Chapter F

Applications of Newton's Laws

Blinn College - Physics 1401 - Terry Honan

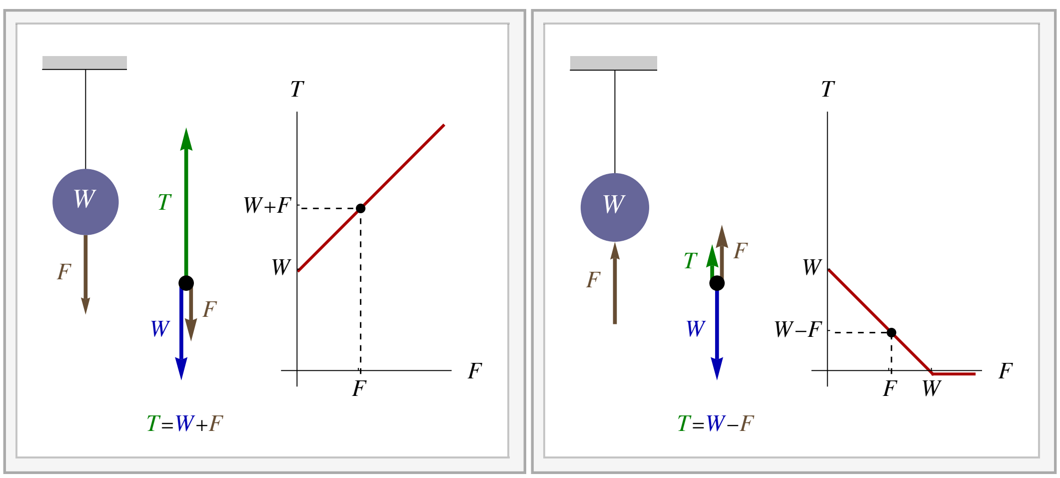
This chapter discusses the kinematics and dynamics of circular motion, and considers the problem of motion in an accelerated frame of reference. More applications on the second law will also be presented.

F.1 - Forces of Constraint

Often in mechanics problems the motion of objects is constrained. A force of constraint is responsible for keeping the constraint; it will take on whatever value is needed to maintain its constraint. Often forces of constraint satisfy inequalities. These vague general comments will become clearer as examples are given. The examples we will discuss here are tension forces, normal forces and static friction forces.

Tension Forces as Constraints

Tension is a force of constraint. The constraint associated with tension is the condition that the length of the rope stays constant. In our examples with ropes and two masses, this condition is what related the accelerations of the masses. In Atwood's machine the string not stretching meant that the two masses moved the same distance, implying the accelerations were the same magnitude. In the example with two pulleys the mass that moved twice the distance of the other had twice the acceleration. These conditions eventually lead to a value for the tension forces.



Interactive Figure

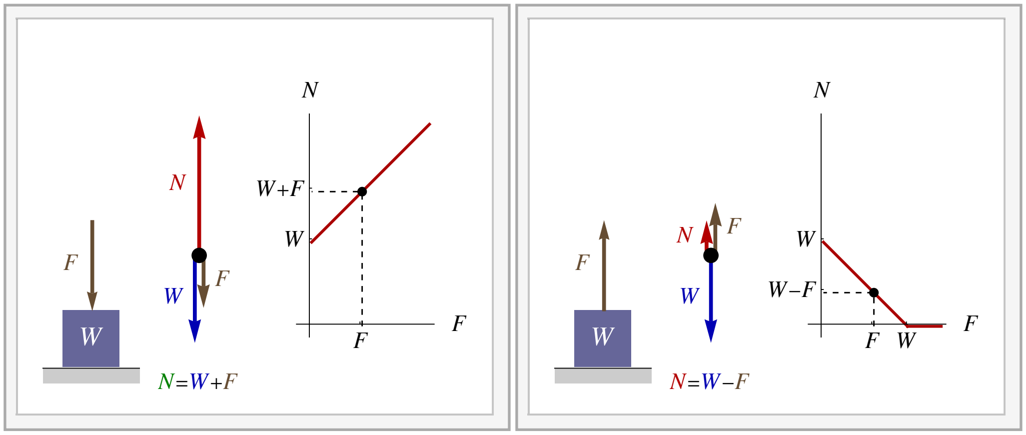
The interactive figures above illustrate tension as a force of constraint. Both show a weight W hanging from a string or rope. On the left an applied force F pulls downward on the rope. The larger F gets the larger $T = W + F$ gets; the tension T will take on whatever value is needed to maintain the constraint. The figure on the right shows the same hanging weight but now the applied force acts upward. As the applied force F gets larger $T = W - F$ decreases. It is clear that if F is larger than the weight W, then the string will lost its tension and the mass will accelerate upward. To maintain the constraint the tension would need to be negative, but it cannot. The tension force satisfies a simple inequality

$$T \geq 0.$$

This is equivalent to saying: "You can't push with a rope." Note that negative tensions are possible for rigid objects, like a stick.

Normal Forces as Constraints

The normal force is also a force of constraint. The constraint is that there is no motion through the surface.



Interactive Figure

Consider a block sitting on a floor as in the interactive figures above. Without an applied force, the normal force is the weight of the book W. If one pushes down on the book with an applied force F, as in the left figure, then the floor pushes up with a larger normal force of $N = W + F$; as F gets larger, N gets larger. If a small force F lifts on the block, as described by the figure on the right, then the normal force $N = W - F$ decreases as F increases. If the lifting force exceeds the weight the normal force the book will lift; N would have to be negative to maintain the inequality. A surface can only push away from it

$$N \geq 0.$$

A negative normal force would describe suction or glue.

Because the normal force is a constraint it is *always* found by solving the second law perpendicular to the surface.

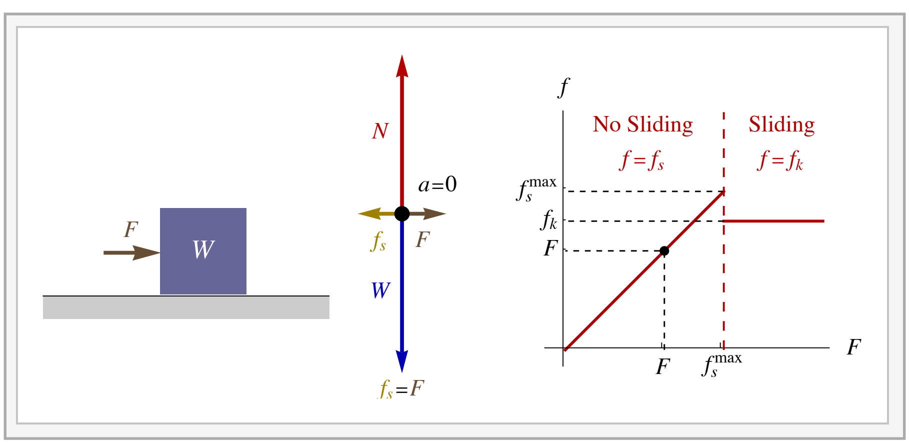
$$F_{\text{net},\perp} = m\, a_{\perp}$$

One does not just insert a value of N into a problem solution; it is something you calculate using the above expression.

Friction Between Surfaces

Recall that forces from surfaces break up into two components: the perpendicular component is the normal force and the parallel component is friction. There are two types of friction between surfaces, static and kinetic. Static friction is the case when there is no sliding. Kinetic friction is when there is sliding. These are very different things. Static friction is a force of constraint. Kinetic friction is a dissipative force.

Static Friction



Interactive Figure

The interactive figure above shows a horizontal pushing force F acting on a block of weight W. The force of static friction f_s will keep the block from sliding. It must then act opposite to F and must have the same magnitude, $f_s = F$. As F increases, so does f_s. It should be clear that if F is large enough the block will slide. There is a largest value of the force of static friction $f_s^{\max}$. When the block slides then there is no longer static friction and we get kinetic friction f_k.

Static friction is a force of constraint; the constraint is that there is no sliding. No sliding implies that the accelerations of the two surfaces are the same. Since it is a constraint we always solve for its value by using

$$F_{\text{net},\parallel} = m\, a_{\parallel}.$$

There is an upper limit on static friction. The constraint is that the maximum value is proportional to the normal force.

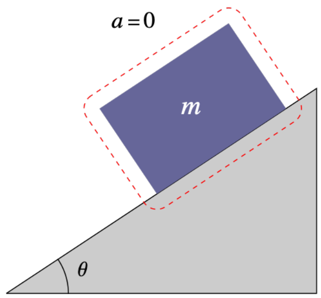
$$f_s \leq \mu_s N$$

μ_s is a dimensionless constant called the coefficient of static friction; it is a property of the two surfaces in contact. At the critical point between sliding and not the inequality is saturated; that is it become equal: $f_s^{\max} = \mu_s N$.

We will demonstrate static friction with two examples: one with no acceleration and another with acceleration.

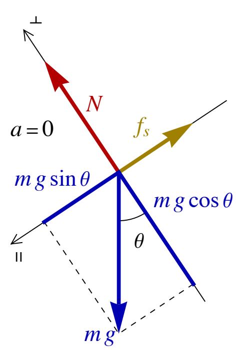
Example F.1 - Stationary Block on an Incline

To measure the coefficient of static friction between a block and wood plank a student finds the largest angle the plane can have while the block does not slide. If this largest angle is measured to be 28°, then what is the coefficient of static friction between the block and the surface.



Solution

First we draw the free-body diagram. All that is touching the block is the surface, so the surface provides the only contact forces. Perpendicular to the surface is the normal force N. Now that we have friction we also must include the parallel component, which is static friction f_s. To find the direction of the static friction force ask what will happen without it; f_s must act up the incline to prevent it from sliding down. We break up the weight mg into components as is typical for problems involving inclined planes. The condition that the block is not sliding implies, since the surface is stationary, that the acceleration is zero.



Applying the second law to our pair of perpendicular directions gives expressions for the static friction force and the normal force.

$$F_{\text{net},\parallel} = m\, a_{\parallel} \implies m\,g \sin\theta - f_s = 0 \implies f_s = m\,g \sin\theta$$

$$F_{\text{net},\perp} = m\, a_{\perp} \implies N - m\,g \cos\theta = 0 \implies N = m\,g \cos\theta$$

We then use our constraint inequality $f_s \leq \mu_s N$, which is the condition for the block not sliding. We then insert the above values for f_s and N. (Be careful when doing algebra with inequalities; multiplying or dividing by a negative changes the direction of the inequality. Here we are diving by only positive quantities.)

$$f_s \leq \mu_s N \implies m\,g \sin\theta \leq \mu_s\, m\,g \cos\theta \implies \tan\theta \leq \mu_s$$

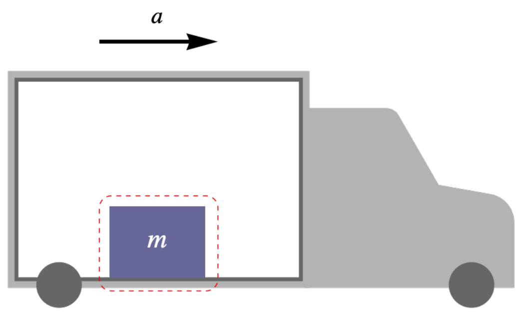
At $\theta_{\max}$, the largest angle without sliding, we saturate the inequality and get our result.

$$\mu_s = \tan\theta_{\max} = \tan 28° = 0.532$$

Note that the mass was not given but that scaled out of the problem.

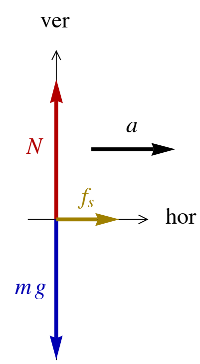
Example F.2 - Crate on Flat Accelerating Surface

A truck with a forward acceleration of $2.5\ \mathrm{m}/\mathrm{s}^2$ carries a crate inside. If coefficient of static friction between the crate and the floor is 0.30, will the crate slide?



Solution

The only thing touching the crate is the truck's floor; the only contact forces are the two surface forces, the normal force and, assuming no sliding, static friction. The direction of the static friction is forward. Without friction, the crate slides backward and friction prevents that. The only other force to include in our free-body diagram is the weight.



Again, applying the second law to our pair of perpendicular directions gives expressions for the static friction force and the normal force.

$$F_{\mathrm{net},\,\parallel} = m\,a_{\parallel} \implies f_s = m\,a$$

$$F_{\mathrm{net},\,\perp} = m\,a_{\perp} \implies N - m\,g = 0 \implies N = m\,g$$

The constraint inequality $f_s \le \mu_s N$, is the condition for the block not sliding. Inserting the information above gives the condition.

$$f_s \le \mu_s N \implies m\,a \le \mu_s\, m\,g \implies a \le \mu_s\, g$$

$$a = 2.5\ \mathrm{m}/\mathrm{s}^2,\ \ \mu_s = 0.30\ \text{ and } g = 9.80\ \mathrm{m}/\mathrm{s}^2 \implies 2.5 \overset{?}{\le} 2.94$$

The question mark above the inequality is used because it is a test, not a mathematical statement. Since the test *is* satisfied the crate *will not slide*.

F.2 - Dissipative Friction

When moving with respect to a surface or medium there is a resistive friction force that opposes the relative motion of the object to the surface or medium. If $\hat{v}$ is the unit vector in the direction of the relative velocity of the object (to the surface or medium) then the friction force is opposite that direction.

$$\vec{f} = -f\,\hat{v}$$

The values of f for different cases will be discussed below.

Kinetic Friction

Kinetic friction is the dissipative force when there is sliding between two surfaces. Its direction opposes the direction of the sliding. The magnitude of the force of kinetic friction is fixed at the value

$$f_k = \mu_k N.$$

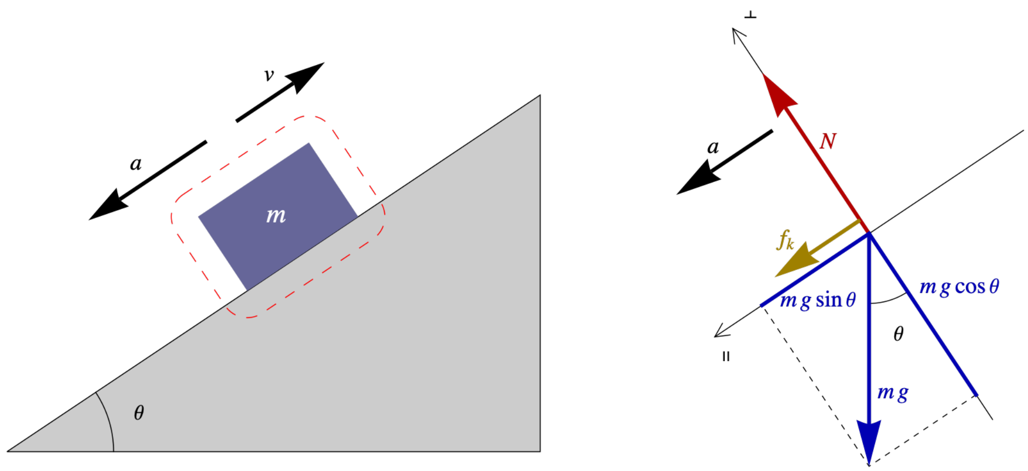
μ_k is the coefficient of kinetic friction. Like the static friction case, it is a property of the two surfaces.

Example F.3 - Block Sliding on an Incline

A block is pushed and released at the bottom of an incline at angle θ. It slides up the incline, stops for an instant and then slides down the incline. The coefficient of kinetic friction between the block and the incline is μ_k.

(a) When sliding up the incline what is its acceleration? Take an acceleration down the incline to be a positive a.

Solution



Applying the second law the direction perpendicular to the surface gives the normal force.

$$F_{\text{net},\perp} = m\,a_\perp \implies N - m\,g\cos\theta = 0 \implies N = m\,g\cos\theta$$

The parallel component equation of the second law gives an expression for a.

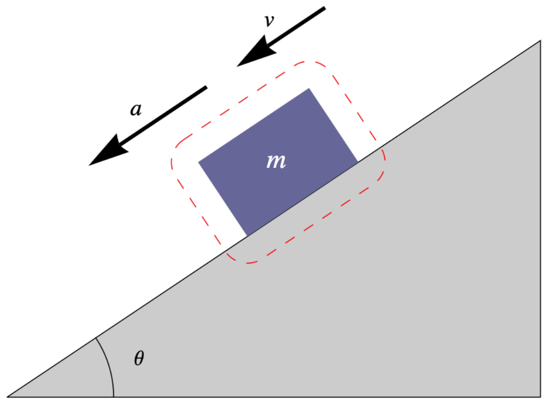
$$F_{\text{net},\parallel} = m\,a_\parallel \implies m\,g\sin\theta + f_k = m\,a$$

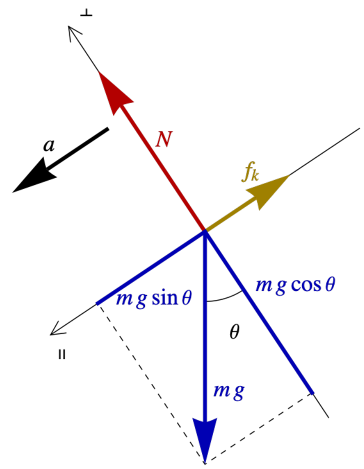
We then use our expression for kinetic friction $f_k = \mu_k N$. Combining this with our expression for N gives $f_k = \mu_k\,m\,g\cos\theta$. We can then solve for a by dividing out the mass.

$$m\,g\sin\theta + \mu_k\,m\,g\cos\theta = m\,a \implies a = g\sin\theta + \mu_k\,g\cos\theta$$

(b) When sliding down the incline what is its acceleration? Take an acceleration down the incline to be a positive a.

Solution





This part is identical to the case above except the kinetic friction is now up the incline and thus comes in with the opposite sign.

$$F_{\text{net},\perp} = m\,a_\perp \implies N - m\,g\cos\theta = 0 \implies N = m\,g\cos\theta$$

$$F_{\text{net},\parallel} = m\,a_\parallel \implies m\,g\sin\theta - f_k = m\,a$$

Since the normal force is the same as before, the kinetic friction force is the same. This gives:

$$a = g\sin\theta - \mu_k\,g\cos\theta$$

Friction in a Medium

When an object moves through a fluid, a liquid or a gas, it experiences a friction force with a magnitude that varies with the speed. This is in contrast to kinetic friction which has a fixed magnitude independent of the speed. There are two simple ways to model friction in a medium: viscous friction, which is proportional to the speed and quadratic drag, which is proportional to the speed squared.

Viscous Friction

Typically, at low speeds when the fluid flow around the object is laminar (smooth) the friction behaves as viscous friction.

$$f = b\,v$$

v is the speed and the constant b depends on the viscosity of the fluid and the geometry and surface texture of the object.

Quadratic Drag

At speeds higher than where the fluid flow around the object becomes turbulent the friction force becomes quadratic drag.

$$f = c\,v^2$$

The constant c depends on the viscosity of the fluid and the geometry and surface texture of the object.

F.3 - Springs and Hooke’s Law

Hooke's law is the force law for a spring. Define $x = 0$ to be the relaxed position of a spring, the equilibrium position, and let x be the amount the spring is stretched from equilibrium. Hooke's law states that there is a proportionality between F and x, $F \propto x$. We can introduce a constant of proportionality k called the spring constant or force constant.

$$F = k\,x \ \text{(Force on spring)}$$

The spring constant is a property of a particular spring; a stiff spring has a large k and a loose spring has a small k. In this expression F is the force stretching the spring. Usually we consider the force of the spring itself when considering Hooke's law. By Newton's third law this is the negative of the force on the spring and we get

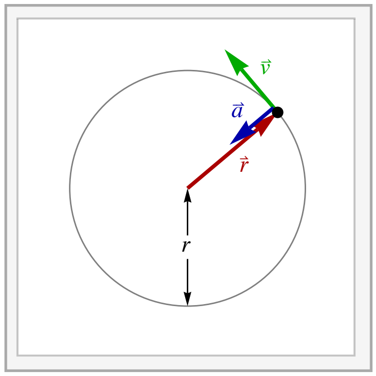
$$F = -k\,x \ \text{(Force of spring).}$$

Note that when the spring is stretched, x is positive and the force is negative. When the spring is compressed the force is positive and x is negative.

A force is said to be *elastic* when it satisfies Hooke's law. We will see that elasticity is quite common for sufficiently small deformations of almost anything. What is special about springs is they maintain their elasticity over large deformations. It should be clear that even a spring will violate Hooke's law when stretched a very large distance.

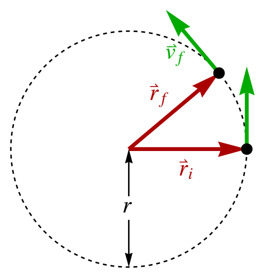
F.4 - Kinematics of Circular Motion

Uniform Circular Motion - Centripetal Acceleration



Interactive Figure

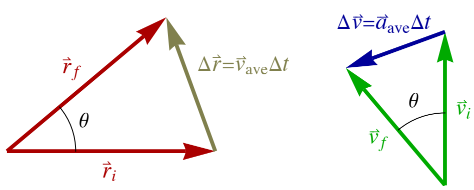
We will first consider the case of uniform circular motion, where uniform means that the speed stays constant. Even with a constant speed there an acceleration, since the direction is changing. To find the acceleration we will first consider a finite time Δt and the average velocity and average acceleration. We will then let the time approach zero; the average values will then approach the instantaneous values.



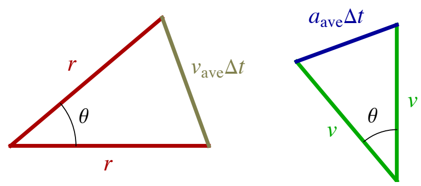
Take $\vec{r}_i$ and $\vec{r}_f$ to be the initial and final position vectors at times t_i and t_f where $\Delta t = t_f - t_i$. The velocities at these times are $\vec{v}_i$ and $\vec{v}_f$; since the speed is constant these have the same magnitude.

Take the angle between the two position vectors as θ. In circular motion the velocity is perpendicular to the position vector; it follows that θ is also the angle between the velocities. The sides opposite the angle θ in the triangles are

$$\Delta \vec{r} = \vec{r}_f - \vec{r}_i = \vec{v}_{\text{ave}}\,\Delta t \text{ and } \Delta \vec{v} = \vec{v}_f - \vec{v}_i = \vec{a}_{\text{ave}}\,\Delta t$$



Looking at just the magnitudes of these vectors we see a pair of isosceles triangles and since the angle is the same for both, we have similar triangles.



Comparing the ratios of similar sides of similar triangles gives

$$\frac{a_{ave}\,\Delta t}{v_{ave}\,\Delta t} = \frac{v}{r}.$$

This can be rewritten as

$$a_{ave} = \frac{v\,v_{ave}}{r}.$$

As $\Delta t \to 0$ the average velocity and average acceleration approach their instantaneous values, so the magnitudes of the average quantities approach the instantaneous magnitudes.

$$v_{ave} \to v \text{ and } a_{ave} \to a$$

With this we can write the value of the instantaneous acceleration as

$$a_c = \frac{v^2}{r}.$$

As $\Delta t \to 0$ it is also true that $\theta \to 0$. It follows that the velocity and acceleration vectors are perpendicular. The direction of the acceleration is toward the center. We will refer to this direction as centripetal and denote the centripetal acceleration as a_c.

Many modern texts avoid the use of the term centripetal and have adopted the term radial instead. The radial direction could be outward or inward and the term centripetal is unambiguously toward the center. The reason for avoiding the word centripetal is because students confuse it with centrifugal. The centrifugal force is the outward false force associated with the centripetal acceleration.

Uniform Circular Motion - Period, Speed and Radius

For uniform circular motion we can relate the period, speed and the radius. This will also give us an alternative expression for the centripetal acceleration in terms of the period. The period T is defined as the time needed for each full revolution.

It is a simple matter to relate the period to the radius and speed. Since the speed is uniform, is the same as the average speed; this is the distance traveled divided by the total time. The distance traveled in one period is one circumference. It follows that

$$v = \frac{2\pi r}{T}.$$

Inserting this into our expression for the centripetal acceleration $a_c = v^2/r$ gives

$$a_c = \left(\frac{2\pi}{T}\right)^2 r.$$

This will prove to be a useful expression. The student should keep in mind that this *only* applies to uniform circular motion.

Example F.4 - Centripetal Acceleration at the Equator

The radius of the Earth is 6.38×10^6 m. What is the acceleration of a point on the equator?

Solution

Since the period of rotation of the Earth is 1 day, we can easily find a_c.

$$r = R_E = 6.38\times10^6 \text{ m and } T = 24\times3600 \text{ s} = 8640 \text{ s} \implies a_c = \left(\frac{2\pi}{T}\right)^2 r = 0.0337 \text{ m/s}^2$$

Note that this is smaller than g but it is not negligible. When g is measured in free fall, the centrifugal contribution is included.

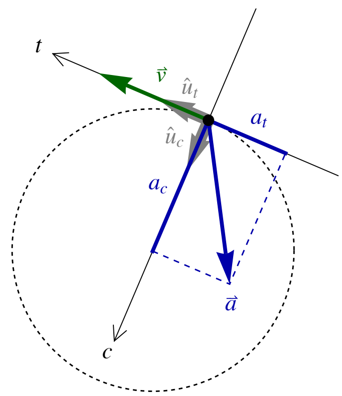
General Circular Motion - Tangential Acceleration

Now consider the more general case of circular motion where we allow the speed to change. Generally, the component of the acceleration perpendicular to the motion is related to the change in direction and the component parallel to the direction of motion is related to the change in speed.

The component of the acceleration in the centripetal direction must be the same as in the uniform case.

$$a_c = \frac{v^2}{r}.$$

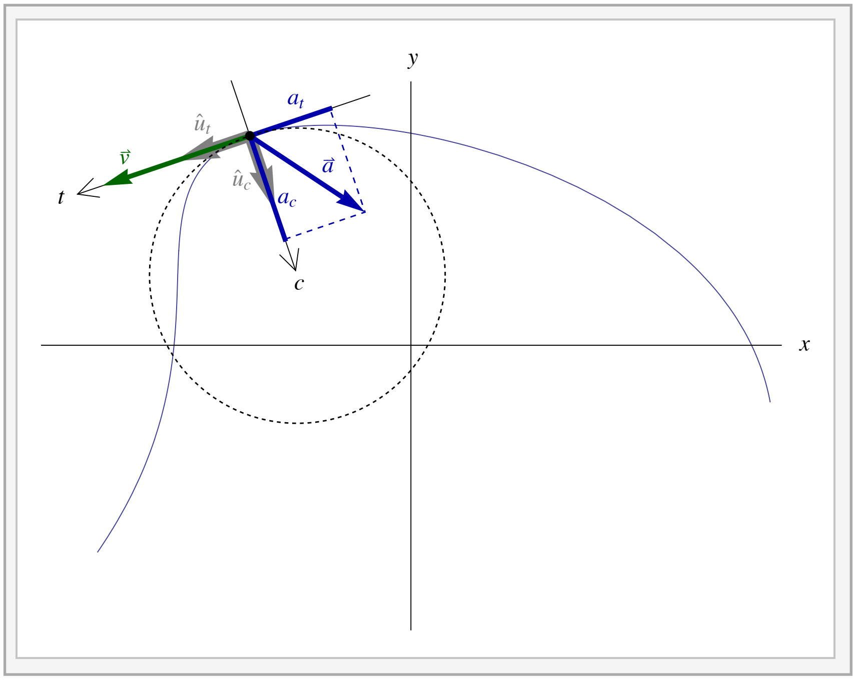
The tangential direction is the direction of the velocity. If the speed is changing then there is also a tangential component of the acceleration a_t. If the speed is increasing the tangential component is positive $a_t > 0$ and if decreasing it is negative $a_t < 0$.



decreasing speed $\Longrightarrow$ $a_t < 0$

General Two Dimensional Motion and the Effective Radius

Now we consider the most general problem of motion in a plane. Any trajectory may be approximated by a circle at some position. We refer to the radius of that circle as the effective radius. We may choose centripetal and tangential directions as before and apply the above expression for the acceleration to the general two dimensional problem. If the direction is not changing at some instant, then the effective radius is infinite; the centripetal acceleration then becomes zero.



Interactive Figure

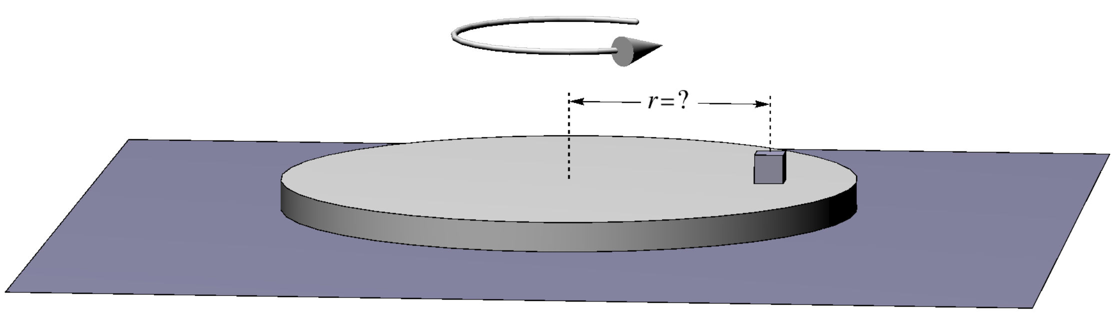
F.5 - Dynamics of Circular Motion

Newton's Second Law $\vec{F}_{\text{net}} = m\vec{a}$ is the foundation of classical mechanics and, of course, it must still apply when we have circular motion. We now have new formulas for the acceleration. We will now consider examples with circular motion.

Examples with Uniform Circular Motion

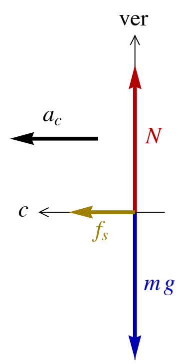
Example F.5 - Block on a Rotating Disk

A block sits on a turntable that rotates about its center once every 2.5 s. If the coefficient of static friction is 0.28, then what is the largest distance r that the block can be from the center without slipping?



Solution

We know the period $T = 2.5$ s and the coefficient of static friction $\mu_s = 0.28$. We are not given the mass so it must cancel out in the algebra; call the mass m.



The only thing touching the block is the surface so we only have the two contact forces of a surface, friction (static here) and the normal force. Now we apply the second law to our perpendicular directions: the centripetal direction is horizontal and parallel to the surface (c = hor = $\parallel$) and vertical is perpendicular to the surface (ver = $\perp$).

$$F_{\text{net},c} = m\,a_c \implies f_s = m\,a_c$$

$$F_{\text{net,ver}} = m\,a_{\text{ver}} = 0 \implies N = m\,g$$

From the period we can find the centripetal acceleration .

$$a_c = \left(\frac{2\,\pi}{T}\right)^2 r$$

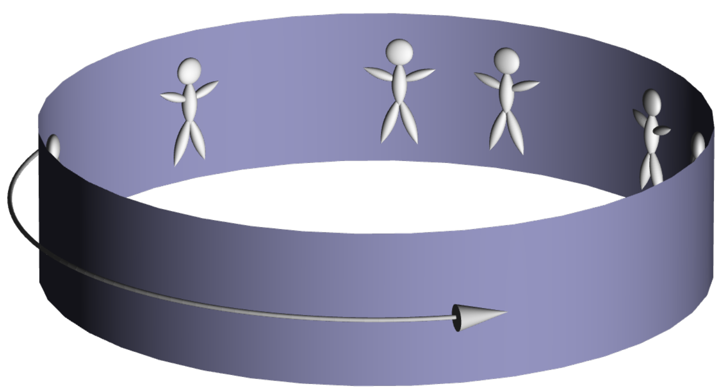
The static friction inequality and the expressions above give us the condition for not sliding.

$$f_s \le \mu_s\,N \implies m\,a_c \le \mu_s\,m\,g \implies a_c \le \mu_s\,g \implies \left(\frac{2\,\pi}{T}\right)^2 r \le \mu_s\,g$$

The maximum radius saturates this inequality. Using $\mu_s = 0.28$, $g = 9.80\ \text{m}/\text{s}^2$ and $T = 2.5\ \text{s}$ gives our result.

$$\left(\frac{2\,\pi}{T}\right)^2 r_{\max} = \mu_s\,g \implies r_{\max} = \frac{\mu_s\,g}{(2\,\pi/T)^2} = 0.434\ \text{m}$$

Example F.6 - Rotating Vertical Cylinder in an Amusement Park

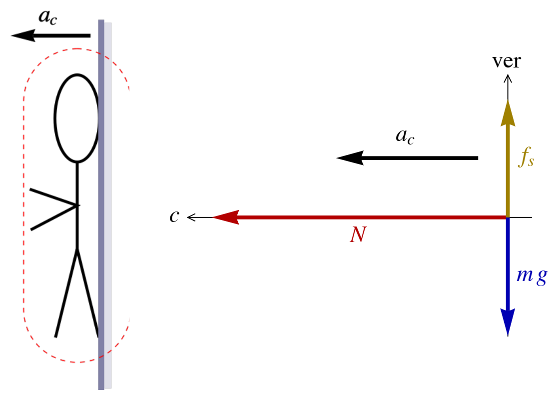


A common amusement park ride consists of vertical cylinder that rotates. Initially, the riders stand on a floor and lean against the inside surface of the cylinder as it starts to rotate. When the rotation is sufficiently rapid, the floor is dropped out and the riders stay pinned against the wall.

To design such a ride you are given the radius of the cylinder R and an estimate of the smallest coefficient of static friction μ_s between a person and the wall. What is the minimum speed of the cylinder's surface for the riders to stay safe and not slide?

Solution

As with the previous example, the only contact forces are due to the surface, but since here the surface is vertical the roles of the normal force and friction are reversed.



$$F_{\text{net},c} = ma_c \implies N = ma_c$$

$$F_{\text{net,ver}} = ma_{\text{ver}} = 0 \implies f_s = mg$$

$$f_s \leq \mu_s N \implies mg \leq \mu_s\, ma_c \implies g \leq \mu_s\, a_c \implies g \leq \mu_s\, \frac{v^2}{R}$$

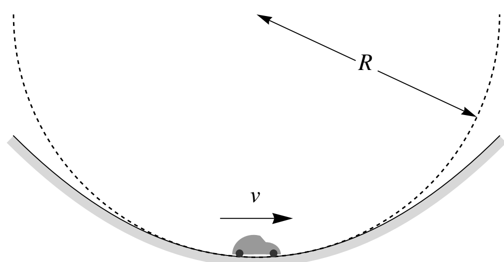
When $v = v_{\text{min}}$ we saturate the inequality and we get:

$$g = \mu_s\, \frac{v_{\text{min}}^2}{R} \implies v_{\text{min}} = \sqrt{\frac{R\,g}{\mu_s}}$$

Nonuniform Examples

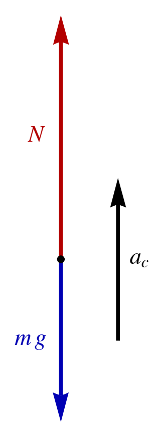
Example F.7 - Car at the Bottom of a Trough

A car of mass m drives with speed v at the bottom of a trough with an effective radius R. What is the normal force of the road on the car?



Solution

The only contact force is the normal force of the road on the car, which acts straight upward. The acceleration is centripetal, which here is upward. Applying the second law in the centripetal direction gives an expression for the normal force.



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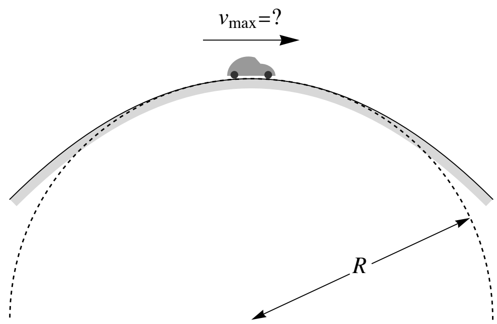
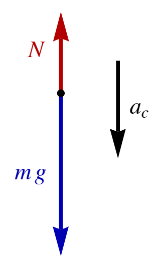
$$F_{\text{net},c} = m a_c \implies N - m g = m \frac{v^2}{R} \implies N = m g + m \frac{v^2}{R}$$

Example F.8 - Car at the Top of a Hill

A car of mass m drives at the top of a hill with an effective radius R. What is the maximum speed the car can have while staying on the road surface?



Solution



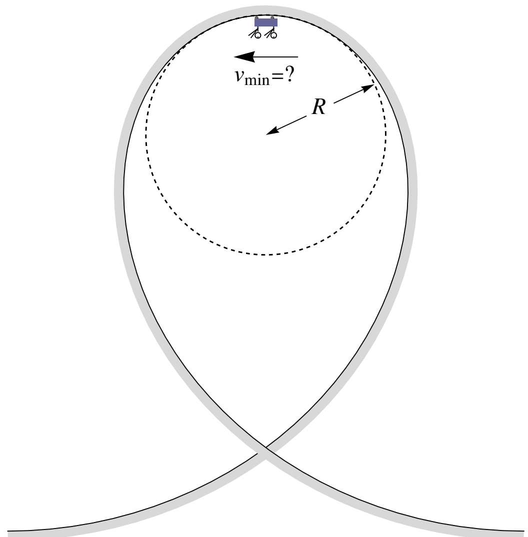
As with the preceding example the only contact force is the upward normal force. The difference is that the centripetal acceleration is now downward. Applying the second law in the centripetal direction to get the normal force.

$$F_{\text{net},c} = m a_c \implies m g - N = m \frac{v^2}{R} \implies N = m g - m \frac{v^2}{R}$$

The normal force constraint inequality $N \geq 0$ gives an inequality for the speed. Saturating these inequalities gives the maximum speed. At a higher speed the car flies off the road surface.

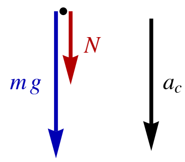
$$N \geq 0 \implies v \leq \sqrt{R g} \implies v_{\text{max}} = \sqrt{R g}$$

Example F.9 - Roller-Coaster Loop-to-Loop



What is the minimum speed of a roller coaster cart at the top of a loop with an effective radius of R?

Solution



The only contact force is the normal force of the roller coaster track on the cart, which acts straight downward. The acceleration is centripetal, which here is also downward. Applying the second law in the centripetal direction gives an expression for the normal force.

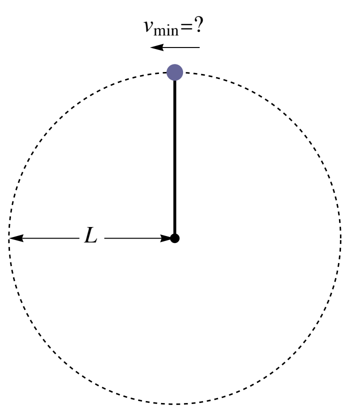
$$F_{\text{net},c} = ma_c \implies mg + N = m\,\frac{v^2}{R} \implies N = m\,\frac{v^2}{R} - mg$$

The normal force constraint inequality $N \geq 0$ gives an inequality for the speed. Saturating these inequalities gives the minimum speed.

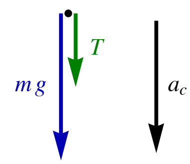
$$N \geq 0 \implies v \geq \sqrt{R\,g} \implies v_{\text{min}} = \sqrt{R\,g}$$

Example F.10 - Ball Swung in Vertical Circle

A small ball of mass m is swung in a vertical circle at the end of a string of length L with a fixed end. What minimum speed is needed at the top of the arc for the ball to not fall out of the circle.



Solution



The only contact force is the tension in the string, which acts straight down. Note that this problem is identical to the roller coaster problem above but the tension plays the role of the normal force.

$$F_{\text{net},c} = m a_c \implies m g + T = m \frac{v^2}{L} \implies T = m \frac{v^2}{L} - m g$$

The tension constraint inequality $T \geq 0$ gives an inequality for the speed. Saturating these inequalities gives the minimum speed.

$$T \geq 0 \implies v \geq \sqrt{L g} \implies v_{\text{min}} = \sqrt{L g}$$