# Chapter N

## *Waves and Sound*

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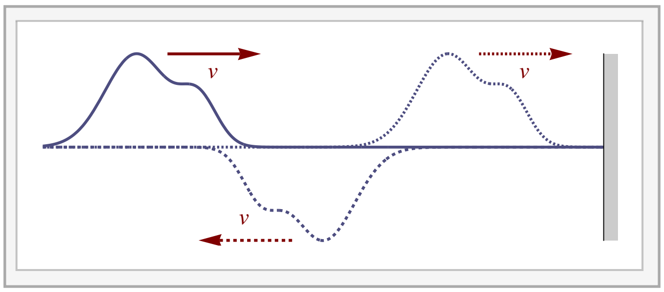
## N.1 - Mechanical Waves

A mechanical wave is a disturbance in a medium that propagates through the medium. A wave carries energy as it propagates. We will generically label this disturbance by $u$. Consider a stretched string. Take $x$ to be the position along the string and $t$ is time. The equilibrium position of the string is its relaxed position. The disturbance $y$ ($u = y$ for a string) is the perpendicular distance of a point on the string from equilibrium. When a pulse is put in the string it maintains its shape and travels the length of the string at a fixed speed of $v = \sqrt{F_T/\mu}$, where $F_T$ is the tension in the string and $\mu$ is the linear density of the string, its mass per length.




Interactive Figure

A wave is said to be *transverse* when the direction of the disturbance is perpendicular to the direction of propagation. Waves on a string are examples of transverse waves. Electromagnetic waves are also transverse; in the electromagnetic case the disturbance is the electric field, which is perpendicular to the direction of propagation. Note that electromagnetic waves are not mechanical waves; there is no medium and it can propagate in a vacuum. There is a plane of possible directions perpendicular to a direction of propagation. Choosing such a direction is choosing a *polarization*. Transverse waves can be polarized.

Now consider a stretched spring. As with the string, perpendicular pulse can also be put into this stretched spring and that will propagate as a transverse wave. If a compression pulse is put into the spring then that pulse will also propagate as a wave. Here a point on the spring moves back and forth a distance $u$ from equilibrium but parallel to the direction of propagation $x$. When the disturbance is parallel to the direction of propagation we call the wave *longitudinal*. Longitudinal waves cannot be polarized.

Another example of a longitudinal wave is sound in a fluid. Here the molecules move back and forth parallel to the direction of wave propagation. We can view the disturbance of sound waves either in terms of displacement or in terms of pressure. Sound waves travel at the speed of sound. This varies with temperature; at 20 °C it is 343 m/s

| **Wave Type** | **Disturbance – $u$** | **Transverse or Longitudinal** | **Wave Speed** |
|---|---|---|---|
| Waves on a string | $u = y(x, t)$ | Transverse | $v = \sqrt{\frac{F_T}{\mu}}$ |
| Electromagnetic Waves | $u = E$ = Electric field | Transverse | $v = c = 3.00\times10^8\ \frac{\text{m}}{\text{s}}$ |
| Compression Waves on a Spring | $u$ = Parallel disp. | Longitudinal | no formula given |
| Sound Waves in a Fluid | $u$ = Pressure or $u$ = Displacement | Longitudinal | $v = v_{\text{sound}} = 343\ \frac{\text{m}}{\text{s}}$ (in air) |

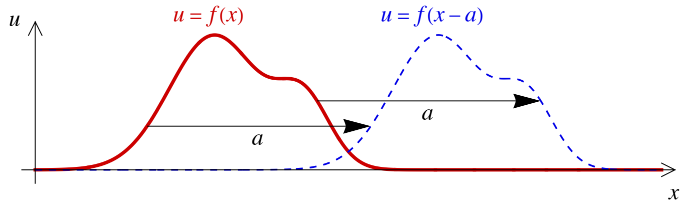
# N.2 - Waves in One Dimension

## Left- and Right-moving Pulses

Consider one-dimensional wave with a disturbance $u(x, t)$ of the following form

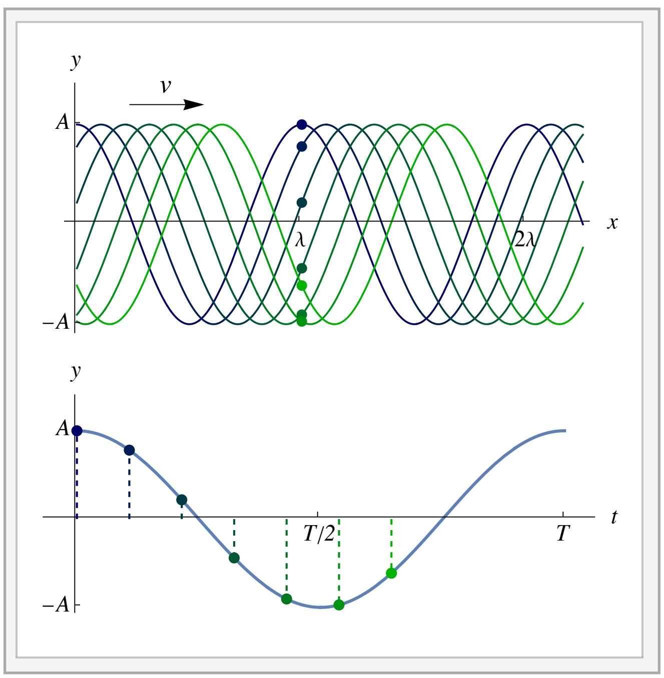
$$u(x, t) = f(x - v\,t) + g(x + v\,t),$$

where $f$ and $g$ are arbitrary functions. To understand this, consider the function $u(x) = f(x)$. If we shift this by $a$ in the positive direction we get $u(x) = f(x - a)$.




We can now see that $u(x, t) = f(x - v\,t)$ describes a pulse of arbitrary shape $u(x) = f(x)$ moving in the positive direction with speed $v$. $u(x, t) = g(x + v\,t)$ corresponds to a pulse of a different arbitrary shape moving in the opposite direction at the same speed.

## Sinusoidal Waves




Interactive Figure

We often consider waves where the shape of the pulse $f$ (or $g$) are sinusoidal.

$$f(x) = A\cos(k\,x)$$

$A$ is called the amplitude. $k$ is called the wave number; this is related to the wavelength $\lambda$, which is the spatial period of the function. Since the period of sine is $2\,\pi$ and the period of the function $f$ is $\lambda$ we get $k\,\lambda = 2\,\pi$ or

$$k = \frac{2\,\pi}{\lambda}.$$

If we take this function $f$ and move it in the positive or negative direction we get $f(x \mp v\,t) = A\cos[k\,(x \mp v\,t)]$ or

$$u(x, t) = A\cos(k\,x \mp \omega\,t),$$

where the angular frequency $\omega$ and wave number are related to the wave speed by $k\,v = \omega$.

If we choose some point on the string $x_0$ then at that position moves as

$$A\cos(\omega\,t + \phi).$$

This is our expression for simple harmonic motion. Since the angular frequency is related to the frequency by $\omega = 2\,\pi\,f$ the wave speed can also be written in terms of the frequency and wavelength.

$$v = \frac{\omega}{k} = f\,\lambda$$

### Dispersion and Generalized Waves

With the wave examples we are discussing in this chapter, the wave speed is independent of frequency. More generally, we can have different physical situations which are described by similar mathematics where we still get sinusoidal waves. These generalized waves are *dispersive*, meaning that different frequency waves will have different speeds. Although we have sinusoidal waves, we no longer have simple pulses that maintain their shape as solutions. If we begin with a pulse then the pulse will spread out with time.

Light in a vacuum is a three-dimensional wave that is not dispersive. Light passing through a medium does have dispersion. A consequence of different frequencies, colors, having different speeds is the splitting of light into its spectrum by a prism.

Another example of dispersive waves is surface water waves; these, it turns out, are a mixture of longitudinal and transverse displacements. In quantum mechanics matter waves are dispersive.

## N.3 - Waves on a String and Power

A rope or a string can be described as having a linear density $\mu$. This is the mass per length of the rope or string.

$$\mu = \frac{m}{L} = \frac{\text{mass}}{\text{length}} \quad \text{(linear density)}$$

A heavy rope has a large linear density and a light string has a small one. Along a rope of string the tension will stay constant. We will use $F_T$ to denote tension, instead of just $T$ to avoid confusion with the period of harmonic motion, which recall is related to the frequency $T = 1/f$. The speed of waves on a string can be shown to be

$$v = \sqrt{\frac{F_T}{\mu}}\,.$$

Waves carry energy. When a sinusoidal wave travels down a string the flow of energy is constant. If some quantity of energy flows in a time then the energy per time or power is constant. Now consider some point on the string. If the wave is moving in some direction at the speed $v$, then the energy moves past the point at the same rate. In a small time $\Delta t$ all the energy in an small segment of width $\Delta x = v\,\Delta t$ will pass the point.

The energy of a particle in simple harmonic motion satisfies

$$E = \frac{1}{2}\,m\,v_{\text{max}}^2$$

Since a point on the string moves in simple harmonic motion we can apply this formula to $\Delta x$. The small mass of the segment is

$$\Delta m = \mu\,\Delta x = \mu\,v\,\Delta t$$

and the small energy in the segment is

$$\Delta E = \frac{1}{2}\,\Delta m\,v_{\text{max}}^2 = \frac{1}{2}\,\mu\,v\,\Delta t\;\; v_{\text{max}}^2.$$

The power is given by $\mathcal{P} = \Delta E/\Delta t$. The maximum speed of a point in simple harmonic motion is $v_{\text{max}} = \omega\,A$. It follow that the power travelling down a string due to a sinusoidal wave is

$$\mathcal{P} = \frac{1}{2}\,\mu\,A^2\,\omega^2\,v.$$

#### Example N.1 - Waves on a Steel Wire

A wave of the form

$$y(x,\,t) = (0.020\ \mathrm{m})\,\sin\left[\left(105\ \mathrm{s}^{-1}\right)t + \left(3.0\ \mathrm{m}^{-1}\right)x\right]$$

travels down a steel wire with a linear density of $0.014226\ \mathrm{kg/m}$.

**(a)** What are the frequency and wavelength of the wave? Also, what is the wave speed and what is the direction of the wave?

**Solution**

From the form of the function we can read off the amplitude $A$, the angular frequency $\omega$ and wave number $k$. Also, the linear density $\mu$ is given.

$$A = 0.020\ \mathrm{m}\ , \quad \omega = 105\ \mathrm{s}^{-1}, \quad k = 3.0\ \mathrm{m}^{-1} \ \text{ and } \ \mu = 0.014226\ \mathrm{kg/m}$$

The frequency, wavelength and speed follow from formulas for sinusoidal waves.

$$f = \frac{\omega}{2\,\pi} = 16.7\ \mathrm{Hz}, \quad \lambda = \frac{2\,\pi}{k} = 2.09\ \mathrm{m} \ \text{ and } \ v = \frac{\omega}{k} = 35\ \frac{\mathrm{m}}{\mathrm{s}} = f\,\lambda$$

The solution for a pulse is $f(x \mp v\,t)$, where the negative sign means the pulse is moving in the positive-$x$ direction and positive implies the negative-$x$ direction. Since the relative sign between $\omega\,t$ and $k\,x$ terms is positive, the wave is moving in the negative-$x$ direction.

**(b)** What is the maximum speed of a point on the wire as the wave passes.

**Solution**

A point on a string (or wire) moves in simple harmonic motion as a sinusoidal wave passes. The maximum speed for simple harmonic motion is

$$v_{\mathrm{max}} = \omega\,A = 2.1\ \mathrm{m/s}\ .$$

Note that the wave speed is quite distinct from the speed of a point on the wire.

**(c)** What is the tension in the wire?

**Solution**

The tension in the wire can be found from the wave speed $v$ and the linear density $\mu$.

$$v = \sqrt{\frac{F_T}{\mu}} \quad \Longrightarrow \quad F_T = \mu\,v^2 = 17.4\ \mathrm{N}$$

**(d)** At what rate does energy flow down the wire?

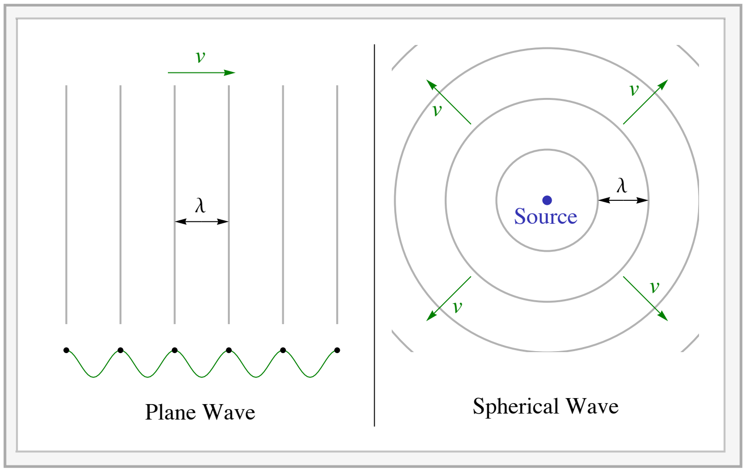
**Solution**

The rate of energy flow is the power. Note that $v$ is the wave speed.

$$\mathcal{P} = \frac{1}{2}\,\mu\,A^2\,\omega^2\,v = 1.10\ \mathrm{W}$$

## N.4 - Waves in Three Dimensions

### Plane Waves and Spherical Waves




Interactive Figure

In three dimensions we define a surface of constant phase, typically the crest of the wave, to be a wave front. An important notion is that of a plane wave; take the wave to move in the positive $x$-direction with the disturbance $u$ being uniform along the $yz$-plane. This turns our disturbance from a genuinely three dimensional function $u(x, y, z, t)$ into being a function of only $x$, spatially.

$$u(x, t) = A\cos(k\,x - \omega\,t) \quad \text{(Plane Wave)}$$

Wave fronts correspond to planes parallel to the $yz$-plane, separated by one wavelength and moving in the positive $x$-direction at speed $v$.

A point source produces a spherical wave. We can write the disturbance $u$ as a function of $r$, the distance form the source and time $t$.

$$u(r, t) = A(r)\cos(k\,r - \omega\,t) \quad \text{(Spherical Wave)}$$

The wave fronts are concentric spheres separated by $\lambda$. The waves fronts move away from the source at the speed $v$.

### Intensity

We saw in the discussion of waves on a string that the average power transmitted down the string was proportional to the wave amplitude squared, $\mathcal{P}_{\text{ave}} \propto \text{Amplitude}^2$. For a wave that moves though three dimensions, what is analogous to power in the string case is the intensity, which we define as power per area.

$$I = \frac{\mathcal{P}_{\text{ave}}}{A} = \frac{E_{\text{ave}}}{A\,\Delta t} \quad \left(\text{Intensity} = \frac{\text{Power}}{\text{Area}} = \frac{\text{Average Energy}}{\text{Area} \times \text{time}}\right)$$

The intensity will always be proportional to the amplitude squared.

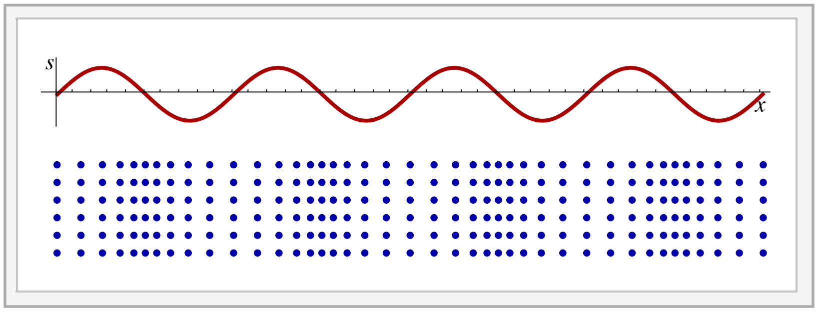
With a plane wave the area stays uniform so the intensity is uniform. For a spherical wave the relevant area is the surface area of a sphere, where $A = 4\,\pi\,r^2$. It follows that the intensity varies with $r$, the distance form the source, by an inverse square law.

$$I = \frac{\mathcal{P}_{\text{ave}}}{4\,\pi\,r^2}$$

$\mathcal{P}_{\text{ave}}$ is the total power output of the source, the rate at which it emits energy.

## Sound Waves in a Fluid

### Displacement and Pressure




Interactive Figure

We now consider sound waves in a fluid (a liquid or a gas.) Our one-dimensional model consists of a plane wave, which is three dimensional but where nothing varies along the $yz$-plane, so it is a function of the one spatial variable $x$. Alternatively, we can describe it as a fluid in a pipe with frictionless walls, making the cross-sectional area $A$ unimportant.

We may view sound waves in a fluid as pressure waves or as displacement waves. Consider the motion of particles (molecules) relative to their equilibrium positions in a fluid. The displacement $s$ is the position of particles relative to their equilibrium position. In our one-dimensional model $s(x, t)$ is the longitudinal displacement of particles as a function of position $x$ and time $t$. Longitudinal means that the displacement is along the direction of propagation, the $\pm x$ direction. Similarly, we can write the pressure as a function of $x$ and $t$, $P(x, t)$. By pressure $P$ we mean the *gauge pressure*, which is the difference between $P_{\text{absolute}}$, the absolute pressure, and $P_0$, the atmospheric pressure or the ambient pressure in the fluid. Although the absolute pressure cannot be negative, the gauge pressure $P$ can.

$$P = P_{\text{gauge}} = P_{\text{absolute}} - P_0$$

## Intensity and Sound Level

We can perceive sound over a wide range of intensities; this makes a logarithmic scale convenient for measuring loudness. We will call our measure of loudness $\beta$, the *sound level*, and measure it in decibels dB. If we define

$$I_0 = 10^{-12}\ \frac{\text{W}}{\text{m}^2}$$

as the *threshold of hearing*, then we can define the sound level (or decibel level) $\beta$, in dB, by:

$$\beta = (10\ \text{dB}) \log\left(\frac{I}{I_0}\right).$$

### Example N.2 - A Very Loud Loudspeaker

At a distance of 6 m from the only speaker at a concert, the sound level is 110 dB. At this very high sound level, one can experience hearing loss in a few minutes. Assume the speaker produces isotropic sound. (Isotropic means the same in all directions. It implies the wave is a spherical wave from a point source.)

**(a)** What is the intensity $I$ of the sound at that 6 m position?

#### Solution

Here we are given $\beta = 110$ dB. We will also need the value of the threshold of hearing $I_0 = 10^{-12}\ \text{W}/\text{m}^2$.

$$110\ \text{dB} = (10\ \text{dB}) \log\left(\frac{I}{I_0}\right) \implies 11 = \log\left(\frac{I}{I_0}\right) \implies 10^{11} = \frac{I}{I_0} \implies I = 10^{11}\ I_0 = 10^{11} \times 10^{-12}\ \frac{\text{W}}{\text{m}^2} = 0.1\ \frac{\text{W}}{\text{m}^2}$$

**(b)** What is the average power of sound produced by the source?

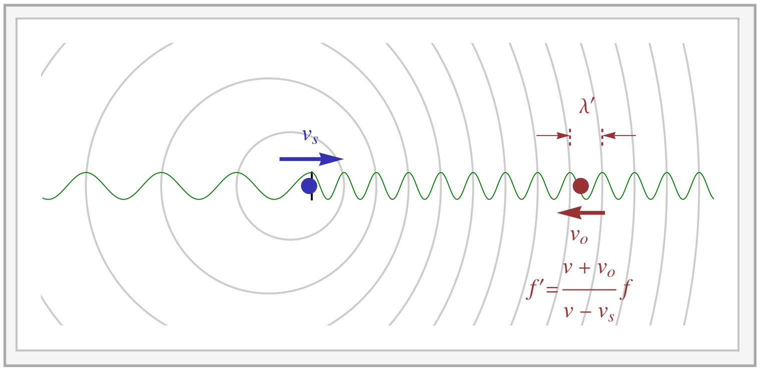
#### Solution

We are given the distance from the source $r$.

$$r = 6\ \text{m}\ \text{ and }\ I = 0.1\ \frac{\text{W}}{\text{m}^2} = \frac{\mathcal{P}_{\text{ave}}}{4\pi r^2} \implies \mathcal{P}_{\text{ave}} = 4\pi r^2 I == \left(452.4\ \text{m}^2\right) I = 45.2\ \text{W}$$

## The Doppler Effect and Shock Waves

When a train is moving toward you with its whistle blowing the observed pitch of the whistle, its frequency, increases. When moving away its pitch decreases. This is known as the Doppler effect; it is a general property of waves but we will discuss it in the context of sound waves. Take the speed of sound to be $v$; this is the speed of sound with respect to the stationary atmosphere. Consider $v_s$ and $v_o$ to be the approaching velocities of the source, producing the sound, and the observer, listening to it. By "approaching velocities" we mean we will take their signs to be positive when moving toward the other. If the source or observer is moving away from the other, we will take the sign of its velocity to be negative.




Interactive Figure - The Doppler Effect: $v$ is the wave speed and $f$ is its frequency; $f'$ is the frequency heard by the observer. $v_s$ is the source velocity and $v_o$ is the observer velocity. Both are positive when moving toward the other.

$\lambda'$ is the wavelength as heard by the observer. The wavelength at the observer $\lambda'$ is the distance between wave fronts; it is smaller than $\lambda = v/f$ by $v_s T$, $\lambda' = \lambda - v_s T$, where $T = 1/f$ is the period. $\lambda'$ is smaller since the second wavefront left the source after it moved $v_s T$ after the first wavefront left the source. The wave moves at speed $v$ relative to the stationary atmosphere, so relative to the moving observer the wave speed is $v + v_o$.

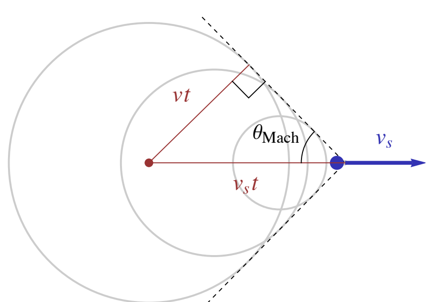
$$\lambda' = \lambda - v_s T = \frac{v}{f} - \frac{v_s}{f} = \frac{v - v_s}{f} \quad \text{and} \quad v' = v + v_o$$

Combining these two expressions we can find the wave frequency as heard by the observer.

$$f' = \frac{v'}{\lambda'} = \frac{v + v_o}{(v - v_s)/f} = \frac{v + v_o}{v - v_s} f$$

When the source travels faster than the speed of sound, the wave fronts meet to form a conical shock wave trailing the source. This shock wave is the sonic boom from a supersonic jet. Another example is the wedge-shaped wake behind a boat; the boat is moving faster than the surface water waves to create a shock wave. The angle of the shock wave behind the source is called the Mach angle, $\theta_{\text{Mach}}$. In a time $t$ the spherical wave front moves by $v\,t$ and the source moves by $v_s\,t$; the Mach angle is found by simple trigonometry.

$$\sin\theta_{\text{Mach}} = \frac{v}{v_s}$$




### Example N.3 - A Train Whistle

A car drives on a road parallel to and near train tracks. The train blows its whistle with a frequency of 720 Hz. Take the speed of the train to be 42 m/s (approximately 95 mi/h) and the speed of the car to be 29 m/s (approximately 65 mi/h). The speed of sound is 343 m/s.

**(a)** If the car is driving toward the approaching train, then what is the frequency heard by the car's driver?

### Solution

The train is the source with $v_s = \pm 42\ \mathrm{m/s}$ and the car is the observer with $v_o = \pm 29\ \mathrm{m/s}$. The convention being used here is the signs are positive when the velocity is toward the other. That is: $v_s > 0$ when the source is moving toward the observer and $v_o > 0$ when the observer is moving toward the source. The frequency of the source is $f$ and the speed of sound is $v$.

$$f = 720\ \mathrm{Hz}\ \text{ and }\ v = 343\ \mathrm{m/s}$$

The train is moving toward the car so $v_s > 0$ and the car is moving toward the train, so $v_o > 0$. Use the expression for the Doppler effect to find the frequency heard by the car $f'$.

$$v_s = +42\ \frac{\mathrm{m}}{\mathrm{s}}\ \text{ and }\ v_0 = +29\ \frac{\mathrm{m}}{\mathrm{s}} \implies f' = \frac{v + v_o}{v - v_s}\, f = 890\ \mathrm{Hz}$$

**(b)** For part **(a)**, what is the frequency heard by the car's driver after the train passes and is receding?

### Solution

Now both the train and car are moving away from each other, so $v_s < 0$ and $v_o < 0$.

$$v_s = -42\ \frac{\mathrm{m}}{\mathrm{s}}\ \text{ and }\ v_0 = -29\ \frac{\mathrm{m}}{\mathrm{s}} \implies f' = \frac{v + v_o}{v - v_s}\, f = 587\ \mathrm{Hz}$$

**(c)** If the car is driving away from the approaching train, then what is the frequency heard by the car's driver?

### Solution

The train is now moving toward the car so $v_s > 0$ and the car is moving away from the train, so $v_o < 0$.

$$v_s = +42\ \frac{\mathrm{m}}{\mathrm{s}}\ \text{ and }\ v_0 = -29\ \frac{\mathrm{m}}{\mathrm{s}} \implies f' = \frac{v + v_o}{v - v_s}\, f = 751\ \mathrm{Hz}$$

**(d)** For part **(c)**, what is the frequency heard by the car's driver after the train passes and is receding?

### Solution

The train is now moving away from the car so $v_s < 0$ and the car is moving toward the train, so $v_o > 0$.

$$v_s = -42\ \frac{\mathrm{m}}{\mathrm{s}}\ \text{ and }\ v_0 = +29\ \frac{\mathrm{m}}{\mathrm{s}} \implies f' = \frac{v + v_o}{v - v_s}\, f = 695\ \mathrm{Hz}$$

### Example N.4 - A Hypersonic Sonic Boom

Supersonic means that something is moving faster than the speed of sound. A rocket or jet is said to be *hypersonic* when it moves faster than the five times the speed of sound. What is the smallest Mach angle for a hypersonic object? (The smallest angle is when $v_s = 5\, v$)

### Solution

This involves just using the formula for the Mach angle.

$$\sin\theta_{\mathrm{Mach}} = \frac{v}{v_s} \implies \theta_{\mathrm{Mach}} = \sin^{-1}\left(\frac{v}{v_s}\right) = \sin^{-1}\left(\frac{v}{5\, v}\right) = \sin^{-1}\left(\frac{1}{5}\right) = 11.5\,°$$

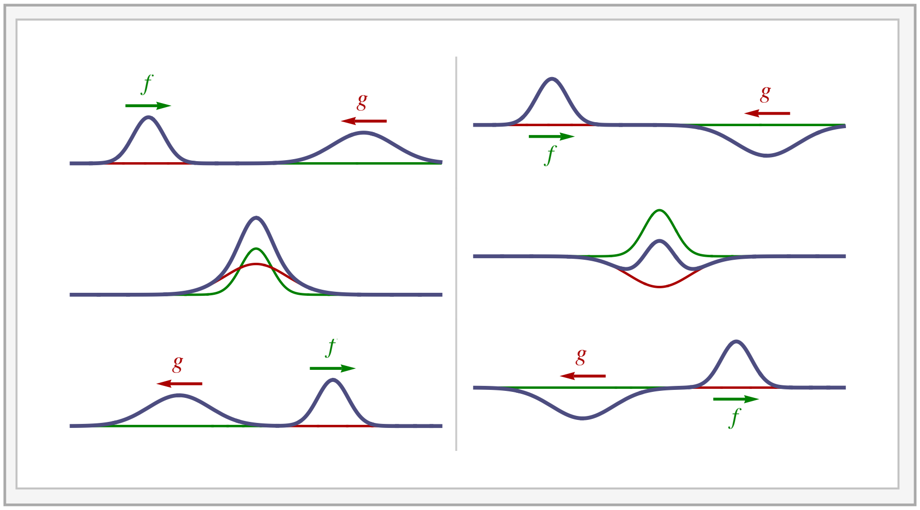
# N.7 - Superposition

## Superposition in One Dimension

For any waves, including generalized dispersive waves, we have the principle of superposition. This means that the sum of two wave solutions is a solution. For strings in one dimension the only way that we can combine different waves is if they are traveling in opposite directions since the general expression for waves is

$$u(x,\, t) = f(x - v\, t) + g(x + v\, t)$$

and two pulses moving in the same direction move at the same speed. Suppose the $f$ and $g$ solutions are pulses moving toward each other. Our general solution shows that when a pulse moving in the positive $x$-direction, $f(x - v\,t)$, meets a pulse moving in the opposite direction, $g(x + v\,t)$, they add where they overlap and move off unchanged by the interaction.

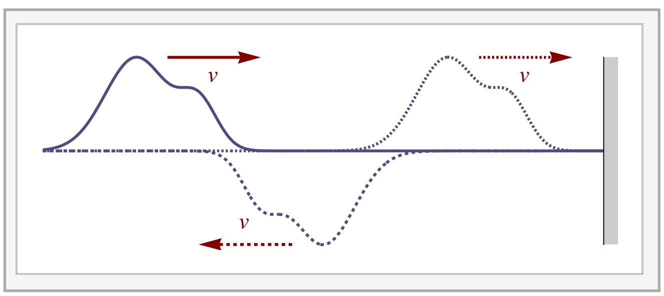



Interactive Figure

## Reflection of Waves

### Reflection from a Fixed Point

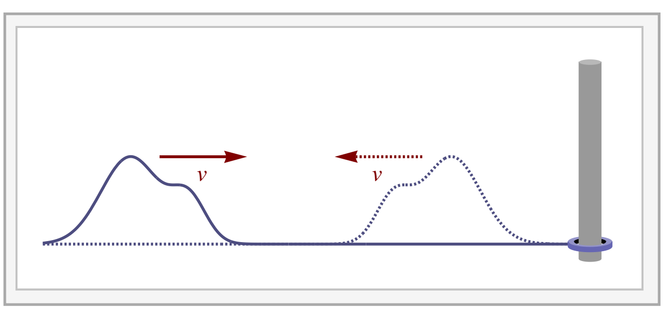
If one end of a string is held fixed then a pulse will flip while being reflected.




Interactive Figure - Reflection of a pulse from a fixed end.

### Reflection from a Free End with Tension

Suppose one end of a string is attached to a ring that slides without friction on a post. In this case, the pulse will reflect without being flipped.

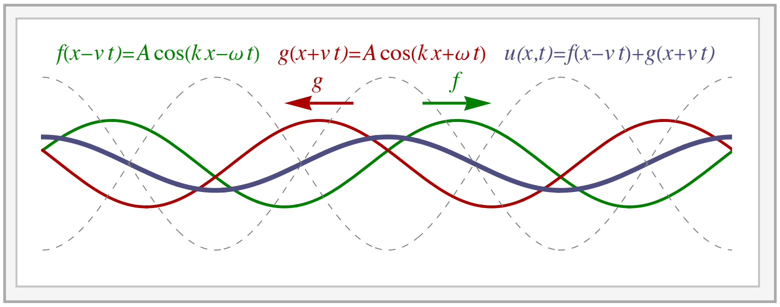



Interactive Figure - Reflection of a pulse from a free end: a ring slides on a post without friction.

## Standing Waves

### Standing Waves

When a mechanical system, say a mass-spring system, is driven by a periodic force it will oscillate at the driving frequency. An extended system, like a building, a bridge or a musical instrument will have many natural frequencies of vibration. We will now study these natural frequencies for several simple cases that involve standing waves.




Interactive Figure

To describe a standing wave consider a one-dimensional case with two sinusoidal waves with the same amplitude and frequency but moving in opposite directions.

$$u(x,\,t) = A\cos(k\,x - \omega\,t) + A\cos(k\,x + \omega\,t)$$

To add these two waves we will use trig. identities. Start with the formulas for cosine of the sum and difference of two angles and add the results.

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \quad\Longrightarrow\quad \cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta.$$

Using $\alpha = k\,x$ and $\beta = \omega\,t$ we get our result.

$$u(x,\,t) = 2\,A\cos(k\,x)\cos(\omega\,t)$$

The positions where the disturbance $u$ is zero at all times are called *nodes*. We refer to the positions where the disturbance has its largest amplitude oscillation as *antinodes*.

$$\cos(k\,x) = 0 \quad\Longleftrightarrow\quad \text{nodes}$$

$$\cos(k\,x) = \pm 1 \quad\Longleftrightarrow\quad \text{antinodes}$$

Note that the distance between nodes is half a wavelength. The wavelength and frequency are the same as for the two left and right moving sinusoidal waves, so it follows the we still have $f\,\lambda = v$.

## Standing Waves on a String

Suppose a string under tension is fixed at either end. If you pluck this string it will vibrate in a fairly complex pattern. This complex pattern can be understood in terms of linear superpositions of some simple vibrational modes, called harmonics. We will now describe these harmonics.

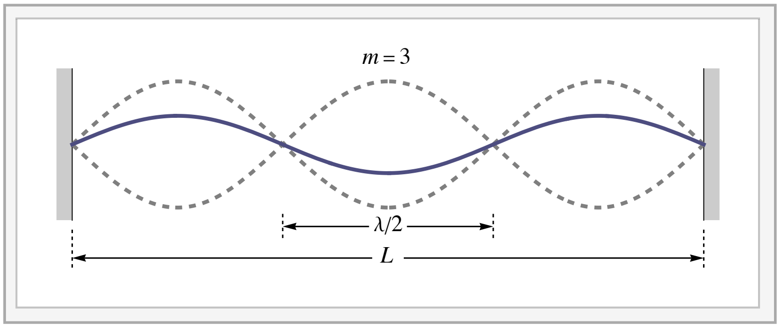
The string has tension $F_T$ and linear density $\mu$ so the wave speed is $v = \sqrt{F_T/\mu}$. It is fixed at $x = 0$ and $x = L$. We insist on a standing wave solution that is zero at $x = 0$; to do this shift the spatial part of the trig. function to $\sin(k\,x)$, since $\sin 0 = 0$. Recall that $u = y$ for a string. A general expression for the standing wave becomes:

$$y(x,\,t) = y_{\max}\sin(k\,x)\cos(\omega\,t + \phi)$$

where $y_{\max}$ is the amplitude of the resulting standing waves and not the amplitudes of the left-right moving waves in the previous subsection. For the solution to be zero at $L$ we have $\sin(k\,L) = 0$. This implies $k\,L = m\,\pi$. Since $k = 2\,\pi/\lambda$ we can find the wavelengths of our harmonic modes.

$$\lambda_m = \frac{2\,L}{m} \text{ where } m = 1,\,2,\,3,\;\ldots$$

Let us get at this expression above in a less mathematical way. The distance between nodes of the resulting standing wave pattern is half a wavelength. $m$ of these half-wavelengths fit into $L$, so $m\,\lambda/2 = L$.




Interactive Figure - A string with both ends fixed.

The frequency follows from $f\,\lambda = v$.

$$f_m = m\,f_1 \text{ where } f_1 = \frac{v}{2\,L} \text{ and } m = 1, 2, 3, \ldots$$

$f_1$ is the fundamental frequency or the first harmonic. The higher $m$ values are the higher harmonics. An instrument will always produce higher harmonics, integer multiples of the fundamentals, but the frequencies we associate with an instrument are the fundamental frequencies. We tune to correct the fundamental frequencies. When we tune a stringed instrument we vary the tension in the string, thus varying the wave speed; given the string's length, we have a unique fundamental frequency for a string. A piano has a separate string for each note, thus tuning a piano is a very tedious procedure. A standard guitar has just six strings but the player's fingers moving on the frets creates many effective lengths for each string, producing many different notes.

## Example N.5 - Guitars Strings

**(a)** The first (bottom) string on a guitar, the "E" string produces a fundamental frequency of 329.63 Hz; this is known in the scientific pitch notation as $E_4$. If the string has a length of 66 cm and has a tension of 43.6 N. What is the linear density $\mu$ of the string?

### Solution

We know the fundamental frequency $f_1$, the length $L$ and the tension $F_T$.

$$f_1 = 329.63\text{ Hz},\ L = 66\text{ cm} = 0.66\text{ m} \text{ and } F_T = 43.6\text{ N}$$

From this information we can find the wave speed.

$$f_1 = \frac{v}{2\,L} \implies v = f_1\,2\,L = 435.11\ \frac{\text{m}}{\text{s}}$$

Using the equation for the speed of waves on a string we can solve for the linear density, or mass/length, $\mu$.

$$v = \sqrt{\frac{F_T}{\mu}} \implies \mu = \frac{F_T}{v^2} = 2.30 \times 10^{-4}\ \frac{\text{kg}}{\text{m}}$$

**(b)** What are the next three higher harmonic frequencies produced by this string.

### Solution

The harmonics are multiples of the fundamental frequency.

$$f_2 = 2\,f_1 = 659.26\text{ Hz},\ \ f_3 = 3\,f_1 = 988.89\text{ Hz} \text{ and } f_4 = 4\,f_1 = 1318.5\text{ Hz}$$

**(c)** The last (top) string on a guitar is also called an "E" string or "low E" string; it produces a fundamental frequency that is two octaves lower in frequency, with scientific pitch notation $E_2$. Each octave represents a factor of two in frequency, so $E_2$ has 1/4 the frequency of $E_4$. This is 82.41 Hz. What are the next three higher harmonic frequencies produced by this string.

### Solution

We know the fundamental frequency $f_1$ and as before we want $f_2$, $f_3$ and $f_4$.

$$f_1 = 82.41\text{ Hz} \implies f_2 = 2\,f_1 = 164.82\text{ Hz},\ \ f_3 = 3\,f_1 = 247.22\text{ Hz} \text{ and } f_4 = 4\,f_1 = 329.63\text{ Hz}$$

Note that the third higher harmonic $f_4$ for the low E string is the same as the fundamental frequency of the E string.

**(d)** The strings on a guitar have the same length. Suppose the same string were used for the low E string, meaning a string with the same linear density, what tension would be needed?

### Solution

The same calculation in part **(a)** could be done in reverse to get the tension. Find the speed of the wave from $f_1 = \frac{v}{2L}$ using the same length as before and then the tension from $v = \sqrt{F_T/\mu}$ using the same $\mu$ as before. A simpler approach will show that the tension would be 1/16 the tension in part **(a)**. Label the new fundamental frequency, speed and tension of the low E as $f_1'$, $v'$ and $F_T'$.

$$f_1 = \frac{v}{2L} \text{ and } f_1' = \frac{v'}{2L} \implies \frac{v'}{v} = \frac{f_1'}{f_1} = \frac{1}{4}$$

Next, write the tension in terms of the speed.

$$v = \sqrt{\frac{F_T}{\mu}} \implies F_T = \mu v^2 \implies \frac{F_T'}{F_T} = \left(\frac{v'}{v}\right)^2 = \frac{1}{16}$$

This is a problem. With too little tension on the low E string it could not be heard or too much tension on the E string would warp the guitar or break the string. This is the reason why the strings get heavier as you go higher in a guitar. By making $\mu$ larger you can get lower frequencies with more reasonable tensions.

## Standing Sound Waves in a Pipe

We will now consider standing waves formed by sound in a pipe. The boundary conditions at the ends of the pipe depend on whether the end is open or closed. At an open end the pressure $P$ is fixed at zero at the end, since the pressure we use in our wave discussion is the gauge pressure which is the pressure difference from that outside the pipe. The displacement $s$ at the end of the pipe is unconstrained. The result is at an open end we have a pressure node and a displacement antinode.

$$\text{Open end at } x_0: \ s(x_0, t) = \pm s_{\max} \text{ (antinode)}, \ \ P(x_0, t) = 0 \ \text{(node)}$$

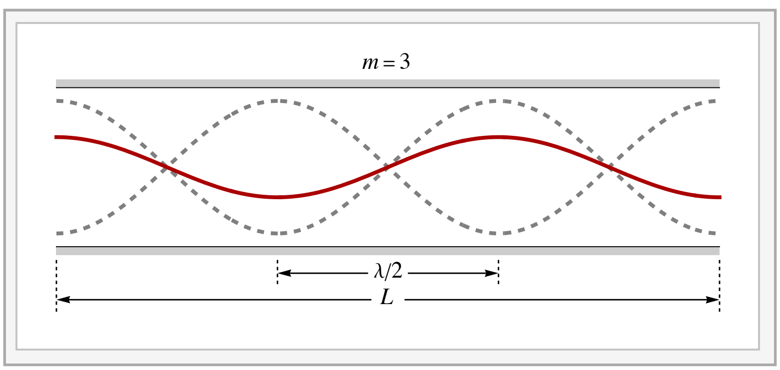
At a closed end of a pipe the displacement is forced to be zero, a node and the pressure is unconstrained, an antinode.

$$\text{Closed end at } x_0: \ s(x_0, t) = 0 \text{ (node)}, \ \ P(x_0, t) = \pm P_{\max} \ \text{(antinode)}$$

When sound waves are viewed as pressure wave, the sine and cosines are swapped and that them swaps nodes and antinodes.

Now consider a pipe of length $L$ with both ends open. From the pressure perspective there is a node at either end, just as the case of a string with both ends fixed. The displacement perspective looks different but the counting is still the same; we need an integer number of half wavelengths over the length $L$.

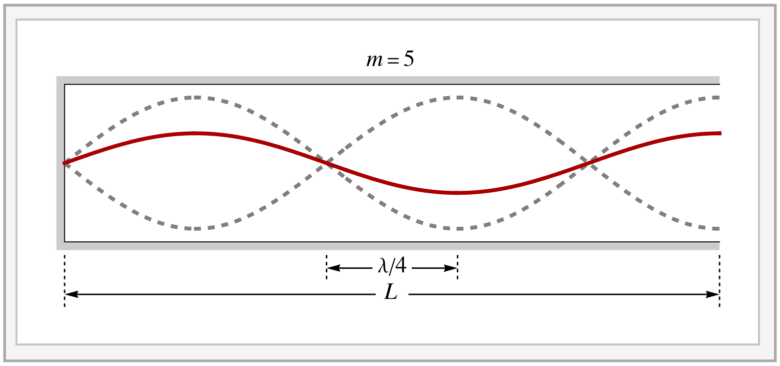
$$\lambda_m = \frac{2L}{m} \text{ and } f_m = m f_1 \text{ with } f_1 = \frac{v}{2L} \quad (m = 1, 2, 3, \ ...)$$




Interactive Figure - A pipe with both ends open

Now we turn to a pipe of length $L$ with one end open and the other closed. From the pressure perspective there is a node at the open end and an antinode at the closed end. The displacement point of view gives the reverse, an antinode at the open end and a node at the closed end. To get the standing waves to fit these boundary conditions we must have an odd number of quarter wavelengths spanning the length $L$, so $L = m\lambda/4$ where $m$ is odd.

$$\lambda_m = \frac{4L}{m} \text{ and } f_m = m f_1 \text{ with } f_1 = \frac{v}{4L} \quad (m = 1, 3, 5, \ ...)$$




Interactive Figure - A pipe with one end closed and one open

### Example N.6 - Organ Pipes

As in the previous example, refer again to the musical note $E_4$ with a frequency of 329.63 Hz.

**(a)** What is the length of an organ pipe with both ends open that produces this note as its fundamental frequency.

#### Solution

For a pipe with both ends open, the formula relating the fundamental frequency to the length is the same as for a string except that here $v$ is the speed of sound.

$$f_1 = 329.63\text{ Hz}\ ,\ \ v = v_{\text{sound}} = 343\ \frac{\text{m}}{\text{s}}\ \text{ and }\ f_1 = \frac{v}{2\,L}\ \implies L = \frac{v}{2\,f_1} = 0.520\text{ m}$$

**(b)** What are the next three higher harmonic frequencies produced by this pipe.

#### Solution

The harmonics are integer multiples of the fundamental frequency, so they are the same as for the guitar string with the same frequency.

$$f_2 = 2\,f_1 = 659.26\text{ Hz},\ \ f_3 = 3\,f_1 = 988.89\text{ Hz}\ \text{ and }\ f_4 = 4\,f_1 = 1318.5\text{ Hz}$$

**(c)** For an organ pipe with one open end and one closed end, what length is need to produce this note as its fundamental frequency.

#### Solution

For a pipe with one open and one closed end, the formula relating the fundamental frequency to the length changes

$$f_1 = 329.63\text{ Hz}\ ,\ \ v = v_{\text{sound}} = 343\ \frac{\text{m}}{\text{s}}\ \text{ and }\ f_1 = \frac{v}{4\,L}\ \implies L = \frac{v}{4\,f_1} = 0.260\text{ m}$$

**(d)** What are the next three higher harmonic frequencies produced by the pipe in **(c)**.

#### Solution

For this case we only get odd harmonics.

$$f_3 = 3\,f_1 = 988.26\text{ Hz},\ \ f_5 = 5\,f_1 = 1648.2\text{ Hz}\ \text{ and }\ f_7 = 7\,f_1 = 2307.4\text{ Hz}$$

## Beats

Consider interference between two sound waves that differ slightly in frequency. The resulting sound is heard to pulse at a frequency much smaller than that of the component waves. At some position from some source a wave will have the form: $u(t) = A\cos(\omega\,t + \phi)$. To make the result its most dramatic we will assume that at the position where the combined waves are heard, the amplitudes of the two waves are the same. Also, for simplicity, we will choose $t = 0$ to be when both waves are at a phase $\phi = 0$. The two waves become:

$$u_1(t) = A\cos(\omega_1\,t)\ \text{ and }\ u_2(t) = A\cos(\omega_2\,t).$$

To add these two waves we will use the same trig. identity we used with standing waves.

$$\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta.$$

Using $\alpha = \frac{1}{2}(\omega_1+\omega_2)\,t$ and $\beta = \frac{1}{2}(\omega_1-\omega_2)\,t$ we get:

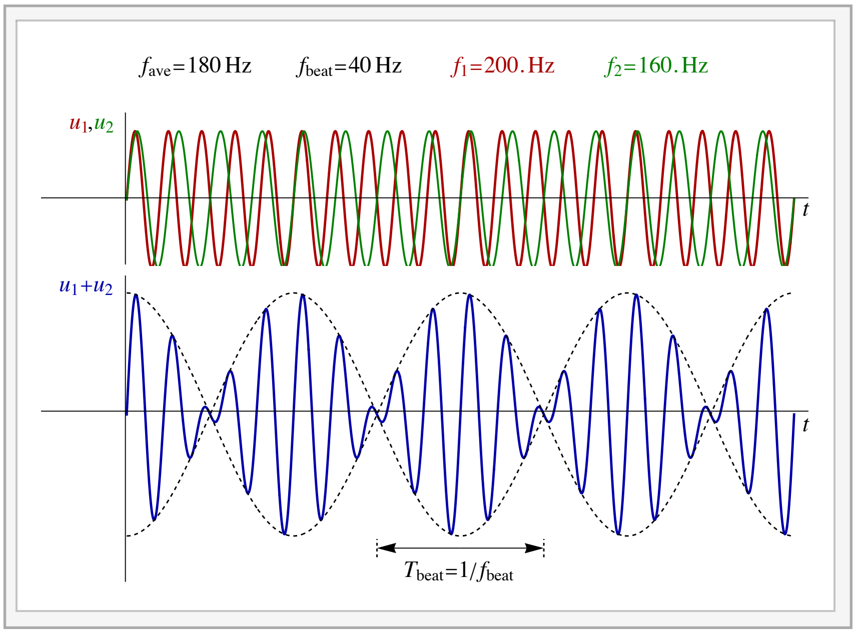
$$\begin{aligned} u(t) &= u_1(t)+u_2(t) = A\cos(\omega_1\,t)+A\cos(\omega_2\,t) \\ &= 2\,A\cos\left(\frac{\omega_1+\omega_2}{2}\,t\right)\cos\left(\frac{\omega_1-\omega_2}{2}\,t\right). \end{aligned}$$

This can be written in terms of the average (angular) frequency $\omega_{\text{ave}}$ and the beat (angular) frequency $\omega_{\text{beat}}$.

$$u(t) = 2\,A\cos(\,\omega_{\text{ave}}\,t\,)\cos\left(\frac{1}{2}\,\omega_{\text{beat}}\,t\right) \quad \text{where} \;\; \omega_{\text{ave}} = \frac{\omega_1+\omega_2}{2} \;\; \text{and} \;\; \omega_{\text{beat}} = |\,\omega_1-\omega_2|$$

Using $\omega = 2\,\pi\,f$ we get the average frequency and beat frequency:

$$u(t) = 2\,A\cos(2\,\pi\,f_{\text{ave}}\,t\,)\cos(\,\pi\,f_{\text{beat}}\,t\,) \quad \text{where} \;\; f_{\text{ave}} = \frac{f_1+f_2}{2} \;\; \text{and} \;\; f_{\text{beat}} = |\,f_1-f_2|$$




Interactive Figure

The frequency $f_{\text{ave}}$ oscillates rapidly. Since the function $\cos(2\,\pi\,f_{\text{ave}}\,t\,)$ varies between $\pm 1$, the function $u(t)$ stays between $\pm 2\,A\cos(\,\pi\,f_{\text{beat}}\,t\,)$, this is called the envelope function,