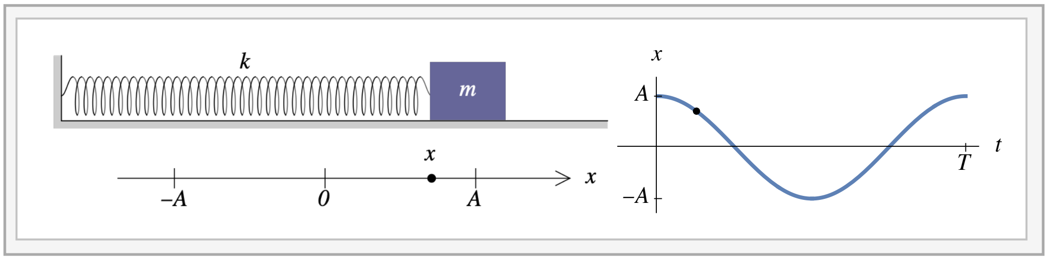
Chapter M

Oscillatory Motion

Blinn College - Physics 1401 - Terry Honan

M.1 - Simple Harmonic Motion

The Mass-Spring System



Interactive Figure

Consider a mass sliding without friction on a horizontal surface. The force of a spring is given by Hooke's law $F = -k\,x$. Applying Newton's second law gives:

$$F_{\text{net}} = m\,a \quad \Longrightarrow \quad -k\,x = m\,a$$

Define the *angular frequency* by

$$\omega = \sqrt{\frac{k}{m}}\,.$$

This gives a simple relation between the acceleration a and the position x.

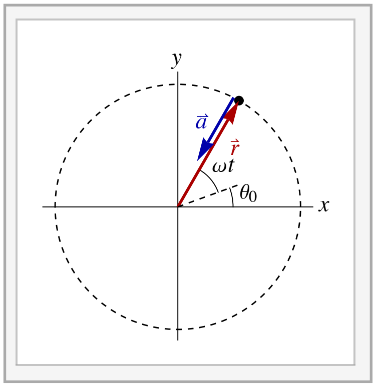
$$a = -\omega^2\,x$$

We will see that this is a quite common expression that occurs in situations that give oscillatory motion.

General Simple Harmonic Motion and Uniform Circular Motion

Whenever the motion of a particle is described by an equation of the form $a = -\omega^2\,x$, we get what is called simple harmonic motion. The mass/spring system is our primary example of simple harmonic motion but we will see other examples this chapter. To solve this equation for the position as a function of time $x(t)$, we note the similarity of this to uniform circular motion. When a particle moves in uniform circular motion of radius r there is a centripetal acceleration of magnitude $\omega^2\,r$. Choose the origin to be the center of the circle; the position vector $\vec{r}$ is radial and since the acceleration is centripetal it is oppose that. With uniform circular motion the particle moves with a constant angular velocity and we get

$$\theta(t) = \theta_0 + \omega\,t$$



It follows then, that $\vec{a} = -\omega^2\,\vec{r}$. If we look at one component of this vector expression we get

$$a_x = -\omega^2\,x$$

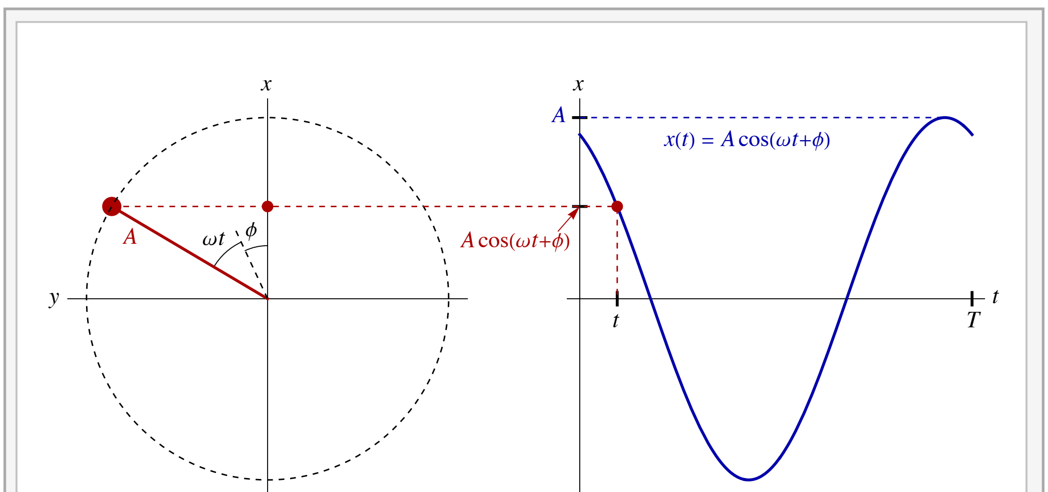
The x-component of the position vector $\vec{r}$ is $x = r\cos\theta$. This gives

$$x(t) = r\cos(\omega\,t + \theta_0)$$

Since the a_x and x in $a_x = -\omega^2\,x$ is the sames as between a_x and x in the original expression for simple harmonic motion, $a = -\omega^2\,x$, we can write the solution for $x(t)$ for simple harmonic motion. We will rename the radius r as A, the *amplitude* of the simple harmonic motion and rename the θ_0 as ϕ, the *phase angle*.

$$x(t) = A\cos(\omega\,t + \phi)$$

The interactive diagram shows how uniform circular motion is related to simple harmonic motion. If we consider just the x-component of the circular motion then that corresponds to a particle in simple harmonic motion. Note in the diagram below the axes are rotated to clarify this connection.



(Note the axes are rotated.)

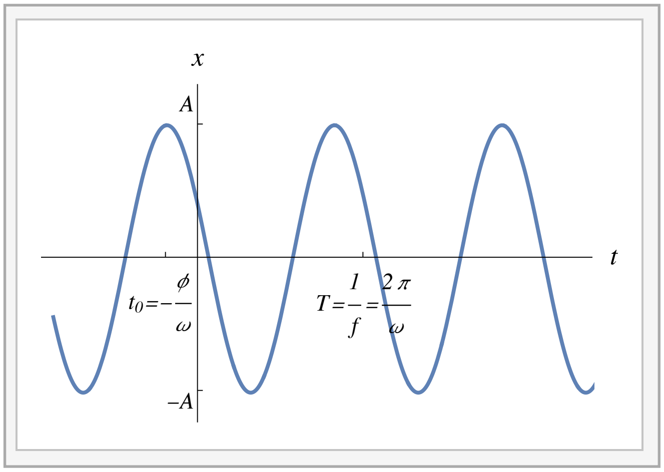
The amplitude A is the largest distance of the particle from its equilibrium position. The angular velocity of the circular motion becomes the angular frequency of simple harmonic motion. The simple harmonic motion repeats itself every period T. Since the cosine function repeats itself every 2π radians, it follows that $\omega T = 2\pi$ and the angular frequency is related to the period just as the angular velocity is to the period

$$\omega = \frac{2\pi}{T}$$

The frequency is the number of cycles per time; since the time per cycle is the period we get $f = 1/T$. Combining this gives

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \text{ and } f = \frac{1}{T} = \frac{\omega}{2\pi}.$$

The phase angle ϕ describes where in the periodic function the motion begins; changing ϕ shifts the graph along the time axis.



Interactive Figure

M.2 - Energy Considerations

The Energy of a Mass-Spring System

The total energy is the sum of kinetic and potential energies. When the mass is at the turning points $x = \pm A$, its speed is zero; we can then write $E = (1/2)\,k\,A^2$. When it passes the equilibrium point $x = 0$ the potential energy is zero and thus its kinetic energy is the maximum; so must the speed be its maximum, $v = \pm v_{\max}$. This allows us to write the energy with two equivalent forms for the total energy.

$$E = \frac{1}{2}\,m\,v^2 + \frac{1}{2}\,k\,x^2 = \begin{cases} \frac{1}{2}\,k\,A^2 \\ \frac{1}{2}\,m\,v_{\max}^2 \end{cases}$$

Equating the two expressions for the total energy gives an expression for the maximum speed in terms of the amplitude.

$$v_{\max} = \sqrt{\frac{k}{m}}\,A = \omega\,A$$

Speed and Position for General Simple Harmonic Motion

For general simple harmonic motion we have

$$x(t) = A\cos(\omega\,t + \phi)$$

The velocity becomes.

$$v(t) = -\omega\,A\sin(\omega\,t + \phi)$$

Since both sine and cosine vary between ± 1 we can identify the maximum speed as

$$v_{\max} = \omega\,A.$$

This is equivalent to what we had for the mass-spring case. The point here is to show that this is generally true. If the mass-spring energy is written in terms of the amplitude A then we can solve for v and, using the mass-spring value of ω, get

$$v = \pm\omega\,\sqrt{A^2 - x^2}\,.$$

This expression is also generally true for simple harmonic motion. To verify that generally, we can write $\cos(\omega\,t + \phi) = x/A$ and $\sin(\omega\,t + \phi) = -v/(\omega\,A)$. Using $\cos^2 + \sin^2 = 1$ we can get the result.

Example M.1 - Simple Harmonic Motion

A particle moves in simple harmonic motion with a frequency of 13 Hz. At $t = 0$ the particle is released from rest from a distance of 2.4 cm from equilibrium.

(a) What is the maximum speed and maximum acceleration of the particle?

Solution

Since the particle is released from rest the amplitude is the initial distance from equilibrium.

$$A = 2.4\ \text{cm} = 0.024\ \text{m}$$

The angular frequency ω is related to the frequency f.

$$f = 13\ \text{Hz} \implies \omega = 2\,\pi\,f = 81.681\ \text{s}^{-1}$$

The maximum speed can now be found.

$$v_{\text{max}} = \omega\,A = 1.96\ \text{m/s}$$

Since the acceleration is the second time-derivative of the position the acceleration follows from the differential equation.

$$a = -\omega^2\,x$$

The magnitude of the maximum acceleration occurs at the largest value of x, which is the amplitude A.

$$a_{\text{max}} = \omega^2\,A = 160.\ \text{m}/\text{s}^2$$

(b) At $t = 0.22$ s, what are the position, velocity and acceleration of the particle?

Solution

Since at time zero the particle is at its maximum displacement we can conclude that the phase angle ϕ is zero.

$$x(t) = A\cos(\omega\,t + \phi) = A\cos(\omega\,t)$$

Taking derivatives gives us the velocity and acceleration.

$$v(t) = -\omega\,A\sin(\omega\,t) \quad \text{and} \quad a(t) = -\omega^2\,x = -\omega^2\,A\cos(\omega\,t)$$

At $t_0 = 0.22$ s we get

$$x(t_0) = 0.0153\ \text{m}\,, \quad v(t_0) = 1.51\ \text{m/s} \;\text{ and }\; a(t_0) = -102\ \text{m}/\text{s}^2$$

(c) When the particle is 1.7 cm from equilibrium, then what are the speed and the magnitude of the acceleration of the particle?

Solution

The distance from equilibrium is $|x|$.

$$|x| = 1.7\ \text{cm} = 0.017\ \text{m}$$

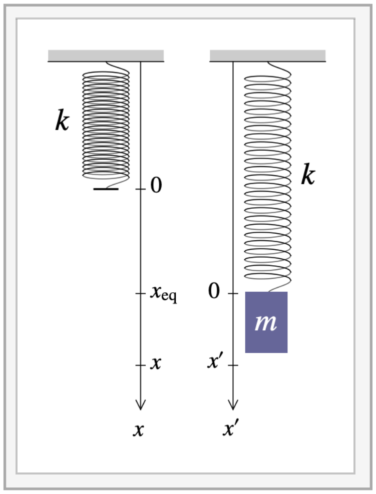
The speed is $|v|$.

$$|v| = \omega\,\sqrt{A^2 - x^2} = 1.38\ \text{m/s}$$

From the discussion in part **(a)** we get the magnitude of the acceleration.

$$|a| = \omega^2\,|x| = 113\ \text{m}/\text{s}^2$$

M.3 - The Vertical Mass-Spring



Interactive Figure

When a mass hangs from a spring we meed to add the effect of gravity. As before, the second law gives our differential equation.

$$F_{\text{net}} = m\,a \implies -k\,x + m\,g = m\,a$$

The equilibrium position of this is when the forces cancel $F_{\text{net}} = 0$. This gives

$$k\,x_{\text{eq}} = m\,g.$$

We can redefine our coordinates relative to the new equilibrium position.

$$x = x' + x_{\text{eq}}$$

If we insert this into our equation we get

$$-k\,x' = m\,a'.$$

Here we have used the value of x_{eq} and shifting the x-coordinate does not affect the acceleration, we have $a = a'$.

The interpretation of the above expression is simple. The effect of gravity is trivial. It just shifts the equilibrium position and we end up with simple harmonic motion about the new equilibrium position.

Example M.2 - Vertical Mass/Spring System

When a mass is hung from a vertical spring, the spring stretches by 8.5 cm. What is the period of oscillation of this system?

Solution

$$x_{\text{eq}} = 8.5\text{ cm} = 0.085\text{ m}$$

When hanging in equilibrium there are two forces acting on the mass, the spring force $k\,x_{\text{eq}}$ acting upward and gravity downward. We do not know the mass m but we will see that it cancels; solve for k in terms of m.

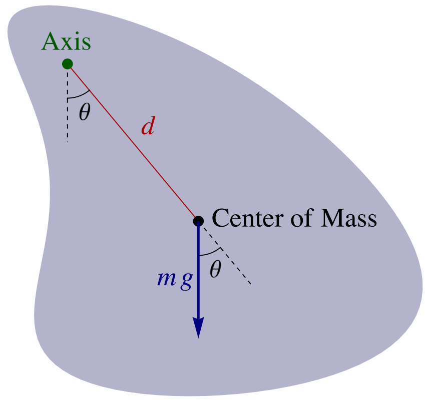
$$k\,x_{\text{eq}} = m\,g \implies k = \frac{m\,g}{x_{\text{eq}}}$$

From the angular frequency we can find the period.

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{m\,g/x_{\text{eq}}}{m}} = \sqrt{\frac{g}{x_{\text{eq}}}} \implies T = \frac{2\,\pi}{\omega} = 2\,\pi\,\sqrt{\frac{x_{\text{eq}}}{g}} = 0.585\text{ s}$$

M.4 - The Physical and the Simple Pendulum

The Physical Pendulum



Consider a rigid body rotating without friction about an axis. The center of mass is a distance d from the axis. At equilibrium, the center of mass will hang below the center. Take the angle θ to be the angle of the line from the axis to the center of mass measured from vertical; note that $\theta = 0$ is the equilibrium position. The only nonzero torque acting on the rigid body is the torque due to gravity. This is

$$\tau_{\text{net}} = \tau_{\text{grav}} = -m\, g\, d \sin\theta.$$

The direction of positive θ gives our sign convention for torque. The reason for the minus sign in the above expression is the torque tends toward smaller angles. The rotational second law gives the expression.

$$\tau_{\text{net}} = I\,\alpha \implies -m\, g\, d \sin\theta = I\,\alpha \implies \alpha = -\frac{m\, g\, d}{I} \sin\theta$$

If we define

$$\omega = \sqrt{\frac{m\, g\, d}{I}}$$

then we get an expression of the form

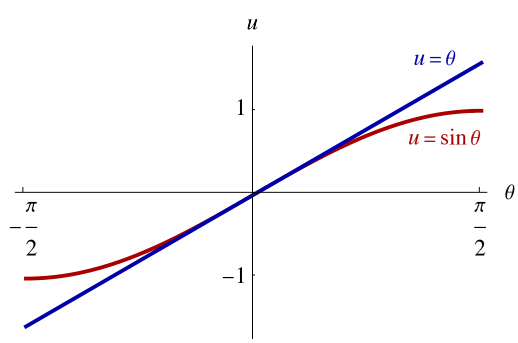
$$\alpha = -\omega^2 \sin\theta.$$

The relation between α and θ in this expression is the same as that between a and x. This is almost of the form of our simple harmonic motion equation $a = -\omega^2 x$, except for the sine function. If we consider small angles then we get

$$\sin\theta \simeq \theta \text{ for small } \theta \text{ in radians}$$

$$\alpha = -\omega^2\, \theta.$$

We can then conclude that for small amplitude oscillations we have simple harmonic motion with an angular frequency ω given by the expression above.



θ in degrees	θ in radians	$\sin\theta$
1	0.0174533	0.0174524
2	0.0349066	0.0348995
5	0.0872665	0.0871557
10	0.174533	0.173648
20	0.349066	0.34202
30	0.523599	0.5
45	0.785398	0.707107
60	1.0472	0.866025
90	1.5708	1.

For small angles in radians, the sine of the angle is approximately equal to the angle in radians. This is shown with a graph on the left and with a table on the right.

Example M.3 - Physical Pendulum

A uniform disk with a radius of 20 cm swings without friction about a perpendicular axis through the rim. What is its period of small oscillations?

Solution

From out table of moments of inertia we get

$$I = \frac{3}{2} m R^2$$

The mass m is not given but will cancel. We are given the radius and need the value for g. The value of d, the distance from the center to the axis is half the radius

$$g = 9.80 \frac{\text{m}}{\text{s}}, \quad d = R = 20 \text{ cm} = 0.20 \text{ m}$$

The distance from the axis to the center of mass is $d = R/2$. The period can then be found from the angular frequency.

$$\omega = \sqrt{\frac{m g d}{I}} \implies T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{\frac{3}{2} m R^2}{m g R}} = 2\pi \sqrt{\frac{3R}{2g}} = 1.10 \text{ s}$$

The Simple Pendulum

The simple pendulum is a special case of the physical pendulum. It is the case where all the mass m is located at a point, the pendulum bob. If the bob is on the end of a string of length L then we get

$$d = L \quad \text{and} \quad I = m L^2.$$

Solving for ω we get

$$\omega = \sqrt{\frac{g}{L}}.$$

Example M.4 - Texas A&M's Foucault Pendulum

A Foucault Pendulum is a pendulum that precesses slowly in a circular path illustrating the earth's rotation underneath; this is due to the earth's Coriolis forces briefly mentioned in Chapter E. The George P. Mitchell Physics Building at Texas A&M University has a large Foucault Pendulum with a period of 10.32 s. What is the length of this pendulum?

Solution

$$\omega = \sqrt{\frac{g}{L}} \implies T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L}{g}} \implies L = g \left(\frac{T}{2\pi}\right)^2 = 26.4 \text{ m}$$