Chapter J

Rotational Kinematics and Energy

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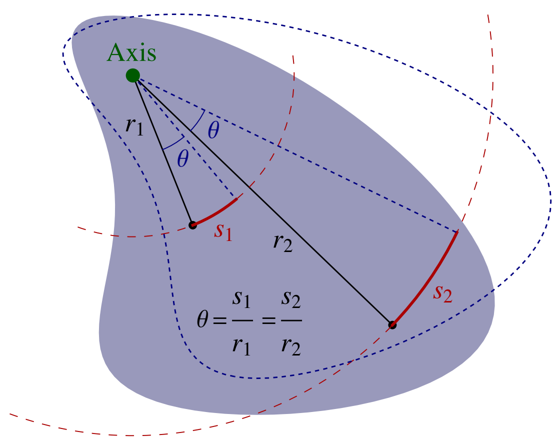
J.1 - Kinematics of Rotations about a Fixed Axis

Rigid Bodies and Rotations in General

The distance between any two positions in a rigid body is fixed. A book can be viewed as a rigid body as long as it is kept closed; when it is opened then the distance between a point on the back cover and a point on the front cover varies and it is not rigid.

A rotation is described by an axis and an angle. An axis is a line. The axis of rotation of a door is its hinge. The axis of a tire is its axle. Often in a planar diagram we will draw an axis as a point. The axis is then the line perpendicular to that plane through the point. A rotation is about some axis and by some angle. Note that when a rigid body rotates different points move different distances. The distance a point moves s is proportional to the (perpendicular) distance from the axis r, but the ratio s/r is the same for any two points. This ratio is just the angle of rotation in radians.



Kinematical Variables

To understand rotational kinematics it is essential to appreciate the analogy to one dimensional kinematics. Recall that a one dimensional vector is a real number and that its direction is given by its sign. The rotational analog of the position is the angle of rotation. The other kinematical variables follow:

	One Dimensional Linear Motion	**Rotations about a Fixed Axes**
Position	x (m)	θ (angle in rad)
Velocity	v (m/s)	ω (angular velocity in rad/s)
Average	$v_{\text{ave}} = \frac{\Delta x}{\Delta t}$	$\omega_{\text{ave}} = \frac{\Delta\theta}{\Delta t}$
Instantaneous	$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$	$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$
Acceleration	a $\left(\text{m}/\text{s}^2\right)$	α $\left(\text{angular acceleration in rad}/\text{s}^2\right)$
Average	$a_{\text{ave}} = \frac{\Delta v}{\Delta t}$	$\alpha_{\text{ave}} = \frac{\Delta\omega}{\Delta t}$
Instantaneous	$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$	$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$

Constant Angular Acceleration

Since the rotational variables θ, ω and α are interrelated the same as x, v and a, we can find expressions for constant angular acceleration.

One Dimensional Linear Motion	Rotations about a Fixed Axes
$v = v_0 + a\,t$	$\omega = \omega_0 + \alpha\,t$
$\Delta\,x = \frac{1}{2}\,(v_0 + v)\;t$	$\Delta\,\theta = \frac{1}{2}\,(\omega_0 + \omega)\;t$
$\Delta\,x = v_0\,t + \frac{1}{2}\,a\,t^2$	$\Delta\,\theta = \omega_0\,t + \frac{1}{2}\,\alpha\,t^2$
$v^2 = v_0^2 + 2\,a\;\Delta\,x$	$\omega^2 = \omega_0^2 + 2\,\alpha\;\Delta\,\theta$

Example J.1 - Decelerating Ceiling Fan

In 8 s, a ceiling fan slows uniformly from 20 rev/min to 8 rev/min.

(a) What is the angular acceleration of the fan?

Solution

We need to convert our angular velocities, the rates of rotation, from rev/min to rad/s.

$$\omega_0 = 20\,\frac{\text{rev}}{\text{min}} \times \frac{2\,\pi\,\text{rad}}{\text{rev}} \times \frac{1\text{ min}}{60\text{ s}} = 2.0944\,\frac{\text{rad}}{\text{s}}$$
$$\omega = 8\,\frac{\text{rev}}{\text{min}} \times \frac{2\,\pi\,\text{rad}}{\text{rev}} \times \frac{1\text{ min}}{60\text{ s}} = 0.83776\,\frac{\text{rad}}{\text{s}}$$

We also know $t = 8$ s and we are looking for α. Use the first equation for constant angular acceleration: $\omega = \omega_0 + \alpha\,t$.

$$\omega = \omega_0 + \alpha\,t \implies \alpha = \frac{\omega - \omega_0}{t} = -0.157\,\frac{\text{rad}}{\text{s}^2}$$

(b) How many times did the fan rotate while slowing?

Solution

The number of rotations is related to the rotation angle $\Delta\theta$.

$$\#\text{ of rotations} = \frac{\Delta\theta}{2\,\pi}$$

We can use any constant angular acceleration equation that involves $\Delta\theta$ to find this, since we have already found α. To find the answer without reference to the α found in part (a) we will use the second equation.

$$\Delta\,\theta = \frac{1}{2}\,(\omega_0 + \omega)\;t = 11.729\text{ rad} \implies \frac{\Delta\theta}{2\,\pi} = 1.87$$

Relation Between Linear and Rotational Variables

In chapter 6 we described circular motion in terms of centripetal and tangential directions, where the centripetal direction is toward the center of the circle and the tangential direction is in the direction of motion, meaning the direction of the velocity. A point on a rigid body a distance r from the center moves in a circle of radius r, so that discussion is relevant here.

The velocity is purely tangential, where v_t is just the speed, which is just $v_t = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$ where Δs is the small arc length traveled in the small time Δt. An angle in radians is defined as the arc length per unit radius, $\theta = s/r$. It follows that the small arc length Δs is related to small angle $\Delta\theta$ by $\Delta s = r\,\Delta\theta$. Since $\omega = \lim_{\Delta\,t \to 0} \frac{\Delta\theta}{\Delta t}$, we get

$$v_t = \lim_{\Delta\,t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta\,t \to 0} \frac{r\,\Delta\theta}{\Delta t} = r\left(\lim_{\Delta\,t \to 0} \frac{\Delta\theta}{\Delta t}\right) = r\,\omega$$

In chapter 6 we saw the acceleration had a centripetal component a_c and a tangential component that depended on the change in the speed. The centripetal component a_c can be written in terms of the radius r and the angular velocity ω.

$$a_c = \frac{v^2}{r} = \frac{(r\,\omega)^2}{r} = \omega^2\, r$$

The tangential component a_t can similarly be written in terms of the radius r and the angular acceleration α.

$$a_t = \lim_{\Delta t \to 0} \frac{\Delta v_t}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta(r\,\omega)}{\Delta t} = r \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = r\,\alpha.$$

The tangential and centripetal directions are perpendicular. It follows that the magnitude of the acceleration can be found by applying the Pythagorean theorem to the perpendicular components a_c and a_t.

$$a = \sqrt{a_c^2 + a_t^2}$$

Example J.2 - Decelerating Ceiling Fan (continued)

(c) While the fan is slowing there is an instant when $\omega = 0.85$ rad/s. At that instant, what are the tangential velocity and the magnitude of the acceleration of the tip of the fan, 0.90 m from the axis.

Solution

$$\omega = 0.85 \text{ rad/s}\,, \quad r = 0.90 \text{ m} \;\text{ and }\; \alpha = -0.157\,\frac{\text{rad}}{\text{s}^2} \;\;\text{(from part (a))}$$

We can solve for the tangential velocity.

$$v_t = r\,\omega = 0.765 \text{ m/s}$$

To find the magnitude of the acceleration we use the Pythagorean theorem with the perpendicular components a_c and a_t.

$$a_c = \omega^2\, r = 0.65025\,\frac{\text{m}}{\text{s}^2} \;\text{ and }\; a_t = r\,\alpha = -0.14137\,\frac{\text{m}}{\text{s}^2} \implies a = \sqrt{a_c^2 + a_t^2} = 0.665\,\frac{\text{m}}{\text{s}^2}$$

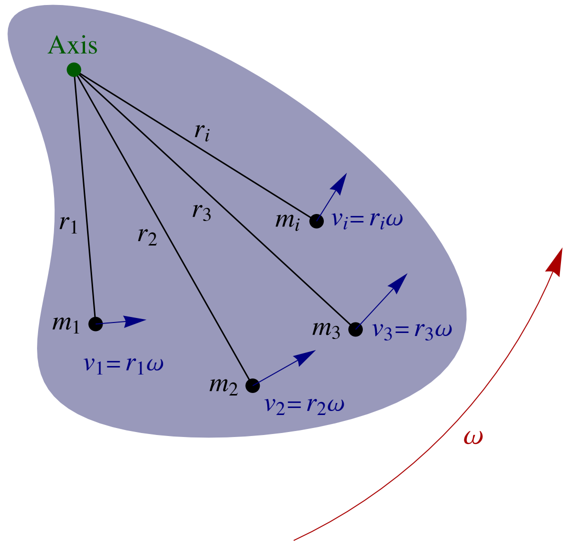
J.2 - Dynamics of Rigid Bodies Rotating about an Axis

Summary and Analogy with One Dimensional Motion

	One Dimensional Linear Motion	Rotations about a Fixed Axes
Kinematics	$x,\ v,\ a$	$\theta,\ \omega,\ \alpha$
Force	F	τ (torque)
Inertia	m	I (moment of inertia)
Momentum	$p = m\,v$	$L = I\,\omega$ (angular momentum)
Second Law	$F_{\text{net}} = m\,a$ $F_{\text{net}} = \lim_{\Delta t\to 0} \frac{\Delta \vec{p}}{\Delta t}$	$\tau_{\text{net}} = I\,\alpha$ $\tau_{\text{net}} = \lim_{\Delta t\to 0} \frac{\Delta L}{\Delta t}$
Conservation of Momentum	$F_{\text{net}}^{\text{ext}} = 0$ $\implies \Delta\, p_{\text{tot}} = 0$	$\tau_{\text{net}}^{\text{ext}} = 0$ $\implies \Delta\, L_{\text{tot}} = 0$
Kinetic Energy	$K = \frac{1}{2}\, m\, v^2$	$K = \frac{1}{2}\, I\,\omega^2$
Work	$W = F\,\Delta x$	$W = \tau\,\Delta\theta$
Work–Energy Theorem	$W_{\text{net}} = \Delta\, K$	$W_{\text{net}} = \Delta\, K$
Power	$\mathcal{P}_{\text{ave}} = \frac{W}{\Delta t} = F\,v$	$\mathcal{P}_{\text{ave}} = \frac{W}{\Delta t} = \tau\,\omega$

This table is an extension of the preceding tables for kinematics. Now we consider dynamics. Dynamical quantities are things like force and mass. The rotational analog of force is called torque and the rotational analog of mass is the moment of inertia. These two quantities are undefined in the table; their definitions follow. For all the other quantities, the above table serves as the definitions of the variables.

Kinetic Energy and the Definition of the Moment of Inertia



Consider a rigid body consisting of point masses m_i. The perpendicular distance from the axis to the i^{th} mass is r_i. If the rigid body rotates with angular velocity ω then the speed of the i^{th} mass is

$$v_i = r_i\,\omega.$$

The total kinetic energy is the sum of the kinetic energies of all the masses. Using the above expression for the speed we get

$$K = \frac{1}{2}\left(m_1\,v_1^2 + m_2\,v_2^2 + \ldots\right) = \frac{1}{2}\left(m_1\,r_1^2\,\omega^2 + m_2\,r_2^2\,\omega^2 + \ldots\right) = \frac{1}{2}\left(m_1\,r_1^2 + m_2\,r_2^2 + \ldots\right)\omega^2.$$

Using our desired expression for the kinetic energy $K = \frac{1}{2}\,I\,\omega^2$ we get the expression for I moment of inertia for a rigid body about some axis.

$$I = m_1\,r_1^2 + m_2\,r_2^2 + \ldots = \sum m\,r^2$$

This expression is for a discrete distribution; this means that the distribution is a collection of point masses.

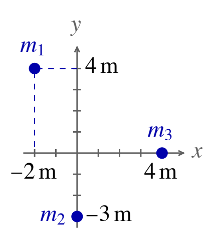
Moment of Inertia

The moment of inertia is a property of a rigid body and an axis. The further the mass is from the axis, the larger the moment.

$$I = m_1\,r_1^2 + m_2\,r_2^2 + \ldots = \sum m\,r^2$$

Example J.3 - A Discrete Distribution

Three masses are attached to a light board (of negligible mass). A $m_1 = 3\text{ kg}$ mass is at $(-2\text{ m}, 4\text{ m})$, a $m_2 = 5\text{ kg}$ mass is at $(0, -3\text{ m})$ and a $m_3 = 2\text{ kg}$ mass is at $(4\text{ m}, 0)$.



(a) What is the moment of inertia about the y-axis?

Solution

The moment of inertia is $I = \sum m\,r^2$, where the perpendicular distance from the y-axis is $|x|$,

$$I = m_1\,x_1^2 + m_2\,x_2^2 + m_3\,x_3^2 = 3\text{ kg}\,(-2\text{ m})^2 + 5\text{ kg}\,0^2 + 2\text{ kg}\,(4\text{ m})^2 = 44\text{ kg m}^2$$

(b) What is the moment of inertia about the x-axis?

Solution

The perpendicular distance from the x-axis is $|y|$,

$$I = m_1\, y_1^2 + m_2\, y_2^2 + m_3\, y_3^2 = 3\text{ kg}\,(-4\text{ m})^2 + 5\text{ kg}\,(-3\text{ m})^2 + 2\text{ kg}\,0^2 = 93\text{ kg m}^2$$

(c) Suppose this rotates about the origin at 3 rad/s. What is its kinetic energy?

Solution

We first need to fine the moment of inertia about the origin and then use $K = \frac{1}{2} I\,\omega^2$. The perpendicular distance from the origin is $r = \sqrt{x^2 + y^2}$.

$$\begin{aligned} I &= m_1\left(x_1^2 + y_1^2\right) + m_2\left(x_2^2 + y_2^2\right) + m_3\left(x_2^2 + y_2^2\right) \\ &= 3\text{ kg}\left((-2\text{ m})^2 + (-4\text{ m})^2\right) + 5\text{ kg}\left(0^2 + (-3\text{ m})^2\right) + 2\text{ kg}\left((4\text{ m})^2 + 0^2\right) \\ &= 137\text{ kg m}^2 \end{aligned}$$

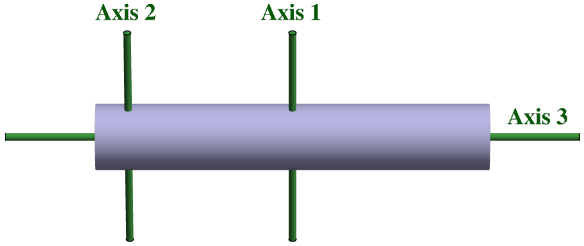
Note that since $\sum m\left(x^2 + y^2\right) = \sum m\,x^2 + \sum m\,y^2$, the moment for part (c) must be the sum of the moments for parts (a) and (b), $137 = 44 + 93$.

To get the kinetic energy use $\omega = 3\text{ rad/s}$.

$$K = \frac{1}{2} I\,\omega^2 = 616.5\text{ J}$$

Example J.4 - Moments for Different Axes

Consider the cylindrical rigid body and the three axes shown.



If I_1 is the moment of inertia for Axis 1, I_2 for Axis 2 and I_3 for Axis 3. Rank the three moments.

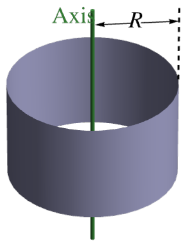
Solution

The distribution of mass is closest to Axis 3, so I_3 is the smallest. The mass is further from Axis 2 than Axis 2, so we get:

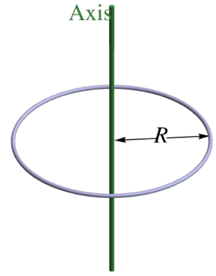
$$I_3 < I_1 < I_2$$

Moments of Inertia for Uniform Bodies

Hoop or Thin-shelled Hollow Cylinder about Perpendicular Axis through Center



Hollow Cylindrical Shell



Hoop

First consider a hoop of mass M and radius R rotating about a perpendicular axis through the center. r is the distance from the axis to the infinitesimal mass dm. All the mass is at the same distance

$$r = R = \text{constant}.$$

It is possible to find I without actually performing an integral.

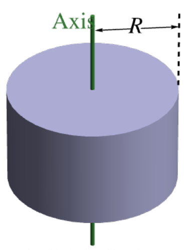
$$I = \sum m\, r^2 = \sum m\ R^2 = R^2 \sum m$$

Since $M = \sum m$ we get

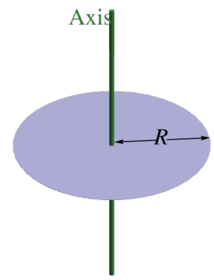
$$I = M\, R^2.$$

Now consider a thin-shelled hollow cylinder about the central axis. It is still true that all the mass is the same perpendicular distance of $r = R$ from the axis and the above formula still applies.

Disk or Solid Cylinder about Perpendicular Axis through the Center



Solid Cylinder



Disk

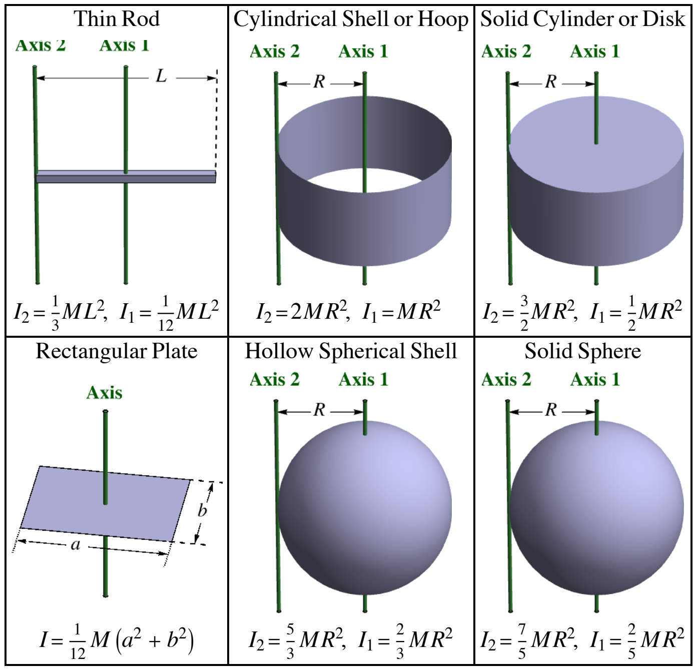
It should now be clear that the moment for a disk should be the same as a solid cylinder. We can break up a disk into concentric thin rings of radius r with thickness dr. The limits of integration become

$$0 \le r \le R.$$

If all the mass were at $r = 0$, then $I = 0$. If all were at $r = R$, then it would be the same as a hoop and $I = M\, R^2$. It then must satisfy: $0 < I < M\, R^2$. It can be shown using calculus that

$$I = \frac{1}{2}\, M\, R^2.$$

Table of Moments of Inertia for Uniform Distributions with Different Geometries and Axes



By a uniform distribution of mass we mean the density is constant throughout the body.

J.3 - Energy and Rigid Bodies

Gravitational Potential Energy

It is a straightforward matter to find the potential energy of a rigid body.

$$U = \sum_i m_i\, g\, y_i = g \sum_i m_i\, y_i = g\, M\, y_{\text{cm}}$$

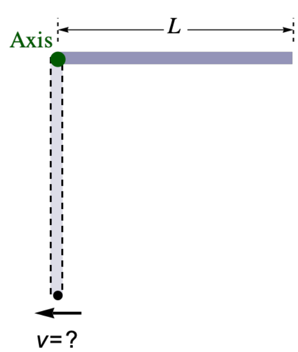
Here M is the total mass and y_{cm} is the height of the center of mass. It follows that the total potential energy of a rigid body is

$$U = M\, g\, y_{\text{cm}}.$$

This is easy to interpret. When calculating the potential energy of a rigid body we treat the body as if all its mass is at its center of mass.

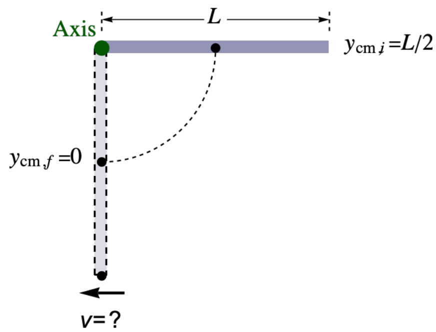
Example J.5 - Swinging Rod

A uniform rod of length L swings without friction about an axis at one end. It is released from rest from a horizontal position. What is the speed of the tip as it swings through the position directly below the axis"



Solution

We will use conservation of energy to find the angular velocity of the rod below and from that find the linear velocity of the tip.



We have kinetic energy $K = \frac{1}{2} I \omega^2$ in the rotating rod and potential energy $U = m g y_{\mathrm{cm}}$, where y_{cm} is the height of the center of the rod. The mechanical energy is conserved and given by.

$$E = \frac{1}{2} I \omega^2 + m g y_{\mathrm{cm}}$$

The initial kinetic energy is zero and the initial and final heights of the center of mass are $y_{\mathrm{cm},i} = L/2$ and $y_{\mathrm{cm},f} = 0$, choosing the lower point as the zero.

$$E_i = E_f \implies 0 + m g \frac{L}{2} = \frac{1}{2} I \omega_f^2 + 0$$

For our uniform thin rod we have $I = \frac{1}{3} m L^2$

$$m g \frac{L}{2} = \frac{1}{2} I \omega_f^2 = \frac{1}{2} \left(\frac{1}{3} m L^2 \right) \omega_f^2 \implies \omega_f = \sqrt{\frac{3g}{L}}$$

The velocity can be found from the angular velocity using the tangent velocity formula, $v_t = r \omega$. Since r is the distance from the axis we have $r = L$.

$$v = r \omega = L \omega_f = L \sqrt{\frac{3g}{L}} = \sqrt{3 g L}$$

Rotation with Translation

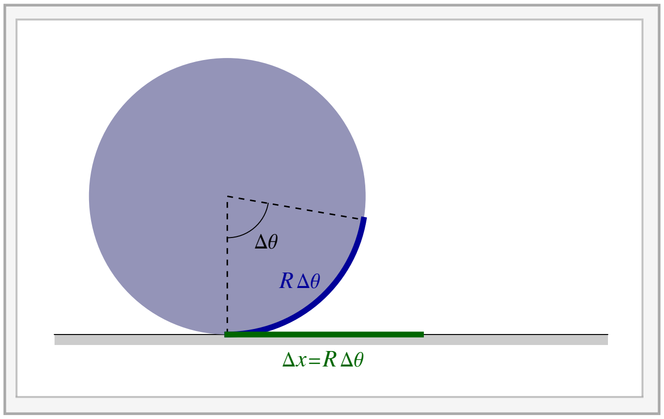
If a body is rotating and translating then there is kinetic energy in the rotation and in the translation. The total kinetic energy K_{tot} is the sum of K_{tran}, the translational kinetic energy and K_{rot}, the rotational kinetic energy. If v is the speed of the center of mass and I is the moment of inertia through the center of mass then

$$K_{\mathrm{tot}} = K_{\mathrm{tran}} + K_{\mathrm{rot}} = \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2$$

Rolling Motion

In the case of a rolling body we have a rolling constraint that a body rolls without slipping. This is: the arc length along the rolling radius of the body is the same as the distance Δx it moves along the surface it rolls on. If it rotates by an angle $\Delta\theta$ then the arc length is $R \Delta \theta$. The constraint becomes

$$R \Delta \theta = \Delta x.$$



Interactive Figure

Since the velocity is $v = \lim_{\Delta t \to 0} \Delta x / \Delta t$ and the angular velocity is $\omega = \lim_{\Delta t \to 0} \Delta\theta / \Delta t$ the rolling constraint becomes

$$R\,\omega = v$$

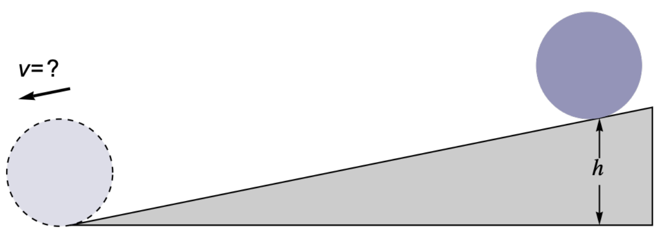
and similarly, with the acceleration and angular acceleration and we get

$$R\,\alpha = a.$$

$$K_{\text{tot}} = K_{\text{tran}} + K_{\text{rot}} = \frac{1}{2}\,M\,v^2 + \frac{1}{2}\,I\,\omega^2 = \frac{1}{2}\left(M + \frac{I}{R^2}\right)v^2$$

The last expression above follows from $v = R\omega$ and replacing the angular velocity ω with v/R. The total kinetic energy depends on the total mass M and the speed of the center of mass. The rotational kinetic energy depends on the moment of inertia about the axis through the center of mass.

Example J.6 - Rigid Body Races



Different objects, a uniform solid sphere, a uniform hollow spherical shell, a uniform solid cylinder and a uniform hollow cylindrical shell, are rolled down an incline. The objects have varying masses and radii. Which will move fastest at the bottom of the incline, which we take to be of height h? Find the speed of each. Assume that they roll without slipping.

Solution

We will solve all four cases by writing $I = \kappa\,M\,R^2$ where the table below gives the different κ values.

Object $I = \kappa\,M\,R^2$	κ
Solid Sphere	2/5
Solid Cylinder	1/2
Hollow Spherical Shell	2/3
Hollow Cylndrical Shell	1

We will see that the mass M and radius R scale out of the problem and the speed at the bottom will only depends on κ and h. The only potential energy here is gravitational potential energy and that is determined by the position of the center of mass.

$$U = M\,g\,y_{\text{cm}}$$

For the total kinetic energy we have translational and rotational terms.

$$K_{\text{tot}} = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2}\,M\,v^2 + \frac{1}{2}\,I\,\omega^2$$

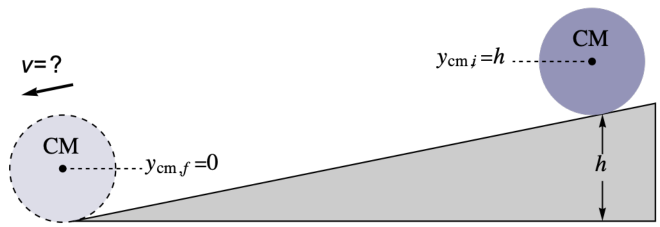
Our rolling without slipping constraint $R\,\Delta\,\theta = \Delta\,x$ implies that $R\,\omega = v$. Using $I = \kappa\,M\,R^2$ and $\omega = v/R$ we get:

$$K_{\text{tot}} = \frac{1}{2}\,M\,v^2 + \frac{1}{2}\,I\,\omega^2 = \frac{1}{2}\,M\,v^2 + \frac{1}{2}\left(\kappa\,M\,R^2\right)\left(\frac{v}{R}\right)^2 = \frac{1}{2}\,(1+\kappa)\,M\,v^2.$$

It follows that our total mechanical energy is

$$E = K_{\text{tot}} + U = \frac{1}{2}\,(1+\kappa)\,M\,v^2 + M\,g\,y_{\text{cm}}.$$

Choosing the lowest position of the center of mass to be $y_{\text{cm}} = 0$ we get $y_{\text{cm},i} = h$ and $y_{\text{cm},f} = 0$. Out initial kinetic energy is zero.



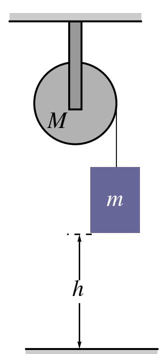
Conservation of mechanical energy gives:

$$E_i = E_f \implies 0 + M\,g\,h = \frac{1}{2}\,(1+\kappa)\,M\,v^2 + 0 \implies v = \sqrt{\frac{2\,g\,h}{1+\kappa}}.$$

It should now be clear that the smaller the κ, the larger the speed at the bottom. The order from fastest to slowest is: solid sphere, solid cylinder, hollow sphere and hollow cylinder.

Previously, when considering pulleys we used *ideal pulleys*, which were frictionless and light, meaning that the mass of the pulley was negligible compared to the other masses in the problem. Now we consider an example with a pulley with mass that is not small.

Example J.7 - A Pulley with Mass



A mass m hangs from a string connected to a pulley as shown above. Take the pulley to be a uniform disk of mass M. The string is tied to the pulley and is wrapped around it many times, to ensure that the string does not slip on the pulley. If m is released from rest at a height h above a floor, then what is its speed when it hits the floor. Assume no friction in the pulley or elzewhere.

Solution

We are not given the radius of the pulley but we will see that it cancels out. Energy is conserved here. The energy consists of the translational kinetic energy of the falling mass, the rotational kinetic energy of the pulley and the gravitational potential energy of the falling mass.

$$E = \frac{1}{2}\,m\,v^2 + \frac{1}{2}\,I\,\omega^2 + m\,g\,y$$

The constraint that the string does not slip along the pulley is equivalent to the rolling constraint. The arc length of rotation must equal the distance the string (and m) move.

$$R\,\Delta\theta = \Delta x$$

and that gives a relation between the angular velocity of the pulley and the linear velocity of the falling mass.

$$R\,\omega = v$$

Because the pulley is a uniform disk we can write.

$$I = \frac{1}{2}\,M\,R^2$$

Using that and writing $\omega = v/R$ we can rewrite the expression for the energy.

$$E = \frac{1}{2}\,m\,v^2 + \frac{1}{2}\times\frac{1}{2}\,M\,R^2\left(\frac{v}{R}\right)^2 + m\,g\,y = \frac{1}{2}\left(m + \frac{1}{2}\,M\right)v^2 + m\,g\,y$$

Taking $y_i = h$, $y_f = 0$ and $v_i = 0$, we can solve for v_f.

$$E_i = E_f \implies 0 + m\,g\,h = \frac{1}{2}\left(m + \frac{1}{2}\,M\right)v_f^2 + 0 \implies v_f = \sqrt{\frac{2\,m\,g\,h}{m + M/2}}$$