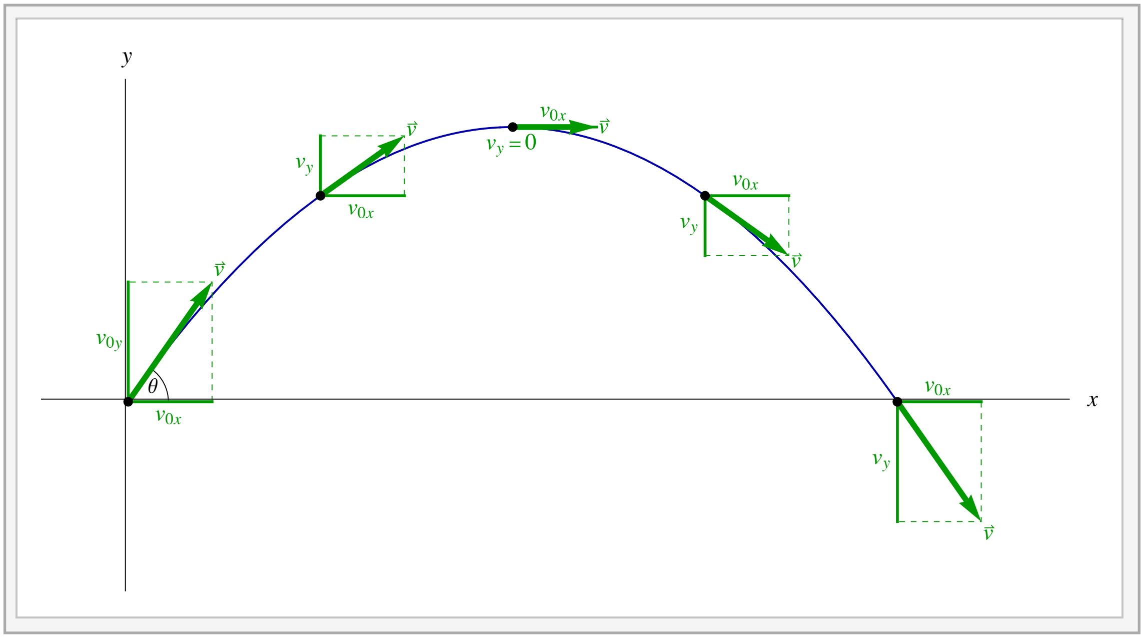
For projectile motion we will take x as the horizontal direction and y as the vertical direction. The acceleration due to gravity is downward and of magnitude g. Writing this as a vector gives

$$\vec{a} = -g\, \hat{y} = \langle 0, -g \rangle \quad \text{or} \quad a_x = 0 \ \text{ and } \ a_y = -g$$

Inserting these components into the two sets of equations above gives:

$$\begin{array}{cc}
\textbf{Horizontal} & \textbf{Vertical} \\
\textbf{Equations} & \textbf{Equations} \\
v_x = v_{0x} & v_y = v_{0y} - g\ t \\
\Delta x = v_{0x}\, t & \Delta y = \frac{1}{2}\,(v_y + v_{0y})\, t \\
 & \Delta y = v_{0y}\, t\ -\ \frac{1}{2}\, g\, t^2 \\
 & v_y^2 = v_{0y}^2 - 2\, g\, \Delta y.
\end{array} \tag{D.6}$$

The horizontal motion is simple. Since there is no horizontal (the x direction) acceleration, the x component of the velocity is constant. The vertical part (the y direction) of the motion is equivalent to free fall. The key to solving projectile motion problems is keeping the two parts separate.



Interactive Figure - Projectile motion with a fixed initial speed and a varying launch angle.

If a projectile is launched at an initial angle of θ with an initial speed v_0 then the components of the initial velocity are given by

$$v_{0x} = v_0 \cos\theta \text{ and } v_{0y} = v_0 \sin\theta.$$

Example D.1 - An Initial Horizontal Velocity

A spring gun shoots a small ball with an initial horizontal velocity from a height of 1.35 m at the same time an identical ball is dropped from the same height. The ball that was shot lands a horizontal distance of 2.42 m from its initial position.

(a) Which ball hits the ground first?

Solution

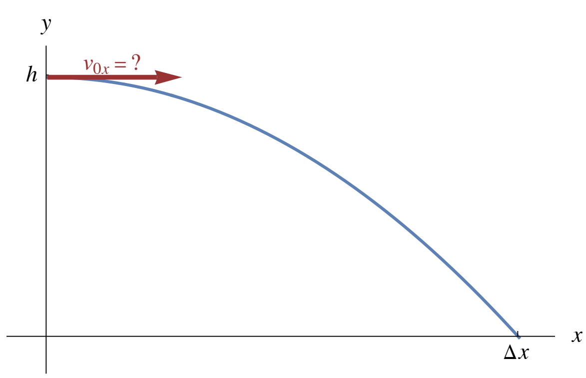
For both balls, the y-component of the initial velocity is zero. The shot ball's velocity has an initial x-component and the dropped ball doesn't. The time of fall is determined only by the vertical equations and the vertical motion of the two is identical. It follows that the two balls hit the floor at the same time.

(b) What is the initial speed of the shot ball?

Solution

First define our variables in terms of the given information.

$$h = 1.35 \text{ m}, \quad \Delta x = 2.42 \text{ m and } g = 9.80 \text{ m}/\text{s}^2$$



The initial speed is also the x-component of the initial velocity.

$$v_0 = \sqrt{v_{0x}^2 + v_{0y}^2} = \sqrt{v_{0x}^2 + 0^2} = v_{0x}$$

First we solve for its time of fall using the vertical equations using $\Delta y = -h$.

$$\Delta y = v_{0y}\, t - \frac{1}{2}\, g\, t^2 \implies -h = 0 - \frac{1}{2}\, g\, t^2 \implies h = \frac{1}{2}\, g\, t^2 \implies t = \sqrt{\frac{2\, h}{g}} = 0.52489 \text{ s}$$

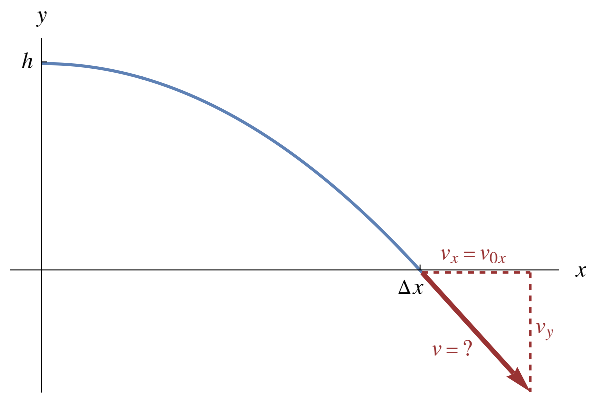
From this we can use the horizontal equation to get v_{0x}.

$$\Delta x = v_{0x}\, t \implies v_{0x} = \Delta x / t = 4.67 \text{ m/s}$$

This is a very standard physics problem. In this class of projectile problems with an initial horizontal velocity, there are three variables: h, Δx and v_{0x}. You are two and asked for the third.

(c) What was the speed of the ball when it hit the floor?

Solution



The x-component of the balls velocity stays constant so $v_x = v_{0x}$. The speed is the magnitude of the velocity.

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{v_{0x}^2 + v_y^2}$$

Using the time and the vertical equations we can find the y-component of the velocity.

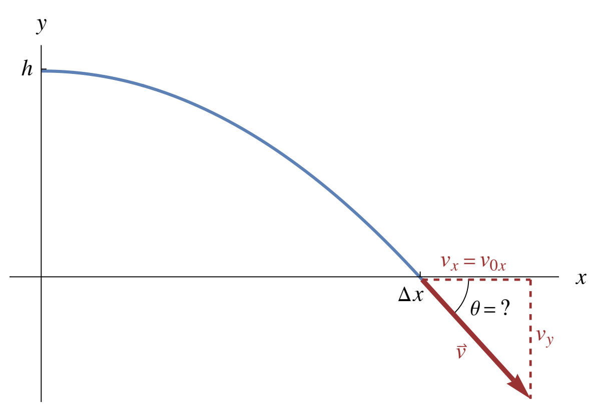
$$v_y = v_{0y} - g\ t = -g\, t = -5.1438 \text{ m/s}$$

and from this we find the speed.

$$v = \sqrt{v_{0x}^2 + v_y^2} = 6.95 \text{ m/s}$$

(d) What was the ball's direction of motion when it hits the floor? Give the angle.

Solution

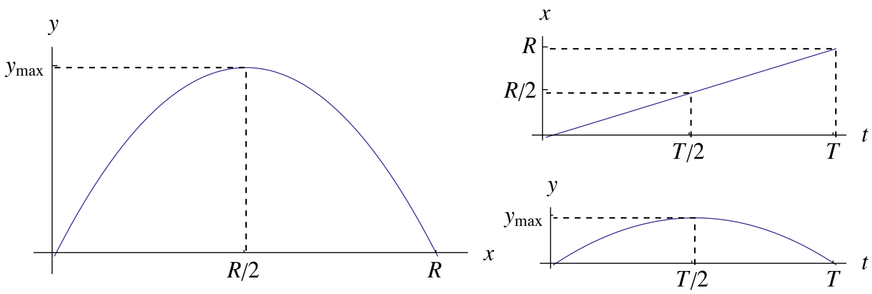


In part (c) we found the magnitude of the velocity. Here we want the direction angle of the same velocity. Recall that the direction of motion is the direction of the velocity vector.

$$\theta = \tan^{-1} \frac{v_y}{v_{0x}} = -47.7\,°$$

Example D.2 - Punted Football

A punter kicks a football from ground level with an initial speed of 27 m/s at an initial angle of 63 °. (Neglect all air resistance.)



(a) What is the maximum height reached by the football?

Solution

Begin with identifying what is given. We know the initial speed and initial angle.

$$v_0 = 27\ \mathrm{m/s} \quad \text{and} \quad \theta = 63\,°$$

We can calculate the components of the initial velocity.

$$v_{0x} = v_0 \cos\theta = 12.258\ \mathrm{m/s} \ \text{ and } \ v_{0y} = v_0 \sin\theta = 24.057\ \mathrm{m/s}$$

The maximum height depends only on v_{0y}. At the highest point the velocity is purely horizontal, so $v_y = 0$. We are looking for y_{max} in the graphs above. $y_{max} = \Delta y$ when $v_y = 0$. We need the fourth of the vertical projectile equations.

$$v_y^2 = v_{0y}^2 - 2\,g\,\Delta y \implies 0 = v_{0y}^2 - 2\,g\,y_{max} \implies y_{max} = \frac{v_{0y}^2}{2\,g} = 29.5\ \mathrm{m}$$

(b) What is the "hang-time" of the punt? (Note that hang-time is the total time the football is in the air, from the ground to the ground.)

Solution

Since the football ends at the same elevation where it began, so $\Delta y = 0$. We need to solve for time so the relevant formula is the third vertical projectile equation.

$$\Delta y = v_{0y}\,t - \frac{1}{2}\,g\,t^2 \implies 0 = v_{0y}\,t - \frac{1}{2}\,g\,t^2 = t\left(v_{0y} - \frac{1}{2}\,g\,t\right)$$

This gives two solutions but $t = 0$ is trivially true. We want the other. We will call this T as shown in the graphs above.

$$0 = v_{0y} - \frac{1}{2}\,g\,T \implies T = \frac{2\,v_{0y}}{g} = 4.9096\ \mathrm{s} = 4.91\ \mathrm{s}$$

(c) What is the horizontal range of the football? The range is the total horizontal distance the football travels in the air.

Solution

So far we have not used any horizontal information or equations. The *range* R is Δx when $\Delta y = 0$. Using the time T we just found in the second horizontal projectile equations.

$$R = \Delta x = v_{0x}\,t = v_{0x}\,T = 60.2\ \mathrm{m}$$

The *range* R of a projectile is the total horizontal distance traveled in the air when it returns to its original level, $\Delta y = 0$. In the punted football example above we saw the the total time to return to the same level was $T = 2\,v_{0y}/g$ and the range R was $R = \Delta x = v_{0x}\,t = v_{0x}\,T$. Combining these expression we get a general expression for the range.

$$T = \frac{2\,v_{0y}}{g} \ \text{ and } \ R = \Delta x = v_{0x}\,t = v_{0x}\,T \implies R = v_{0x}\,\frac{2\ v_{0y}}{g} = \frac{2\,v_{0x}\,v_{0y}}{g} = \frac{2\,v_0\cos\theta\,v_0\sin\theta}{g} = \frac{v_0^2}{g}\,2\cos\theta\sin\theta.$$

There is a trig identity for the sine of twice an angle: $\sin(2\,\theta) = 2\sin\theta\cos\theta$. This gives an expression for the range.

$$R = \frac{v_0^2}{g}\sin(2\,\theta) \tag{D.7}$$

Remember that this *only* applies when $\Delta y = 0$.

Example D.3 - Punted Football (continued)

(d) Apply the range formula to the football example to get the same answer as in part (c).

Solution

$$R = \frac{v_0^2}{g}\sin 2\,\theta = 60.2\text{ m}$$