Chapter H

Potential Energy and Energy Conservation

Blinn College - Physics 1401 - Terry Honan

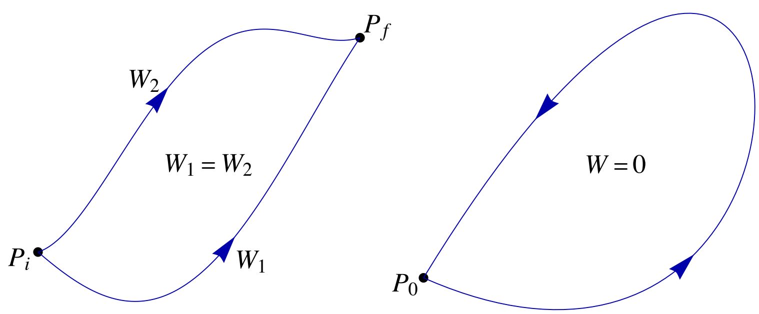
H.1 - Conservative Forces and Potential Energy

When a body is moved in a uniform gravitational field $\vec{g}$ the work done by gravity is given by: $W = -m\,g\,\Delta y$ Note that if the body is moved along different paths with the same endpoints (starting and stopping points) the Δy is the same and thus the work done by gravity is the same. In stretching a spring the work of the spring $W = -\frac{1}{2}\,k\left(x_f^2 - x_i^2\right)$ depends only on the endpoints x_i and x_f and not on the details of the path.

We will define such forces (where the work is independent of path) as conservative and we will then be able to define a new type of energy, called *potential energy* or U, for these forces. The entire effect of the work of these forces will be incorporated into considering the potential energy functions at the endpoints of the paths. Potential energy will be an easy and useful bookkeeping tool for keeping track of the work contributions for conservative forces in the work-energy theorem.

Conservative and Nonconservative Forces

A force is defined to be a conservative force when its work is independent of the path taken; for any two paths with the same endpoints $W_1 = W_2$. Equivalently, we can say that for a conservative force the work around any closed path (a path that ends where it begins) is zero.



Equivalent definitions of a conservative force: On the left, any two paths with the same endpoints will have the same work. On the right, the work for any closed path is zero.

Conservative Examples

Examples of conservative forces have already been mentioned. These are uniform gravitational forces and the elastic force of a spring. Other examples of conservative forces that will be encountered are nonuniform gravitational forces, which will be discussed later this semester, and the electrostatic force, which will be considered in the second semester course.

Dissipative friction is nonconservative.

Consider an object being dragged along a horizontal floor by a horizontal rope. If the dragging force is horizontal, the normal force is the weight and the magnitude of the force of kinetic friction is $f_k = \mu_k\,m\,g\; = \;\text{constant}$. The direction of the force of friction opposes the direction of motion so the work done by friction for a small displacement d is $-f_k\,d$, since $\cos 180\,° = -1$. For a long path, summing over all then small displacements gives

$$W_f = -f_k\,(\text{path length})$$

Since it depends on the path length, it is clear that kinetic friction is not conservative.

Driving forces are nonconservative.

By driving forces we mean the force of a car's engine propelling a car or the force of a cyclist propelling a cycle. Consider a car that begins at some location, drives on some path and then returns to the same point. The car's engine does work in this process and thus cannot be conservative. A conservative force must give zero work for a closed path.

Other Nonconservative Forces

Any force that is not conservative is nonconservative, where our two conservative examples are gravity or the elastic force of a spring. If a tension force or normal force acts on a body then those also represent nonconservative forces; in some examples the tension and normal force will be present but do no work. If there is some external applied force acting on a body then that is nonconservative.

Potential Energy

For any conservative force we can define a potential energy, U. This idea is this: since the work depends only on the endpoints of a path and not the details of the path then we can write the work as the difference of some function evaluated only at the endpoints. We will define this function as the negative of the potential energy function. The reason for the sign will become clear later.

The definition of potential energy is

$$\Delta U = -W$$

The zero of potential energy is arbitrary. In some cases there will be standard choices of the zero position.

Potential Energy for Uniform Gravity

Since for gravity we have $W = -m\,g\,\Delta y$, we define gravitational potential energy by $\Delta U = m\,g\,\Delta y$. We can choose the zero of potential energy to be where $y = 0$ and then define the potential energy function as

$$U = m\,g\,y.$$

The zero of y is still arbitrary.

Elastic Potential Energy

The work done by a spring is given by $W = -\frac{1}{2}\,k\left(x_f^2 - x_i^2\right)$. When we take $\Delta U = -W$ we get

$$\Delta U = -W = \frac{1}{2}\,k\left(x_f^2 - x_i^2\right).$$

We want to find a function $U(x)$ that satisfies $\Delta U = U(x_f) - U(x_i)$. The easiest choice is

$$U = \frac{1}{2}\,k\,x^2.$$

In making this choice we take the zero position of potential energy to be the equilibrium position $x = 0$.

Work and Mechanical Energy

Let us now apply these ideas to the work-energy theorem. We begin by writing all forces acting on a body as

$$\vec{F}_{\text{net}} = \underbrace{\vec{F}_{\text{nc}}}_{\substack{\text{all nonconservative}\\ \text{forces}}} + \underbrace{\vec{F}_1 + \vec{F}_2 + \ldots}_{\text{conservative forces}}.$$

$\vec{F}_{\text{nc}}$ represents the sum of *all* nonconservative forces. The other forces are the conservative forces listed separately. We now make the same decomposition of the corresponding works.

$$W_{\text{net}} = W_{\text{nc}} + W_1 + W_2 + \ldots$$

Now we make the replacements $W_i = -\Delta U_i$. Plugging the above expression for W_{net} into the work-energy theorem $W_{\text{net}} = \Delta K$ and moving the ΔU_i terms to the right hand side gives:

$$W_{\text{nc}} = \Delta K + \Delta U_1 + \Delta U_2 + \ldots.$$

This result applies to a single mass. To apply it to a system containing multiple masses, like for instance Atwood's machine, we can sum this over every mass in the system. Now take W_{nc} to be the sum of the W_{nc} for all the masses. Call K_{tot} the sum of the kinetic energies of all masses and U_{tot} the sum over all the U_i for all the masses. We end up with the result

$$\begin{aligned} W_{\text{nc}} &= \Delta K_{\text{tot}} + \Delta U_{\text{tot}} \\ &= \Delta E_{\text{mech}} \end{aligned}$$

where we have defined the total mechanical energy of the system as $E_{\text{mech}} = K_{\text{tot}} + U_{\text{tot}}$. Usually the mechanical energy will just be written as E.

Nonconservative forces will usually consist of friction forces, which remove mechanical energy from a system, and driving forces like the work done by car's engine or a cyclist, which add mechanical energy.

H.2 - Conservation of Energy

There are two notions of conservation of energy we will consider. One will be of practical importance for problem solving. The other is a very fundamental notion.

Conservation of Mechanical Energy

The conservation of mechanical energy is a principle that will prove very useful in problem solving. Begin with the fundamental expression $W_{\text{nc}} = \Delta E_{\text{mech}}$. If in some problem there are no nonconservative forces (i.e. no dissipative friction or driving force) then we can conclude that $\Delta E_{\text{mech}} = 0$ or that

$$E_{\text{mech},i} = E_{\text{mech},f} \text{ or } E_i = E_f.$$

To solve such a problem we need to find an expression for E, the total mechanical energy. To do this we add a kinetic energy for each mass in the problem, add in gravitational potential energies for each mass and add an elastic potential energy for each spring.

Energy as a Fundamentally Conserved Quantity

Energy is a fundamentally conserved quantity. This means that it cannot be created or destroyed; we can just convert it from one form to another.

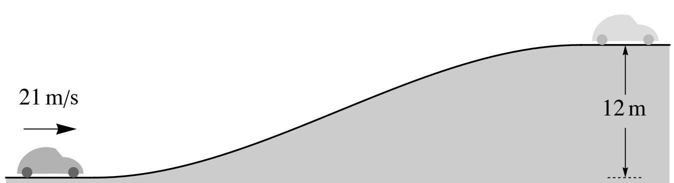
Consider the mechanical energy of a car $W_{\text{nc}} = \Delta E_{\text{mech}}$. The mechanical energy is not conserved due to the W_{nc} of friction and the engine. The energy lost to friction goes into heat. The energy from the engine comes from the energy stored in the chemical bonds of the fuel.

When solving problems that involve nonconservative forces we can rewrite $W_{\text{nc}} = \Delta E_{\text{mech}} = E_f - E_i$ as

$$E_i + W_{\text{nc}} = E_f$$

Written this way, problem solving is more similar for both types of problems, where W_{nc} is zero or not.

Example H.1 - A Rolling Car



A 1500kg car rolls in neutral up a 12m high hill. The car's speed at the bottom of the hill is 21m/s.

(a) Suppose there is no friction. What is the speed of the car at the top of the hill?

Solution

$$m = 1500\text{kg}, \quad h = 12\text{m} \text{ and } v_i = 21\text{m/s}.$$

For a car we have $W_{\text{nc}} = W_{\text{friction}} + W_{\text{engine}}$ but since it is in neutral $W_{\text{engine}} = 0$. For part (a) we also have $W_{\text{friction}} = 0$.

Since there is just one mass and no springs, the mechanical energy is $E = \frac{1}{2} mv^2 + mgy$, and since $W_{\text{nc}} = 0$ mechanical energy is conserved. It is most convenient to choose the lowest point to be the zero of potential energy so we take $y_i = 0$ and $y_f = h$.

$$E = \frac{1}{2} mv^2 + mgy \text{ and } E_i = E_f \implies \frac{1}{2} mv_i^2 + 0 = \frac{1}{2} mv_f^2 + mgh$$

It follows that the speed at the top is

$$v_f = \sqrt{v_i^2 - 2gh} = 14.3 \text{ m/s}$$

(b) Suppose now that there is friction and the speed of the car at the top is 9.5m/s . What is the work done by friction?

Solution

The car is still in neutral so we again have $W_{\text{engine}} = 0$.

$$W_{\text{nc}} = W_{\text{friction}} + W_{\text{engine}} = W_{\text{friction}} + 0$$

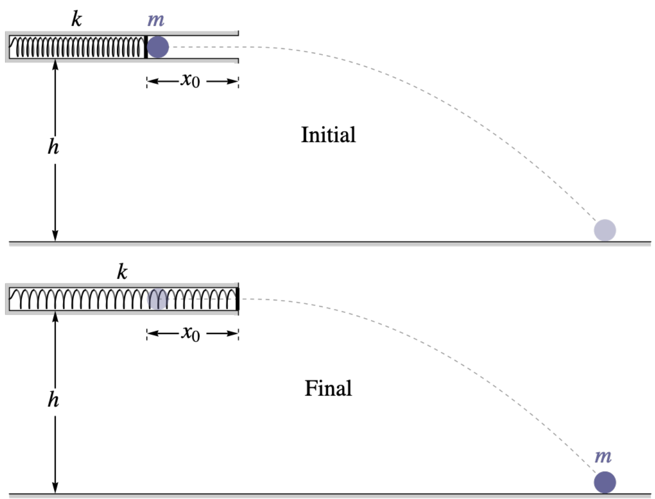
Since there is just one mass and no springs, the mechanical energy is $E = \frac{1}{2}mv^2 + mgy$, and since $W_{\text{nc}} = W_{\text{friction}}$ mechanical energy is conserved.

$$E = \frac{1}{2}mv^2 + mgy \quad \text{and} \quad E_i + W_{\text{nc}} = E_f \implies \frac{1}{2}mv_i^2 + 0 + W_{\text{friction}} = \frac{1}{2}mv_f^2 + mgh$$

We can now find W_{friction}, which we expect to be negative. $v_f = 9.5\ \text{m/s}$.

$$W_{\text{friction}} = \frac{1}{2}mv_f^2 + mgh - \frac{1}{2}mv_i^2 = -86\,700\ \text{J}$$

Example H.2 - A Spring Gun



A ball of mass m is shot from a horizontal spring gun at a height h above the floor. The spring has a force constant k and is compressed by x_0 when cocked. What is the speed of the ball when it hits the floor?

Solution

There is just one mass and thus one kinetic energy term $K = \frac{1}{2}mv^2$. There are two potential energy terms gravitational $U_{\text{grav}} = mgy$ and elastic $U_{\text{elastic}} = \frac{1}{2}kx^2$. The total energy is thus:

$$E = \frac{1}{2}mv^2 + mgy + \frac{1}{2}kx^2.$$

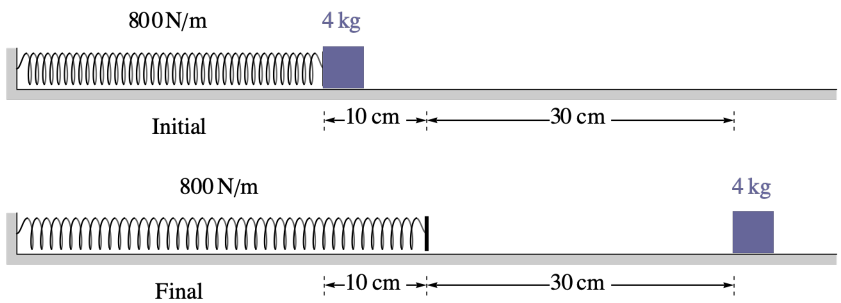
Since there is no friction and no non-conservative energy source then $W_{\text{nc}} = 0$ and mechanical energy is conserved. It is most convenient to choose the lowest point to be the zero of potential energy so we take $y_i = h$ and $y_f = 0$. The initial and final values of x, the compression of the spring from equilibrium are $x_i = x_0$ and $x_f = 0$.

$$E_i = E_f \implies 0 + mgh + \frac{1}{2}kx_0^2 = \frac{1}{2}mv_f^2 + 0 + 0$$

Solving for the final velocity gives

$$v_f = \sqrt{\frac{k}{m}x_0^2 + 2gh}\,.$$

Example H.3 - A Spring and a Block



A 4-kg block is pushed along a (level) floor by a spring with a force constant of 800 N/m as shown. Initially the spring is compressed by 10 cm. After leaving the spring the block slides an additional distance of 30 cm before coming to a stop. What is the coefficient of kinetic friction between the block and the floor.

Solution

We are given the mass, the force constant of the spring, the amount the spring was compressed initially and the distant is slides after leaving the spring.

$$m = 4\text{kg}, \quad k = 800\ \text{N/m}, \quad x_0 = 0.10\ \text{m} \text{ and } d = 0.30\ \text{m}$$

Since the floor is level we can set $y = 0$ and omit the gravitational potential energy. This leaves just kinetic energy and elastic potential energy.

$$E = \frac{1}{2}\, m\, v^2 + \frac{1}{2}\, k\, x^2$$

Because there is friction we have $W_{\text{nc}} = W_{\text{friction}}$. The normal force on the block is just its weight, $N = mg$. The total distance the block slides is $\Delta x = x_0 + d = 0.40$ m.

$$W_{\text{nc}} = W_{\text{friction}} = -f_k\, \Delta x = -\mu_k\, N\, \Delta x = -\mu_k\, mg\, (x_0 + d)$$

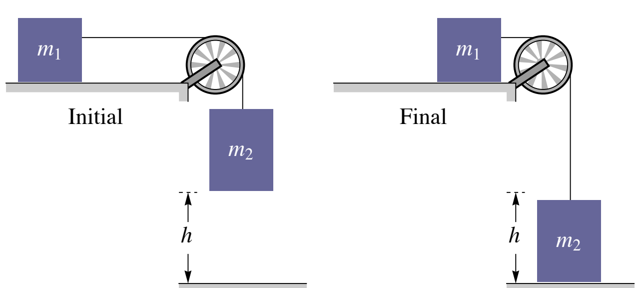
Note that the sign above follows from $\cos 180° = -1$. Both initial and final velocities are zero, $v_i = 0 = v_f$. We also have $x_i = x_0$ and $x_f = 0$.

$$E_i + W_{\text{nc}} = E_f \implies 0 + \frac{1}{2}\, k x_0^2 - \mu_k\, mg\, (x_0 + d) = 0 + 0$$

Solving for the coefficient of kinetic friction we get

$$\mu_k = \frac{k\, x_0^2}{2\, mg\, (x_0 + d)} = 0.255\ .$$

Example H.4 - Two Connected Masses



Two blocks of masses m_1 and m_2 are connected by a light string over an ideal pulley as shown. m_1 slides on a horizontal table and m_2 is initially a height h above the floor.

(a) Suppose there is no friction between m_1 and the table. What is the speed of m_2 when it hits the floor?

Solution

We have potential energies of $U = m_1 g y_1$ and $U = m_2 g y_2$ for the two masses. We can choose our zero value for y differently for each mass. Let us choose $y_1 = 0$ along the tabletop; this removes that potential energy term completely. For the hanging mass choose its lowest point to be the zero. So, $y_2 = h$ initially and $y_2 = 0$ at the floor. Both masses have kinetic energies. Because of our simple pulley arrangement both masses move the same distances, $\Delta x_1 = \Delta x_2 = \Delta x$, and thus will have the same speed: $v = v_1 = v_2$. The total kinetic energy becomes

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v^2 + \frac{1}{2} m_2 v^2 = \frac{1}{2} (m_1 + m_2) v^2$$

and the total mechanical energy is

$$E = \frac{1}{2} (m_1 + m_2) v^2 + m_2 g y_2$$

Since there is no friction for part (a) we have $W_{\text{nc}} = 0$ and then conservation of mechanical energy.

$$E_i = E_f \implies 0 + m_2 g h = \frac{1}{2} (m_1 + m_2) v_f^2 + 0$$

The final speed follows.

$$v_f = \sqrt{2 \left(\frac{m_2 g}{m_1 + m_2} \right) h}$$

The expression was written so that the part inside the brackets is just the linear acceleration that we could have found with a more involved force analysis in Chapter D.

(b) Suppose now that there is a coefficient of kinetic friction μ_k between m_1 and the table. What is the speed of m_2 when it hits the floor?

Solution

Now that we have friction mechanical energy is no longer conserved. W_{nc} is just the work done by friction.

$$W_{\text{nc}} = W_{\text{friction}} = -f_k \, \Delta x = -(\mu_k N) \, \Delta x = -(\mu_k m_1 g) h$$

We can then write

$$E_i + W_{\text{nc}} = E_f \implies (0 + m_2 g h) - \mu_k m_1 g h = \frac{1}{2} (m_1 + m_2) v_f^2 + 0$$

Solving for v_f with the acceleration again in brackets gives

$$v_f = \sqrt{2 \left(\frac{m_2 g - \mu_k m_1 g}{m_1 + m_2} \right) h}$$