Chapter G

Work and Kinetic Energy

Blinn College - Physics 1401 - Terry Honan

G.1 - Introduction to Work

Mechanical Advantage

In Chapter D we considered the example of a pulley system lifting a weight. We saw that using multiple pulleys one can lift a heavy object with a smaller force. In the example, a weight W could be lifted by a tension $T = W/5$, but the smaller force must act over a larger distance. To lift the weight by Δy one must pull on the rope by $\Delta x = 5\,\Delta y$. This suggests that force times distance is an important quantity; we will define it as the work.

One Dimensional Work by a Constant Force

If in one dimension we move something by a displacement Δx with a constant force F. We will define the work done in this case by

$$W = F\,\Delta x.$$

Note that even in one dimension force and displacement are vector quantities; a one dimensional vector is a real number and the sign gives its direction. It follows that if the force and displacement are in the same direction then the work is positive and if opposite it is negative. The work is a scalar quantity.

Units: The SI unit for work and energy is: J = joule = N m

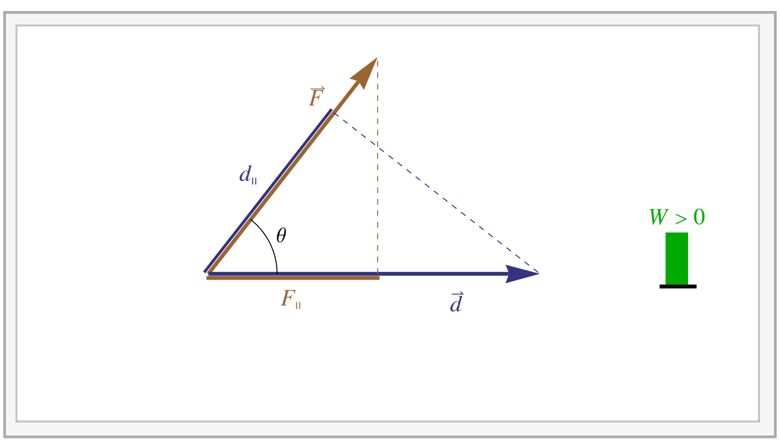
Two and Three Dimensional Work by a Constant Force

We now need to generalize this to the case of a constant force $\vec{F}$ and a straight-line path in two or three dimensions. Here we have a displacement $\vec{d}$ and we will define θ to be the angle between the two vectors $\vec{F}$ and $\vec{d}$. What is important is the component of the force along the direction of the displacement, which will be written $F_{\parallel}$ and is given by $F_{\parallel} = F\cos\theta$.

The definition of work becomes

$$W = F_{\parallel}\,d = F\,d\cos\theta = F\,d_{\parallel}$$

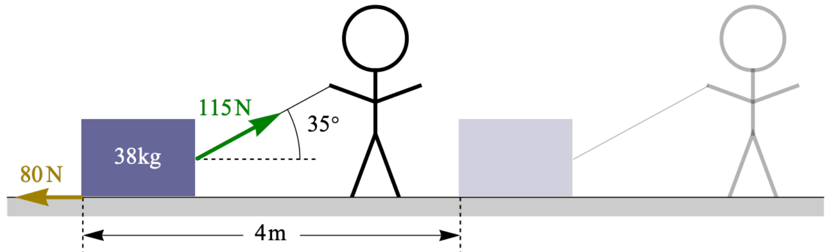
where $d = \|\vec{d}\|$ is the magnitude of the displacement $\vec{d}$ and $d_{\parallel}$ is the component of $\vec{d}$ parallel to the force $\vec{F}$.



Interactive Figure

Example G.1 - Dragging a Crate

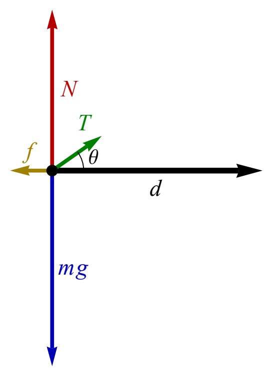
A 38-kg crate initially at rest is dragged by a rope a distance of 4m along a horizontal floor. The rope has a tension of 115 N and makes an angle of 35° from horizontal. There is a backward friction force of 80 N acting on the crate. There are four forces acting on the crate: tension, friction, the normal force and gravity.



(a) What is the work done by each force?

Solution

The free-body diagram for the crate shows the four forces and their directions. The displacement $\Delta\vec{r}$ is shown for reference, where $\|\Delta\vec{r}\| = \Delta x$, but $\Delta\vec{r}$ is not a force and not part of the free-body diagram.



$$T = 115\text{N},\ f = 80\text{N},\ m = 38\text{kg},\ \theta = 35\,°\ \text{and}\ d = 4\text{m}$$

Since we have constant forces with a straight-line path, we can write the work for each force as:

$$W = F\,d\cos\theta.$$

For the tension we have

$$W_T = T\,d\cos\theta = 377.\,\text{J}$$

For the friction force the angle is $\theta = 180\,°$.

$$W_f = f\,d\cos 180\,° = -f\,d = -320\ \text{J}$$

Both the normal force and gravity (the weight) are perpendicular to the displacement. Since $\cos 90\,° = 0$ both forces give zero work.

(b) Find the acceleration of the crate and its speed after moving 4m.

Solution

To find the acceleration, which is horizontal, we only need to consider the horizontal components of forces. The horizontal component of the tension is $T\cos\theta$ and friction is backward and negative.

$$F_{\text{net,hor}} = T\cos\theta - f = m\,a \implies a = \frac{1}{m}(T\cos\theta - f) = 0.374\ \frac{\text{m}}{\text{s}^2}$$

Using constant acceleration kinematics and that the initial velocity is zero, $v_0 = 0$, allows us to find the final speed.

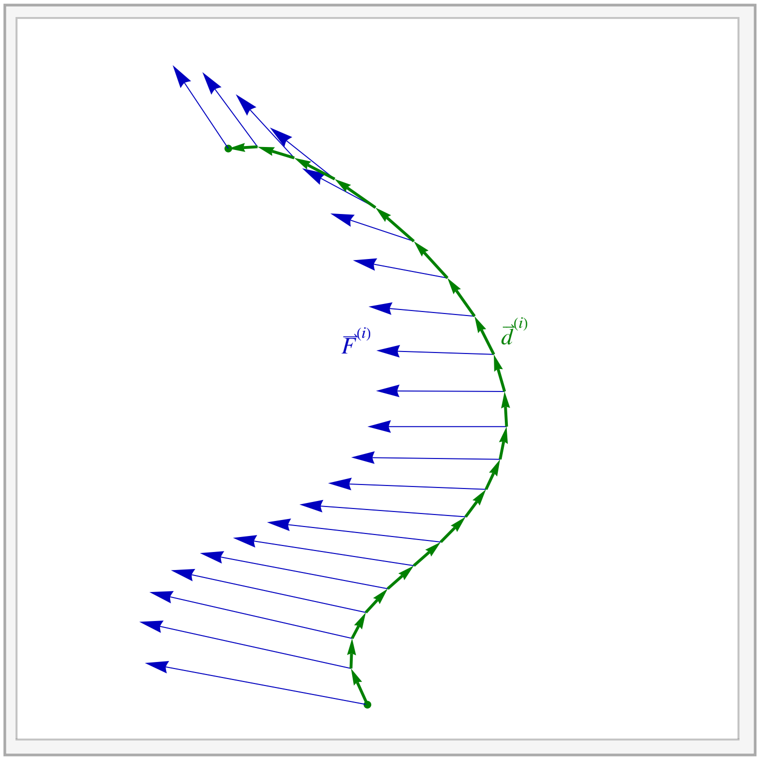
$$v^2 = v_0^2 + 2\,a\,\Delta x \implies v^2 = v_0^2 + 2\,a\,d \implies v = \sqrt{2\,a\,d} = 1.73\ \text{m/s}$$

G.2 - Work in General

So far we have the definition of work for a constant force and a straight line path to be $W = F_{\parallel}\, d = F\, d \cos\theta = F\, d_{\parallel}$. We need to generalize this to a varying force acting on a general path that is allowed to curve. Break up the path into many small segments $\vec{d}^{(i)}$; the force at that small segment $\vec{F}^{(i)}$ will be essentially constant if the segment is sufficiently small. The work over that small segment is then $F_{\parallel}^{(i)}\, d^{(i)}$ The generalized definition of work will then be summing the work over all the small segments

$$W = \sum_i F_{\parallel}^{(i)}\, d^{(i)} \quad \left(\text{small } \vec{d}^{(i)}\right)$$

This expression will not be used in calculations this semester. It will be used for definitions.



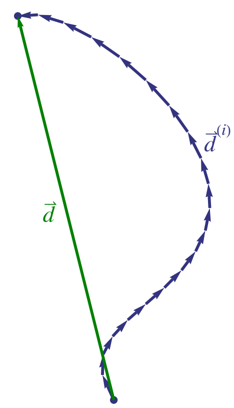
Interactive Figure

Constant Force

If a force is constant then $\vec{F} = \vec{F}^{(i)}$ and them we can take it out of the above sum.

$$W = \sum_i F_{\parallel}^{(i)}\, d^{(i)} = \sum_i F\, d_{\parallel}^{(i)} = F \sum_i d_{\parallel}^{(i)} = F\, d_{\parallel} = F_{\parallel}\, d = F\, d \cos\theta \quad \left(\text{small } \vec{d}^{(i)}\right)$$

Here $\vec{d}$ is the displacement vector from the start of the path to its end: $\vec{d} = \sum_i \vec{d}^{(i)}$.



This means the the work for constant force is the same as the work for a constant force over the straight-line path, where $\vec{d}$ is the displacement vector from the start to the end of the path.

Work Done by Gravity

An important special case of the previous result is the work done by gravity. The force $\vec{F} = -m\,g\;\hat{y}$ is a constant and the component of a displacement $\vec{d}$ in the direction of gravity is: $d_{\parallel} = -\Delta y$.

$$W_{\text{grav}} = F\,d_{\parallel} = -m\,g\,\Delta y.$$

Example G.2 - A Box on a Table

Consider a box of mass m and a table of height h.

(a) What is the work done by gravity when the box is moved from the table top to the floor?

Solution

We choose positive y to be upward so we have $\Delta y = -h$. It follows that

$$W_{\text{grav}} = -m\,g\,\Delta y = m\,g\,h.$$

(b) What is the work done by gravity when the box is moved from the floor back to the table top?

Solution

Now we have $\Delta y = +h$ and

$$W_{\text{grav}} = -m\,g\,\Delta y = -m\,g\,h.$$

(c) What is the total work done by gravity when the box is moved from the table top to the floor and then back to the table top?

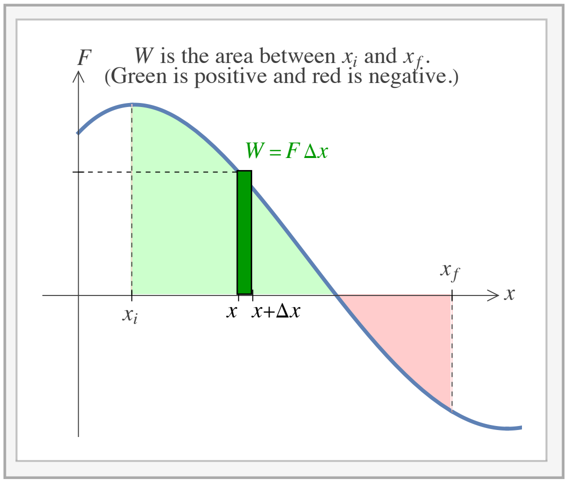
Solution

The net vertical displacement is zero. $\Delta y = 0$. So

$$W_{\text{grav}} = -m\,g\,\Delta y = 0.$$

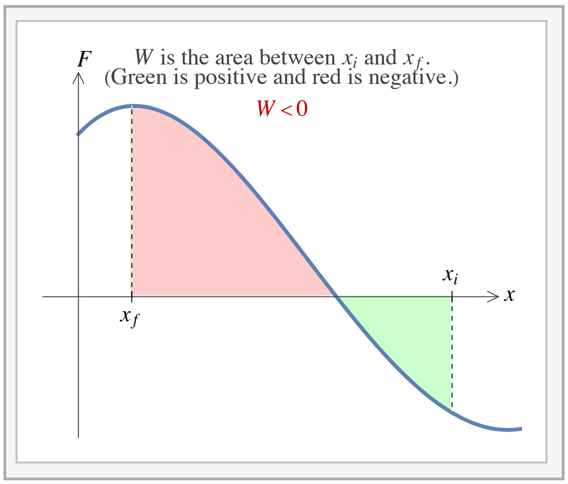
One Dimensional Work by a Varying Force

If $F(x)$ is a one dimension force as a function of position x, the work when moving from position x_i to x_f is the area under the F vs. x graph between x_i and x_f. The convention when discussing area under a curve is the when $F(x)\,\Delta x$ is negative, the contribution to the area is negative.



Work is the area under an F vs. x graph, where the contribution to the area for negative $F(x)\,\Delta x$ is negative, as shown in red.

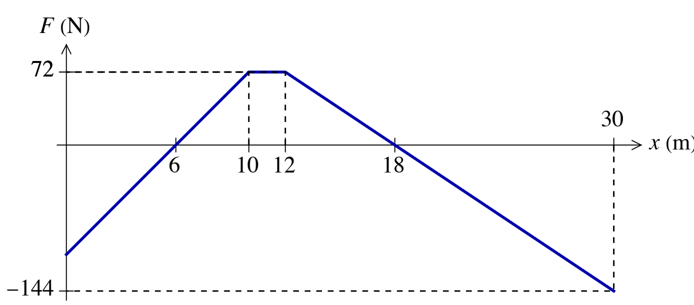
Note that when $x_f < x_i$, then $\Delta x < 0$ and the sign of $F(x)\,\Delta x$ is reversed; positive F then gives a negative area and negative F gives a positive area.



When $x_f < x_i$ the signs are reversed: positive forces do negative work and negative forces do positive work.

Example G.3 - Work Done by a Varying Force

A particle moves along the x-axis under the force represented by the graph below.

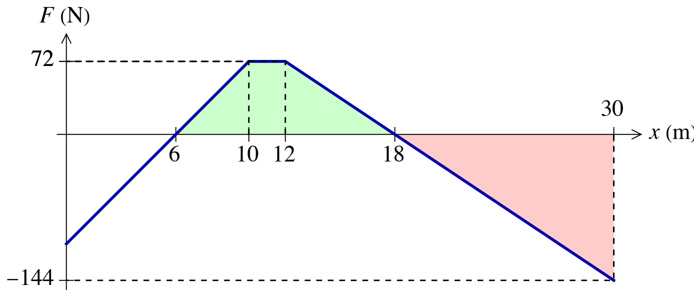


(a) What is the work done by the force on the particle between 6 m and 30 m?

Solution

The mass of the particle is unimportant for this part of the problem; we will return to this example later and then the mass will be needed. Work is the area under the F vs. x graph. To find the area, break it into segments.

$$W_{6\to30} = W_{6\to10} + W_{10\to12} + W_{12\to18} + W_{18\to30}$$



For this we have three triangles, where area = $\frac{1}{2}$ base × height and a rectangle for $W_{10\to12}$. For the last triangle, since the force is negative, the contribution to the area will be negative.

$$W_{6\to10} = \frac{1}{2}\,(4\text{ m}\times72\text{ N}) = 144\text{ J}$$

$$W_{10\to12} = 2\text{ m}\times72\text{ N} = 144\text{ J}$$

$$W_{12\to18} = \frac{1}{2}\,(6\text{ m}\times72\text{ N}) = 1216\text{ J}$$

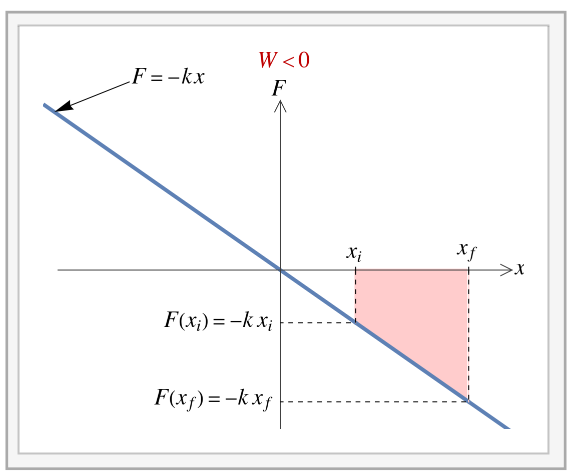
$$W_{18\to30} = -\frac{1}{2}\,(12\text{ m}\times120\text{ N}) = -864\text{ J}$$

Summing these gives.

$$W_{6\to30} = W_{6\to10} + W_{10\to12} + W_{12\to18} + W_{18\to30} = -360\text{ J}$$

Work Done by a Spring

In chapter 6, we discussed Hooke's law, the force law of a spring. The force of a spring acting on something is $F = -k\,x$. We may now obtain an expression for the work done by a spring. Apply the preceding discussion of the work done in one dimension by a varying force $F(x)$ using the Hooke's law $F(x) = -k\,x$.



The negative area between x_i and x_f is just the negative area of the triangle from the origin to x_f to $F(x_f)$ subtracting the missing part, the negative area of the triangle from he origin to x_i to $F(x_i)$

$$W_{\text{spring}} = \frac{1}{2}\,x_f\,F(x_f) - \frac{1}{2}\,x_i\,F(x_i) = \frac{1}{2}\,x_f\,(-k\,x_f) - \frac{1}{2}\,x_i\,(-k\,x_i)$$

This gives the expression.

$$W_{\text{spring}} = -\frac{1}{2}\,k\left(x_f^2 - x_i^2\right).$$

Example G.4 - Hooke's Law

It takes a force of magnitude 60 N to compress a spring by 4 cm.

(a) What is the force constant of the spring?

Solution

$$F = 60\,\text{N} \quad \text{and} \quad x = 0.04\,\text{m}$$

We will use Hooke's Law to find the force constant. We will ignore the sign because only magnitudes are given.

$$F = k\,x \implies k = F/x = 1500\,\text{N/m}.$$

(b) How much work is done compressing the spring? What is the work done *by* the spring?

Solution

The work done *by* the spring is

$$W_{\text{spring}} = -\frac{1}{2}\,k\left(x_f^2 - x_i^2\right) = -\frac{1}{2}\,k\left(x^2 - 0^2\right) = -\frac{1}{2}\,k\,x^2 = -1.2\,\text{J}\,.$$

G.3 - Power

Power, in the most general sense, is the rate that something uses or provides energy.

$$\mathcal{P}_{\text{ave}} = \frac{\text{Energy}}{\Delta t}.$$

The power delivered by a motor or engine is the rate that it can do work

$$\mathcal{P}_{\text{ave}} = \frac{W}{\Delta t}.$$

For the one-dimensional case we can write the work in terms of the force and velocity. For a small $d = \Delta x$, since $v = \Delta x/\Delta t$ we get:

$$\mathcal{P}_{\text{ave}} = \frac{W}{\Delta t} = \frac{F\Delta x}{\Delta t} = F\,v$$

Units: The SI unit for power is: W = watt = J/s

G.4 - Kinetic Energy and the Work-Energy Theorem

The Net Work

Newton's second law isn't a general statement about forces but is about the net force acting on a body, $\vec{F}_{\text{net}} = m\,\vec{a}$. $\vec{F}_{\text{net}}$ is the vector sum of all forces acting on a body; we use a free-body diagram to help us sum these forces. Let us symbolically write the net force in terms of all the forces acting on a body (all the forces in the free-body diagram) as

$$\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 + \ldots.$$

Each force acting on a body does work on the body. (Some of these works may be zero, however.) If the work done by $\vec{F}_i$ is labeled W_i then we can define the net work as the sum of all these works.

$$W_{\text{net}} = W_1 + W_2 + \ldots.$$

The Work-Energy Theorem

The work-energy theorem is a very important result. It is where the idea of energy comes into physics and it explains why work is a useful notion. We define the *kinetic energy* by

$$K = \frac{1}{2}\, m\, v^2.$$

The theorem is quite simple to state; it equates the net work to the change in the kinetic energy.

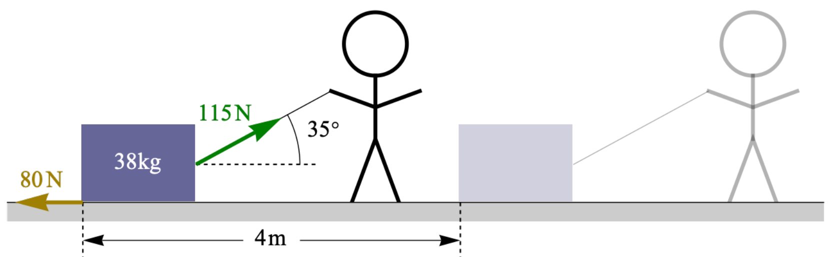
$$W_{\text{net}} = \Delta K$$

where the change is kinetic energy is $\Delta K = K_f - K_i = \frac{1}{2}\, m\, \left(v_f^2\ -\ v_i^2\right)$.

Example G.5 - Dragging a Crate (continued)

Before discussing the theorem let us consider an example. We will continue the "Dragging a Crate" example.

A 38-kg crate initially at rest is dragged by a rope a distance of 4m along a horizontal floor. The rope has a tension of 115 N and makes an angle of 35° from horizontal. There is a backward friction force of 80 N acting on the crate. There are four forces acting on the crate: tension, friction, the normal force and gravity.



In the earlier example we found the work done by each force

$$W_T = 376.8\ \text{J}\ ,\quad W_f = -320.0\ \text{J}\ \text{ and } W_N = 0 = W_{\text{grav}}$$

and we solved for the speed of the crate after moving.

(a) What is the net work?

Solution

$$W_{\text{net}} = W_T + W_f + W_N + W_{\text{grav}} = 376.8\ \text{J} - 320.0\ \text{J} + 0 + 0 = 56.8\ \text{J}$$

(b) Using the Work-Energy theorem find the speed of the crate after moving 4m.

Solution

$$W_{\text{net}} = \Delta K = \frac{1}{2}\, m \left(v_f^2 - v_i^2\right)$$

Using $m = 38$ kg and $v_i = 0$ we get the same value for the speed.

$$v_f = \sqrt{\frac{2}{m}\, W_{\text{net}}} = 1.73\ \text{m/s}$$

Understanding the Work-energy Theorem

Consider a single mass moving along a general path. The net work is the sum of the works done by each force acting on a body. Since the sum of all the forces is the net force, the net work becomes the work done by the net force. We can understand this theorem using Newton's second law and considering a small displacement $\vec{d}$.

$$W_{\text{net}} = F_{\text{net},\parallel}\, d = m\, a_\parallel\, d = \frac{m}{2} \times 2\, a_\parallel\, d$$

Choose coordinates so that this small displacement is in the x-direction, then we can write $d = \Delta x$ and $a_{\parallel} = a_x$. For a small displacement, the force is constant and then so is the acceleration. The kinematics equations for constant acceleration gives

$$2\,a_{\parallel}\,d = 2\,a_x\,\Delta x = v_x^2 - v_{0,x}^2 = v_{f,x}^2 - v_{i,x}^2 = v_f^2 - v_i^2$$

The reason for the final equality in the expression is that at that instant the particle is moving in the x-direction only. Inserting this into the previous result gives the Work-Energy theorem, at least for a small displacement.

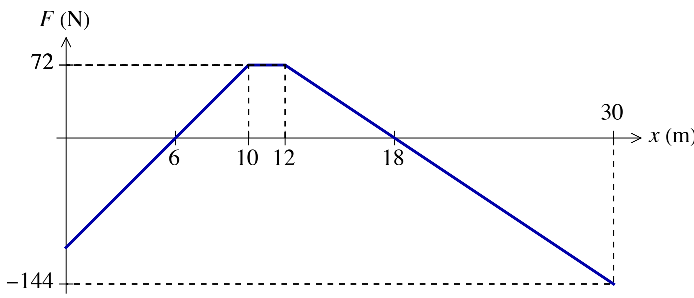
$$W_{\text{net}} = \frac{m}{2}\left(v_f^2 - v_i^2\right) = \Delta K$$

For motion along a general path, the net work will be the sum over the net work of each small segment making the path. Adding all the ΔK for each small displacement gives the total ΔK for the entire path. The work-energy theorem follows.

$$W_{\text{net}} = \Delta K$$

Example G.6 - Work Done by a Varying Force (continued)

A particle moves along the x-axis under the force represented by the graph below.



(b) Suppose this is the only force acting on a 30.0-kg particle. If the particle has a speed of 7.00 m/s at $x = 6$ m, then what is its speed at $x = 30$ m?

Solution

We are now also given the mass and the initial velocity.

$$m = 30.0\text{ kg} \;\text{ and }\; v_i = v(6\text{ m}) = 7.00\text{ m/s}$$

Since there is only one force acting, that force is the net force and its work is the net work. This is what we found in part **(a)**.

$$W_{\text{net}} = W_{6\to 30} = -360\text{ J}$$

The work-energy theorem gives us the final velocity.

$$W_{\text{net}} = \frac{1}{2}\,m\left(v_f^2 - v_i^2\right) \implies v_f^2 - v_i^2 = \frac{2\,W_{\text{net}}}{m} \implies v_f = v(30\text{ m}) = \sqrt{v_i^2 + \frac{2\,W_{\text{net}}}{m}} = 5.00\,\frac{\text{m}}{\text{s}}$$

Note that by the work-energy theorem, whenever the net work is negative the speed will decrease.