Chapter I

Linear Momentum

Blinn College - Physics 1401 - Terry Honan

I.1 - Momentum and Newton's Second Law

Definition of Momentum

So far we have written the second law in terms of the acceleration of a particle. It turns out that Newton wrote it differently; his preferred form of the second law was written in terms of the time rate of change of the momentum of the body.

Momentum is a vector quantity describing the dynamics of a moving body. It is defined simply as the mass times the velocity and we use the symbol $\vec{p}$ to denote it.

$$\vec{p} = m\,\vec{v}$$

We will often refer to this as *linear momentum*; this will distinguish it from angular momentum which will be discussed later.

The Second Law

Newton's second law can be written in terms of the momentum. If the mass of a body is constant then $\Delta\vec{p} = m\left(\vec{v}_f - \vec{v}_i\right) = m\,\Delta\vec{v}$ and we can write

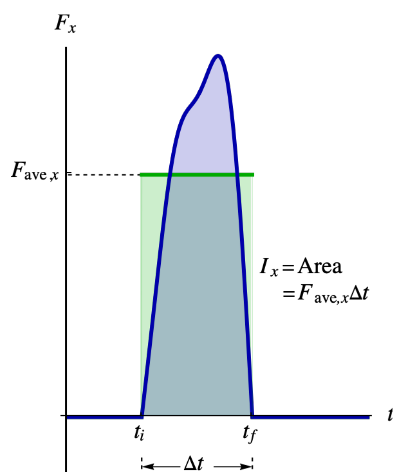
$$\frac{\Delta\,\vec{p}}{\Delta t} = m\,\frac{\Delta\,\vec{v}}{\Delta t} = m\,\vec{a}_{\text{ave}} \implies \vec{F}_{\text{net,ave}} = \frac{\Delta\,\vec{p}}{\Delta t} \quad \text{and} \quad \vec{F}_{\text{net}} = \lim_{\Delta t\to 0}\frac{\Delta\,\vec{p}}{\Delta\,t}$$

This form of the second law was Newton's preferred way to write it. Although this is equivalent to the $\vec{F}_{\text{net}} = m\,\vec{a}$ form when the mass is constant, when the mass is changing this new momentum form is the proper one.

I.2 - Impulse and Momentum

Impulse and Average Force

Collisions are usually quick things but they are not instantaneous. When a bat hits a baseball, the ball rides along the bat for a period of time. An impulsive force is a large force acting over a short period of time. We will define the impulse as the area under the force versus time graph. Suppose there is a collision between t_i and t_f the force is zero other than in that time interval; the x-component of the impulse I_x is the area under the graph of F_x vs. time between t_i and t_f.



The average force is defined as the impulse divided by the time Δt, where the width of the interval is $\Delta t = t_f - t_i$. It follows that the area under the force F_x versus time curve is also the area of the rectangle with Δt at its base and with a height of $F_{\text{ave},x}$

$$\vec{I} = \vec{F}_{\text{ave}}\,\Delta t \quad \text{and} \quad I_x = F_{\text{ave},x}\,\Delta t = (\text{Area under } F_x \text{ vs. } t)$$

The Impulse-Momentum Theorem

The impulse-momentum theorem is an immediate consequence of Newton's second law.

$$\vec{I}_{\text{net}} = \Delta\,\vec{p}$$

Here the net impulse $\vec{I}_{\text{net}}$ is the impulse of the net force $\vec{F}_{\text{net}}$ and the change in the momentum can be written $\Delta\,\vec{p} = m\left(\vec{v}_f - \vec{v}_i\right)$. Usually, the "net" subscript is omitted when writing the impulse-momentum theorem. The physical significance of this is when there is an impulsive force, it typically is much larger than any other forces acting over the short time of the collision and the other forces can be neglected. For example, when a bat hits a baseball, that force is much larger than gravity or some other force during the collision.

Example I.1 - Hitting a Baseball

A 0.145 kg baseball thrown at 40 m/s is hit straight back at the pitcher at 50 m/s. If the bat is in contact with the ball for 0.035 s, then what is the average force of the bat on the ball?

Solution

As a vector the impulse-momentum theorem in one dimension becomes:

$$F_{\text{ave}}\,\Delta\,t = I = \Delta\,p = m\,(v_f - v_i).$$

Remembering that a one-dimensional vector is a real number, where the sign gives the direction we can write the given information as:

$$m = 0.145\text{ kg},\quad v_i = -40\text{ m/s},\quad v_f = 50\text{ m/s} \text{ and } \Delta t = 0.035\text{ s}\,.$$

Because the ball changes direction, one of the velocities must be negative. If we choose the direction of the force of the bat on the ball to be positive then that makes the initial velocity negative. Solve for the average force.

$$F_{\text{ave}} = \frac{m}{\Delta\,t}\,(v_f - v_i) = 373\text{ N}$$

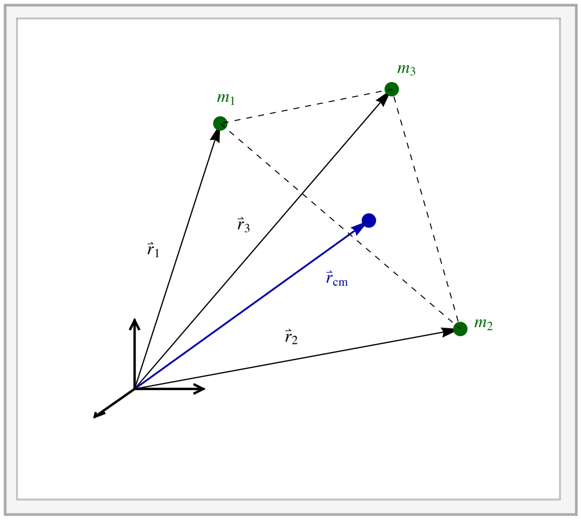
I.3 - A System of Particles

Introduction

So far our discussion of dynamics has applied only to point particles. When we discussed the dynamics of extended objects we treated them as particles. Why was this proper? In all the examples considered the body was not rotating, so each point on it had the same acceleration. This allowed us to treat it as a particle. If a body rotates then different points have different accelerations and we must be more careful. We can treat it as a system of particles.

A system is a collection of point particles. This could represent a huge number of particles, like every atom in a solid or fluid or it could be a small number like the earth, moon and sun. We arbitrarily divide the world into a system and everything else. We then break up the forces into internal forces, which are between particles of our system and external force between particles of our system and outside.

Center of Mass



We will now define the center of mass of a system of particles as a weighted average of the positions. Consider the system to be a *discrete distribution*, that is a collection of point masses, which consist of masses m_1 at postion vector $\vec{r}_1$, m_2 at $\vec{r}_2$, etc. The total mass of our system is M:

$$M = m_1 + m_2 + \ldots$$

We define the position vector of the center of mass by

$$\vec{r}_{cm} = \frac{1}{M}\left(m_1\,\vec{r}_1 + m_2\,\vec{r}_2 + \ldots\right)$$

Note that the position vectors $\vec{r}_i$ point from the origin of our coordinate system to the mass m_i. Since the x component of the position vector is just x, the x component of the center of mass is just

$$x_{cm} = \frac{1}{M}\left(m_1\,x_1 + m_2\,x_2 + \ldots\right)$$

where the y and z components satisfy similar expressions.

Because of how the instantaneous velocity is related to the position vector and how the instantaneous acceleration is related to the velocity, we can write similar expressions for the velocity and acceleration of the center of mass.

$$\vec{v}_{cm} = \frac{1}{M}\left(m_1\,\vec{v}_1 + m_2\,\vec{v}_2 + \ldots\right) \text{ and } \vec{a}_{cm} = \frac{1}{M}\left(m_1\,\vec{a}_1 + m_2\,\vec{a}_2 + \ldots\right)$$

Example I.2 - The Earth-Moon System

The masses of both the earth and moon and the earth-moon distance are given by

$$M_E = 5.97\times10^{24}\text{ kg},\;\; M_M = 7.35\times10^{22}\text{ kg and } R_{EM} = 3.85\times10^{8}\text{ m}.$$

Where is the center of mass of the earth-moon system? Give its distance from the center of the earth.

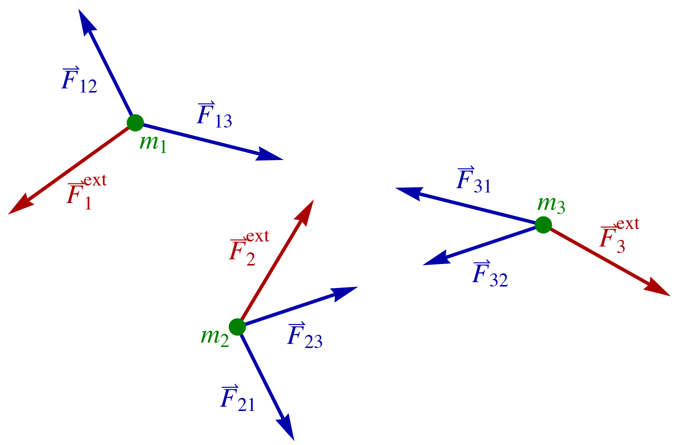
Solution

The center of mass must be along the line between the two centers. Take the origin to be at the center of the earth with the x-axis directed toward the moon.

$$x_{cm} = \frac{M_E\,x_E + M_M\,x_M}{M_E + M_M} = \frac{M_E\,0 + M_M\,R_{EM}}{M_E + M_M} = \frac{M_M}{M_E + M_M}\,R_{EM}$$
$$= 0.0122\,R_{EM} = 4.68\times10^{6}\text{ m}$$

Compare this to the earth's radius, $R_E = 6.38 \times 10^6$ m. The center of mass is below the earth's surface.

A Three Particle System



Consider a three particle system with masses m_1, m_2 and m_3. For the forces on $m_1$can be written as a sum of internal forces $\vec{F}_{12}$ and $\vec{F}_{13}$ and external forces $\vec{F}_1^{\text{ext}}$, representing everything outside our system acting on m_1. The forces for m_2 and m_3 break up similarly giving

$$\vec{F}_{\text{net},1} = \vec{F}_1^{\text{ext}} + \vec{F}_{12} + \vec{F}_{13} = \lim_{\Delta t \to 0} \frac{\Delta \vec{p}_1}{\Delta t} = m_1 \vec{a}_1$$

$$\vec{F}_{\text{net},2} = \vec{F}_2^{\text{ext}} + \vec{F}_{21} + \vec{F}_{23} = \lim_{\Delta t \to 0} \frac{\Delta \vec{p}_2}{\Delta t} = m_2 \vec{a}_2$$

$$\vec{F}_{\text{net},3} = \vec{F}_3^{\text{ext}} + \vec{F}_{31} + \vec{F}_{32} = \lim_{\Delta t \to 0} \frac{\Delta \vec{p}_3}{\Delta t} = m_3 \vec{a}_3.$$

To concentrate on the bulk motion of our system we sum over these expressions. The crucial point is that the internal forces cancel by Newton's third law. $\vec{F}_{12} + \vec{F}_{21} = \vec{0}$, $\vec{F}_{13} + \vec{F}_{31} = \vec{0}$ and $\vec{F}_{23} + \vec{F}_{32} = \vec{0}$.

$$\vec{F}_1^{\text{ext}} + \vec{F}_2^{\text{ext}} + \vec{F}_3^{\text{ext}} = \lim_{\Delta t \to 0} \frac{\Delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)}{\Delta t} = m_1 \vec{a}_1 + m_2 \vec{a}_2 + m_3 \vec{a}_3$$

The General System

For a general system we define $\vec{F}_{\text{net}}^{\text{ext}}$ as the net force

$$\vec{F}_{\text{net}}^{\text{ext}} = \vec{F}_1^{\text{ext}} + \vec{F}_2^{\text{ext}} + \ldots$$

$\vec{p}_{\text{tot}}$ as the total momentum and M as the total mass

$$\vec{p}_{\text{tot}} = \vec{p}_1 + \vec{p}_2 + \ldots$$

$$M = m_1 + m_2 + \ldots$$

Taking two derivatives of our definition of the center of mass, $M\,\vec{r}_{\text{cm}} = m_1\,\vec{r}_1 + m_2\,\vec{r}_2 + \ldots$, gives:

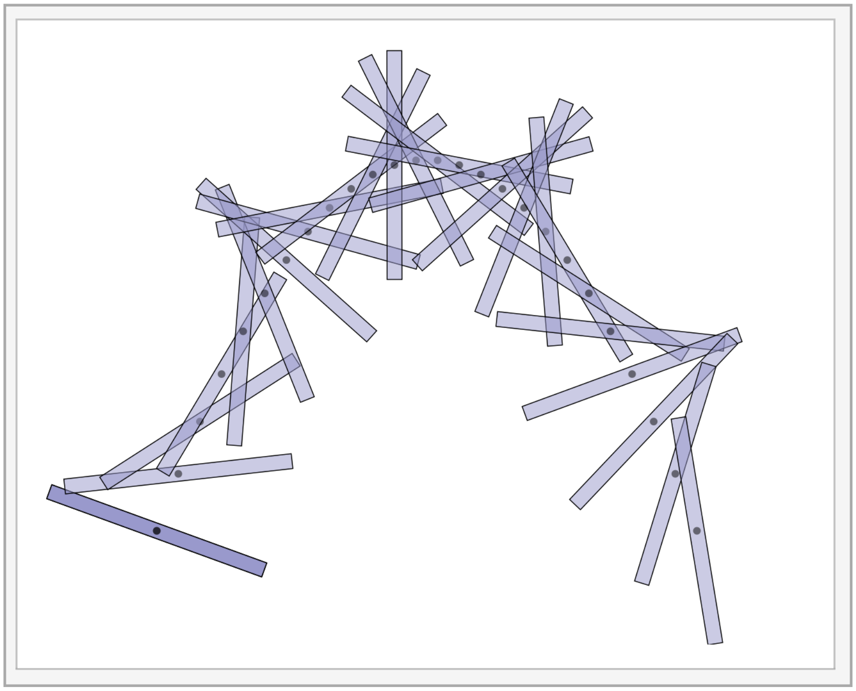
$$M\,\vec{a}_{\text{cm}} = m_1\,\vec{a}_1 + m_2\,\vec{a}_2 + \ldots,$$

since the second derivative of the position vector $\vec{r}$ is the acceleration $\vec{a}$. With this we get the two expressions for the second law for a system of particles.

$$\vec{F}_{\text{net}}^{\text{ext}} = \lim_{\Delta t \to 0} \frac{\Delta \vec{p}_{\text{tot}}}{\Delta t} \quad \text{or} \quad \vec{F}_{\text{net,ave}}^{\text{ext}} = \frac{\Delta \vec{p}_{\text{tot}}}{\Delta t}$$

$$\vec{F}_{\text{net}}^{\text{ext}} = M\,\vec{a}_{\text{cm}}$$

Suppose a uniform stick is thrown so that it has both translational and rotational motion. The net external force acting on it is $\vec{F}_{\text{net}}^{\text{ext}} = M\,\vec{g}$, The second expression above implies that $\vec{a}_{cm} = \vec{g}$, so the stick's center of mass will follow the parabolic path of a projectile while the stick rotates about that path.



A stick thrown with both translational and rotational motion. The center of mass follows the arc of a projectile while the stick rotates about that.

Conservation of Linear Momentum

The conservation of momentum is a consequence of the momentum form of the second law for a system.

$$\vec{F}_{net}^{ext} = \lim_{\Delta t \to 0} \frac{\Delta \vec{p}_{tot}}{\Delta t}$$

If there are no external forces acting on a system them the total momentum of the system is conserved.

$$\vec{F}_{net}^{ext} = \vec{0} \implies \Delta \vec{p}_{tot} = \vec{0}$$

This is the second fundamentally conserved quantities encountered in our course. To see how this is fundamental, imagine enlarging the system to include everything; there is then, by definition, no external force and thus the total momentum is conserved.

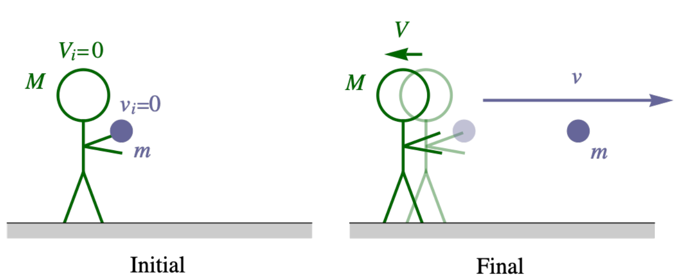
The conservation is also true component by component. If there is no external force in some direction, say the x direction, then the x component of the total momentum is conserved.

$$F_{net,x}^{ext} = 0 \implies \Delta p_{tot,x} = 0$$

Example I.3 - Throwing and Catching a Ball on Frictionless Ice

We will consider a man of mass M, initially at rest on perfectly frictionless ice.

(a) Suppose the man throws a ball of mass m forward at speed v. What is his recoil speed, V?

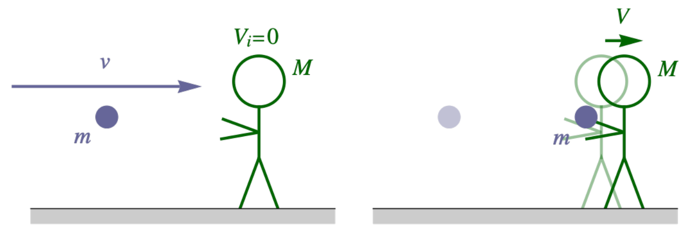


Solution

The man will recoil backward and his final velocity is negative, $-V$, where V is the speed. All velocities are horizontal here so we will take this to be a purely one-dimensional problem. We take the system to be the man and the ball, so there is no external force and the total momentum is conserved. We can then solve for the recoil speed.

$$F_{\text{net}}^{\text{ext}} = 0 \implies \Delta p_{\text{tot}} = 0 \implies p_{\text{tot},i} = p_{\text{tot},f} \implies 0 = M(-V) + m\,v \implies V = \frac{m}{M}\,v$$

(b) Suppose now that the man, still at rest initially, catches a ball of mass m thrown toward him. What is the final velocity of both the man and ball after the catch.

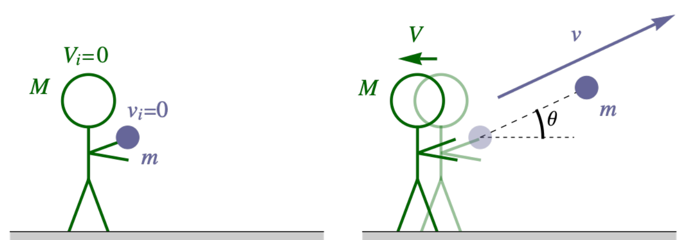


Solution

Again, we take this to be a purely one-dimensional problem and there is no external force on the man-ball system. The total momentum is still conserved. The final momentum will use the combined mass of $M + m$.

$$F_{\text{net}}^{\text{ext}} = 0 \implies \Delta p_{\text{tot}} = 0 \implies p_{\text{tot},i} = p_{\text{tot},f} \implies 0 + m\,v = (M + m)\,V \implies V = \frac{m}{M + m}\,v$$

(c) Now consider the two-dimensional modification of part (a) where the ball is thrown at an angle θ above vertical at speed v. Find the recoil speed, V.



Solution

To conserve momentum the man would need to recoil opposite the ball's velocity into the ice. Clearly this cannot happen because of the normal force of the ice on the man. The normal force is a net external force. Since there is no friction there is no horizontal net force, $F_{\text{net},x} = 0$, so the horizontal component of the total momentum is conserved. The horizontal component of the ball's velocity is $v_x = v\cos\theta$.

$$F_{\text{net},x}^{\text{ext}} = 0 \implies \Delta p_{\text{tot},x} = 0 \implies p_{\text{tot},x,i} = p_{\text{tot},x,f} \implies 0 = M(-V) + m\,v\cos\theta \implies V = \frac{m}{M}\,v\cos\theta$$

Symmetries and Conservation Laws - Noether's Theorem

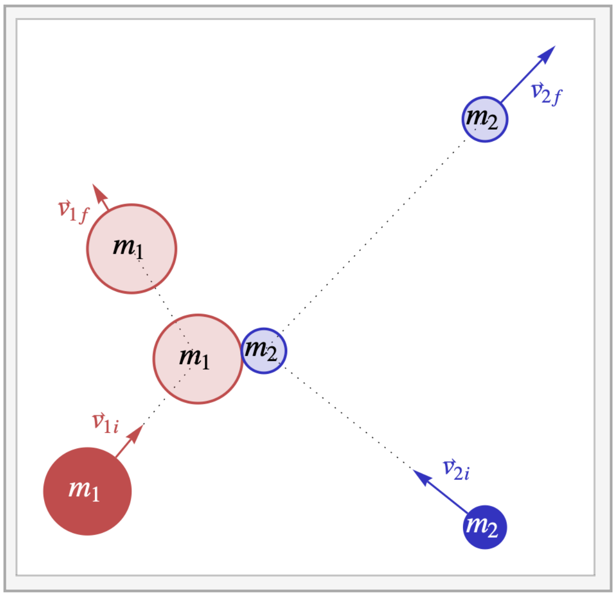
It is a very deep and fundamental matter that symmetries give rise to conserved quantities. This result is known as Noether's theorem. The mathematician Amalie Noether demonstrated around 1920 that to every symmetry there is a conservation law. For example, the invariance or symmetry of the laws of physics under time translations, that the laws are the same now as a few minutes from now, implies that there is a conserved quantity; this is energy! The symmetry that the laws of physics are invariant under spatial translations implies a conserved quantity, in this case linear momentum. Rotational symmetry implies conservation of angular momentum.

I.4 - Two-Body Collisions

Introduction

We now consider two-body collisions as a special case of our more general discussion. If there are no external forces act while two bodies collide, then the total momentum of the two-body system is conserved. Even if there are external forces, often we can neglect them and consider momentum conserved. Consider a mid-air collision between two bodies. Gravity acts as an external force during the collision but usually, to a good approximation, the collision is so fast that the large impulsive internal forces dominate the gravity force and gravity can be neglected. We can equate the total momentum just before and just after the collision.

Momentum Conservation



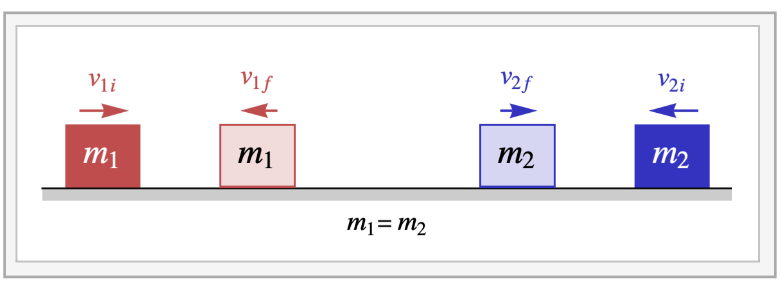
Interactive Figure

Mass m_1 moving at $\vec{v}_{1i}$ collides with mass m_2 moving at $\vec{v}_{2i}$. After the collision the velocities are $\vec{v}_{1f}$ and $\vec{v}_{2f}$. The conservation of momentum for a two-body collision has the form

$$m_1\,\vec{v}_{1i} + m_2\,\vec{v}_{2i} = m_1\,\vec{v}_{1f} + m_2\,\vec{v}_{2f}.$$

The left-hand side is the total initial momentum and the right hand side is the total final momentum.

In the case of a one dimensional collision then the above expression applies but we may omit the vector arrows. In one dimension a vector is a real number and the sign gives the direction. The vector nature of momentum and velocity is reflected in their signs.



Interactive Figure

Elastic Collisions - Kinetic Energy Conservation

Typically energy is lost in a collision. Often to a reasonable approximation we can consider conservation of energy. The relevant energy is kinetic.

$$\frac{1}{2}\, m_1\, v_{1i}^2 + \frac{1}{2}\, m_2\, v_{2i}^2 = \frac{1}{2}\, m_1\, v_{1f}^2 + \frac{1}{2}\, m_2\, v_{2f}^2 \quad \text{(elastic)}$$

In other words

$$K_{\text{tot},i} = K_{\text{tot},f}. \quad \text{(elastic)}$$

Inelastic and Totally Inelastic Collisions

When we say a collision is inelastic we mean merely that it is not elastic.

$$K_{\text{tot},i} \neq K_{\text{tot},f} \quad \text{(inelastic)}$$

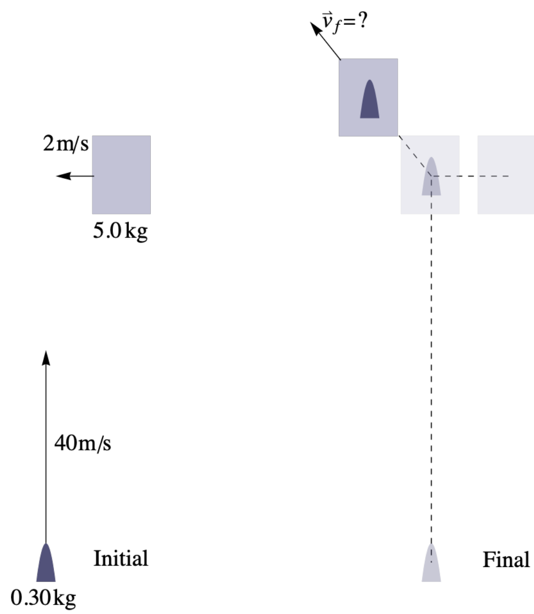
The extreme case of an inelastic collision is called totally inelastic. The most energy that can be lost in a collision is when the two colliding objects stick together. This means that the two objects have the same final velocity.

$$\vec{v}_{1f} = \vec{v}_{2f} = \vec{v}_f \quad \text{(totally inelastic)}$$

The conservation of momentum formula then has the simple form:

$$m_1\, \vec{v}_{1i} + m_2\, \vec{v}_{2i} = (m_1 + m_2)\, \vec{v}_f. \quad \text{(totally inelastic)}$$

Example I.4 - A Projectile embeds in a Block



(a) A projectile with a mass of 0.30 kg moving in the y-direction at 40 m/s collides with and embeds in a 5.0-kg mass moving in the negative x-direction at 2.0 m/s. What is the combined final velocity of the block and projectile after the collision?

Solution

$$m_1 = 0.30\text{ kg},\ m_2 = 5.0\text{ kg},\ \vec{v}_{1i} = \langle 0, 40\rangle\text{ m/s},\ \vec{v}_{2i} = \langle -2.0, 0\rangle\text{ m/s}$$

This is a totally inelastic collision. We can solve for the final velocity.

$$m_1\, \vec{v}_{1i} + m_2\, \vec{v}_{2i} = (m_1 + m_2)\, \vec{v}_f \implies \vec{v}_f = \frac{m_1\, \vec{v}_{1i} + m_2\, \vec{v}_{2i}}{m_1 + m_2} = \langle -1.89, 2.26\rangle\text{ m/s}$$

(b) A one-dimensional version of the previous problem follows: A projectile with a mass of 0.30 kg moving in the x-direction at 40 m/s collides with and embeds in a 5.0-kg mass moving in the negative x-direction at 2.0 m/s. What is the combined final velocity of the block and projectile after the collision?

Solution

One-dimensional vectors are real numbers where the sign gives the direction. We must be careful with signs.

$$m_1 = 0.30\text{ kg},\ m_2 = 5.0\text{ kg},\ v_{1i} = 40\text{ m/s},\ v_{2i} = -2.0\text{ m/s}$$

We can similarly solve for the final velocity.

$$m_1\, v_{1i} + m_2\, v_{2i} = (m_1 + m_2)\ v_f \quad\Longrightarrow\quad v_f = \frac{m_1\, v_{1i} + m_2\, v_{2i}}{m_1 + m_2} = +0.377\ \mathrm{m/s}$$

Example I.5 - A Bullet Passes through a Block

A 17.5-gram bullet moving at 355 m/s passes through an initially stationary 1.55-kg block. If the bullet passes exits the block at 125 m/s, then what is the final velocity of the block?

Solution

The bullet's mass must be converted to kilograms. We are given the initial and final velocities of the bullet and the initial velocity of the block is zero. The final velocity of the block is the only unknown in the momentum conservation equation and we can then solve for it.

$$m_1 = 0.0175\ \mathrm{kg}\ ,\quad v_{1i} = 355\ \mathrm{m/s}\ ,\quad v_{1f} = 125\ \mathrm{m/s}\ ,\quad m_2 = 1.55\ \mathrm{kg}\ ,\quad v_{2i} = 0\ ,\quad v_{2f} = ?$$

This is a one-dimensional problem.

$$m_1\, v_{1i} + m_2\, v_{2i} = m_1\, v_{1f} + m_2\, v_{2f} \quad\Longrightarrow\quad m_1\, v_{1i} + 0 = m_1\, v_{1f} + m_2\, v_{2f} \quad\Longrightarrow\quad v_{2f} = \frac{m_1}{m_2}\,(v_{1i} - v_{1f}) = 2.60\ \mathrm{m/s}$$

One Dimensional Elastic Collisions

For the case of a one dimensional elastic collision we can solve for the final velocities in terms of the initial velocities and the masses.

$$m_1\, v_{1i} + m_2\, v_{2i} = m_1\, v_{1f} + m_2\, v_{2f} \quad \text{(momentum eq.)}$$

$$\frac{1}{2}\, m_1\, v_{1i}^2 + \frac{1}{2}\, m_2\, v_{2i}^2 = \frac{1}{2}\, m_1\, v_{1f}^2 + \frac{1}{2}\, m_2\, v_{2f}^2 \quad \text{(energy eq.)}$$

Given the two initial velocities and the masses, we can solve for the final velocities. This algebra is awkward, so the solutions will be given.

$$v_{if} = \frac{m_1 - m_2}{m_1 + m_2}\, v_{1i} + \frac{2\, m_2}{m_1 + m_2}\, v_{2i} \quad\text{and}\quad v_{2f} = \frac{2\, m_1}{m_1 + m_2}\, v_{1i} + \frac{m_2 - m}{m_1 + m_2}\, v_{2i}$$

Example I.6 - One-Dimensional Elastic Collision

A car rolls at speed v toward a truck with twice the car's mass rolling in the opposite direction at the same speed. The collision is elastic and head-on, so that both vehicles stay on the same line. What are both final velocities?

Solution

We must write our answers in terms of the speed v. Since the car and truck are moving in opposite directions their velocities must have opposite signs. We will take the car's direction as positive. Take m as the mass of the car, so the track has mass $2m$. Since m is not given the answer cannot depend on it; we will see is cancels.

$$m_1 = m,\ m_2 = 2m,\ v_{1i} = v,\ v_{2i} = -v$$

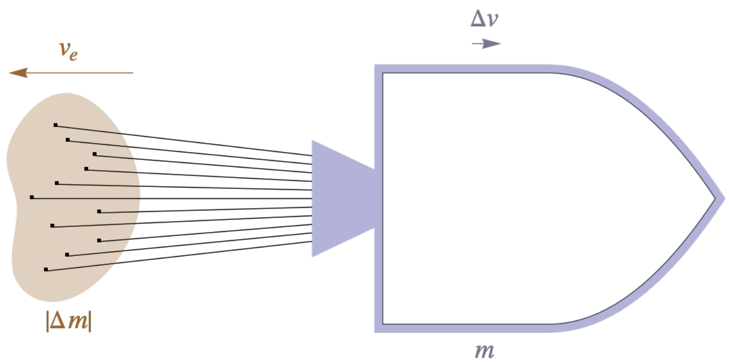
This is now a straight-forward application of the solutions given above.

$$v_{if} = \frac{m_1 - m_2}{m_1 + m_2}\, v_{1i} + \frac{2\, m_2}{m_1 + m_2}\, v_{2i} = \frac{m - 2\, m}{3\, m}\, v + \frac{4\, m}{3\, m}\,(-v) = -\frac{1}{3}\, v - \frac{4}{3}\, v = -\frac{5}{3}\, v$$

$$v_{2f} = \frac{2\, m_1}{m_1 + m_2}\, v_{1i} + \frac{m_2 - m}{m_1 + m_2}\, v_{2i} = \frac{2\, m}{3\, m}\, v_{1i} + \frac{2\, m - m}{3\, m}\,(-v) = \frac{2}{3}\, v - \frac{1}{3}\, v = \frac{1}{3}\, v$$

I.5 - Rocket Propulsion

When moving with respect to a medium one can propel something forward by pushing backward. To walk, you push backward in the floor and the floor then pushes forward on you. A boat pushes backward on the water and the water pushes forward in it. A plane or jet propels itself similarly by pushing backward on the air. This leads to an obvious question for rocket propulsion. How does a rocket propel itself forward in the vacuum of space? The answer is that the rocket throws part of itself, its spent fuel, backward and thus propels the rest of the rocket forward. The mass change of a rocket is an essential part of its propulsion.



Consider a rocket of mass m moving with a velocity v. The rocket propels itself forward by shooting spent fuel backward at a speed of v_e, the exhaust speed, relative to the rocket. In doing this the mass of the rocket changes. The (positive) mass of ejected fuel is $|\Delta m|$. Ejecting the fuel backward makes the rocket recoil forward with a small Δv. Looking at conservation of momentum in the frame where the rocket was initially at rest gives

$$0 = m\,\Delta v - v_e\,|\Delta m| \quad \Longrightarrow \Delta v = v_e\,\frac{|\Delta m|}{m}$$

The force propelling a rocket forward due to this expelled fuel is known as the thrust. From the momentum form of the second law the force on the expelled fuel is

$$F = \frac{\Delta p}{\Delta t} = v_e\,\frac{|\Delta m|}{\Delta t}$$

By Newton's third law, the thrust, the forward force on the rocket is

$$\text{thrust} = v_e\,\frac{|\Delta m|}{\Delta t}$$