Chapter C

Current and DC Circuits

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C.1 - Current and Current Density

Basic Definitions

If ΔQ is the charge that passes through some surface, usually a cross-section of a wire, in the small time Δt then the current I is defined by

$$I = \frac{\Delta Q}{\Delta t}$$

Units: The SI unit for Current is: ampere = A = C/s

Example C.1 - Battery on a Notebook Computer

A notebook computer has a battery with a capacity of 6.41 A·h and a voltage of 20 V.

(a) What is the total charge and the corresponding number of electrons stored in this?

Solution

Since an ampere is a coulomb/second then an A·h is a measure of charge. This problem is just a conversion of units

$$Q = 6.41\ \text{A}\cdot\text{h} = 6.41\ \text{A} \times 3600\ \text{s} = 23\,000\ \text{C}$$

Since charge is quantized we can find the number of elementary charges which is just the number of electrons stored.

$$Q = n e \implies n = \frac{Q}{e} = \frac{Q}{1.60 \times 10^{-19}\ \text{C}} = 1.44 \times 10^{23}$$

(b) What is the total energy capacity of this battery?

Solution

$\Delta U = q\,\Delta V$ is the energy when a charge q moves across a potential difference ΔV. Here we have $V = |\Delta V| = 20$ V and $q = Q$.

$$\Delta U = q\,\Delta V = Q\,V = 461\,000\ \text{J}$$

C.2 - Resistance

Ohm's Law

In electrostatics, currents are not allowed. We saw that the electrostatic electric field inside a conductor had to be zero and that implied that the voltage across a conductor also became zero. An electric field in a conductor or a voltage across one will necessarily produce a current.

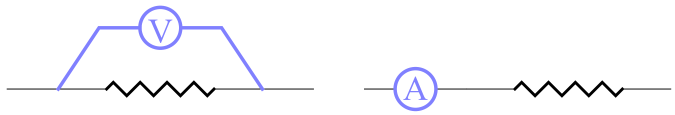
The conductivity and resistivity are properties of a material. For an object, like a wire, we can define a quantity called the resistance R by the macroscopic form of Ohm's law

$$V = I\,R$$

$$R = \text{Resistance}$$

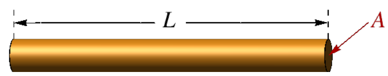
Units: The SI unit for Resistance is: ohm = Ω = V / A

$V = I\,R$ relates the voltage *across* a resistor to the current *through* it. When passing through the resistor in the direction of the current, it is a voltage drop, a decrease in potential. To measure the voltage across a resistor connect the leads of the voltmeter to either side of the resistor. To measure the current through a resistor connect the ammeter in line with the resistor.

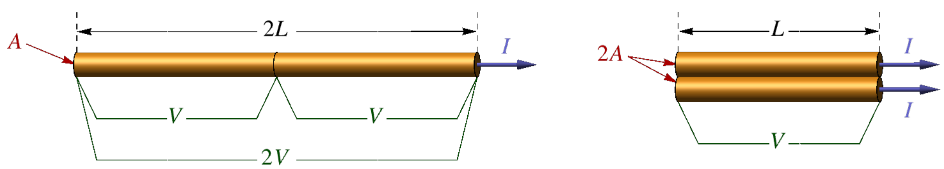


Use of Voltmeter   Use of Ammeter

How does the resistance of a wire vary with its dimensions. Take the wire to be a long cylinder with length L and cross-sectional area A.



Double the length of a wire by putting identical wires end to end as shown in (a) below. Keep the current through the long wire the same; the voltage across each short segment will stay the same and the total voltage across the long wire will double. This shows its resistance, the voltage per current, doubles and that resistance is proportional to the length of a wire: $R \propto L$. Next, model doubling the area by placing two wires side-by-side as in (b) below. If we keep the voltage across both the same the the current through each will stay the same and the total current will then double. Since resistance is voltage per current, the resistance is halved; the resistance is inversely proportional to the cross-sectional area: $R \propto 1/A$.



(a)   (b)

Combining these proportionalities we get: $R \propto L/A$. We introduce a material-dependent constant of proportionality ρ, the resistivity. A good conductor has a small resistivity and a poor conductor has a resistivity with a high value.

$$R = \frac{\rho\,L}{A}.$$

Units: The SI unit for resistivity is: Ω·m

Material	Resistivity – ρ (Ω m)
Copper	1.68×10^{-8}
Aluminum	2.65×10^{-8}
Silver	1.47×10^{-8}
Gold	2.22×10^{-8}
Glass	$10^5 - 10^8$

Resistivities for Different Materials at 20°C

Example C.2 - Resistance of a Copper Wire

The resistivity of copper is $1.68 \times 10^{-8}\ \Omega\cdot\text{m}$. Consider a copper wire with a length of 750 m and a diameter of 3 mm.

(a) What is the resistance of this wire?

Solution

$$\rho = 1.68 \times 10^{-8}\ \Omega\cdot\text{m}\,,\quad L = 750\ \text{m} \ \text{ and } \ d = 3 \times 10^{-3}\ \text{m} \Longrightarrow A = \pi\,r^2 = \pi\left(\frac{d}{2}\right)^2 = 7.0686 \times 10^{-6}\ \text{m}^2$$

We can now find the wire's resistance.

$$R = \frac{\rho L}{A} = 1.78\ \Omega$$

(**b**) What is the current through the wire if it is connected across a 1.5 V battery?

Solution

$$V = 1.5\ \text{V} \quad \text{and} \quad V = I R \implies I = \frac{V}{R} = 0.841\ \text{A}$$

Variation of Resistance with Temperature

Resistance in a metal is caused by collisions between the moving electrons with the vibrating atoms. If there were no vibration in the atoms there would be no collisions and the resistance would be zero. As the temperature is increased the vibrational motion of the atoms increases and the collisions increase. Because of this, resistance in metals increases with temperature.

As temperatures approach absolute zero, the vibrational motion of atoms approaches its minimum value consistent with quantum physics. We would expect the resistivity to go to zero as temperatures approach absolute zero. But what we observe in some materials is much more dramatic; below some low critical temperature T_C, the resistivity abruptly goes to exactly zero. This is known as superconductivity. Superconductivity was first discovered in 1911 in a solid mercury wire at $T_C = 4.2$ K. In 1986 a new class of materials was discovered; these new materials have critical temperatures greater than 90 K.

Semiconductors have different temperature behavior than metals; the resistivity of pure semiconductors decreases with temperature. This is because, unlike metals, more charge carriers are available for conduction at higher temperatures.

C.3 - Power and DC Voltage Sources

Power in General

Power is generally defined as the rate that work is done or, more generally, the rate that energy is used or provided.

$$\mathcal{P} = \frac{\Delta \text{Energy}}{\Delta t}$$

When a charge Q is moved across a potential difference ΔV the potential energy difference is $\Delta U = Q\,\Delta V$. It follows that when an small charge ΔQ moves across a voltage of V the infinitesimal energy change is $\Delta U = V\,\Delta Q$. Writing $\mathcal{P} = \Delta U / \Delta t$ and using $I = \Delta Q / \Delta t$ gives

$$\mathcal{P} = V\,I.$$

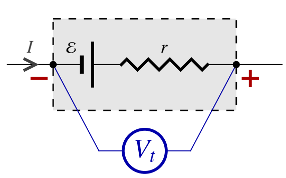
Power Dissipated in a Resistor

Ohm's law $V = I\,R$ relates the voltage drop across a resistor to the current through it. Using it we can write equivalent expressions for the power dissipated in a resistor.

$$\mathcal{P} = V\,I = I^2\,R = \frac{V^2}{R}$$

The energy lost to resistance is dissipated as heat. This is called *Joule heating*.

Terminal Voltage



Treat every DC voltage source as an ideal voltage source with EMF (electromotive force) $\mathcal{E}$ in series with its internal resistance r. The voltage across the terminals V_t of the source is then

$$V_t = \mathcal{E} - I\,r.$$

When there is no load, $I = 0$, the terminal voltage V_t is the same as the EMF $\mathcal{E}$. With a load the terminal voltage drops.

Example C.3 - Terminal Voltage

The measured voltage across a D-cell battery is 1.486 V when no current is drawn. When the battery produces a 250 mA current, its measured voltage is 1.454 V. What is the internal resistance of the battery?

Solution

The voltage with no load (current) is the EMF.

$$\mathcal{E} = 1.486\ \text{V}$$

The other voltage, with the current, is the terminal voltage.

$$I = 0.250\ \text{A} \quad \text{and} \quad V_t = 1.454\ \text{V}$$

We then solve for the internal resistance.

$$V_t = \mathcal{E} - I\,r \quad \Longrightarrow \quad r = \frac{\mathcal{E} - V_t}{I} = 0.128\ \Omega$$

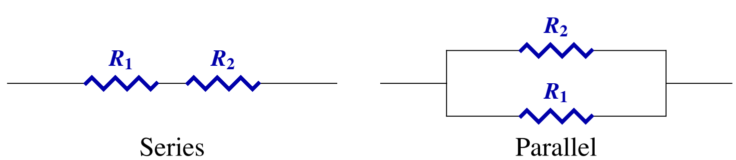
Circuit Diagrams and Nodes

A real-world wire has resistance. When we draw circuit diagrams we always consider the wires to be perfect conductors. Since $R = 0$, the voltage drop across a wire is zero. A wire in a circuit diagram is a point of constant voltage; this is what we call a node. The most effective way to analyze complex circuit diagrams is in terms of nodes and the circuit elements (voltage sources, resistors, capacitors, etc.) connected between nodes.

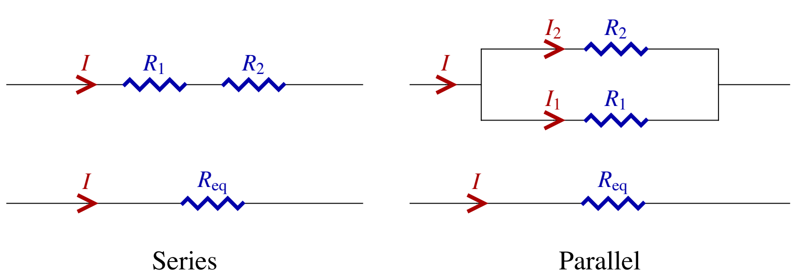
When it is necessary to consider the real-world resistance of a wire one can simply view it as an ideal conductor with a resistor with $\rho\,\ell/A$ of resistance placed in line.

Voltages in circuits are always differences. If we choose some node to be zero voltage then we can assign a voltage to each node in a circuit. A point of zero voltage in a circuit is called a *ground*.

C.4 - Combinations of Resistors



Any combination of resistors with one wire in and one wire out can be reduced to its equivalent resistance. If the combination were placed inside some black box then outside the box the combination would look like a single resistor, which we call its equivalent resistance. For series and parallel resistor combinations, there are simple formulas for finding these equivalent resistances.



Series

Resistors are in series when all the current through one passes through the others; there is no branching between them. The total voltage is the sum of the voltages.

$$I = I_1 = I_2 = \ldots \quad \text{and} \quad V = V_1 + V_2 + \ldots$$

Using $V = I\,R$ gives $I\,R_{\text{eq}} = I\,R_1 + I\,R_2 + \ldots$. The equivalent resistance of series resistors is given by

$$R_{eq} = R_1 + R_2 + \ldots.$$

Parallel

Resistors are in parallel when the voltage across the one is the same as the voltage across the others. Resistors are in parallel when they are connected between the same two nodes, where a node is a point of constant voltage in a circuit. The current branches and the total current is the sum of the currents.

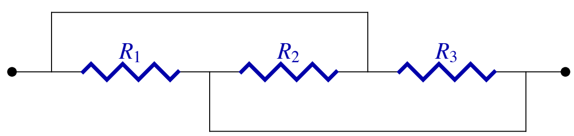
$$V = V_1 = V_2 = \ldots \quad \text{and} \quad I = I_1 + I_2 + \ldots$$

Using $I = V/R$ gives $V/R_{eq} = V/R_1 + V/R_2 + \ldots.$ The equivalent resistance of series resistors is given by

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots.$$

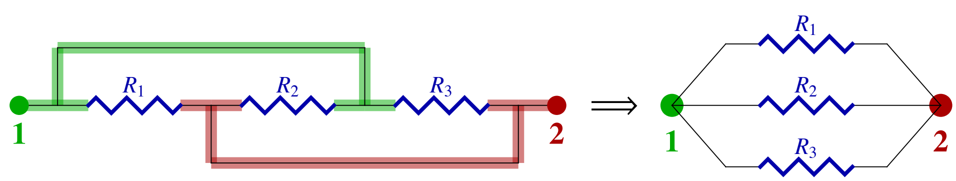
Example C.4 - Identifying Nodes

What is the equivalent resistance of this network of resistors?



Solution

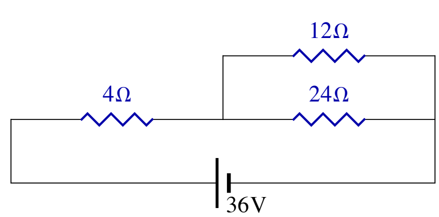
Recall that a wire in a circuit diagram is a perfect conductor, so there is no voltage drop across wires in circuit diagrams. A node is a point of constant potential in a circuit. To identify nodes follow along a wire as far as possible without hitting some circuit element, like a resistor, voltage source or capacitor. There are two nodes, the points drawn in the diagram; label the node on the left as 1 and the node on the right as 2. It is useful, conceptually, to mark nodes with colors. Using green for node 1 and red for node 2, then trace as far as you can go along perfect conductors until you hit a resistor. This gives the picture on the left below.



The next step is to rewrite the nodes as points and draw the resistors as they are connected between the nodes. This is the picture on the right, above. All three resistors are connected across the same two nodes so they are in parallel. We can then write down the equivalent resistance.

$$R_{eq} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1}$$

Example C.5 - A Resistor Circuit



The diagram shows three resistors connected across a 36V battery. Complete the table with the voltage across, the current through and the power dissipated in each of the three resistors.

	4 Ω	12 Ω	24 Ω
V			
I			
$\mathcal{P}$			

Solution

First find the equivalent resistance across the battery. The 12 Ω and 24 Ω resistors are in parallel. Call their equivalent $R_{\parallel}$.

$$\frac{1}{R_{\parallel}} = \frac{1}{12\,\Omega} + \frac{1}{24\,\Omega} = \frac{1}{8\,\Omega} \implies R_{\parallel} = 8\,\Omega$$

This resistor is then in series with the remaining 4 Ω resistance. This gives the overal equivalent resistance.

$$R_{\text{eq}} = 4\,\Omega + 8\,\Omega = 12\,\Omega$$

This equivalent resistance determines I_{battery} the current that the battery will provide. The 4 Ω resistor is in series with the battery so the current provided by the battery will pass through the 4 Ω resistor. This allows us to begin filling in the table with the current through the 4 Ω resistor.

$$I_{4\,\Omega} = I_{\text{battery}} = \frac{V_{\text{battery}}}{R_{\text{eq}}} = \frac{36\text{ V}}{12\,\Omega} = 3\text{ A}$$

With a table like this, once one thing in a column is known, we can find the rest of that column using $V = I\,R$ and $\mathcal{P} = I\,V$

$$V_{4\,\Omega} = I_{4\,\Omega}\,4\,\Omega = 12\text{ V}$$

We can also find the power.

$$\mathcal{P}_{4\,\Omega} = I_{4\,\Omega}\;V_{4\,\Omega} = 36\text{ W}$$

The voltages across the 12 Ω and 24 Ω resistors are equal, because they are in parallel and that voltage added to $V_{4\,\Omega}$ will give the total voltage V_{battery}.

$$V_{12\,\Omega} = V_{24\,\Omega} = V_{\text{battery}} - V_{4\,\Omega} = 36\text{ V} - 12\text{ V} = 24\text{ V}$$

$I = V/R$ allows us to find the currents through the 12 Ω and 24 Ω resistors.

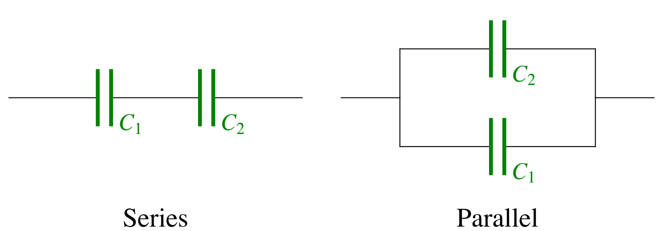
$$I_{12\,\Omega} = \frac{V_{12\,\Omega}}{12\,\Omega} = \frac{24\text{ V}}{12\,\Omega} = 2\text{ A} \quad\text{and}\quad I_{24\,\Omega} = \frac{V_{24\,\Omega}}{24\,\Omega} = \frac{24\text{ V}}{24\,\Omega} = 1\text{ A}$$

We can also use $\mathcal{P} = I\,V$ to finish the table with the last power values.

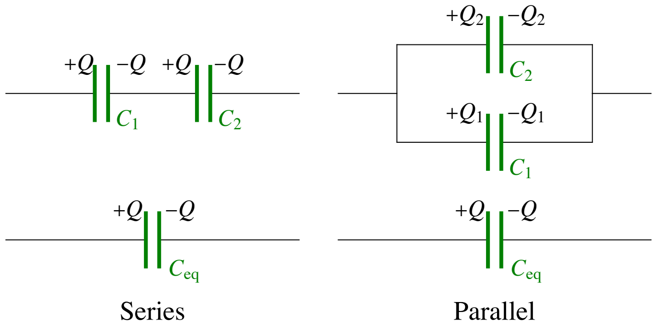
$$\mathcal{P}_{12\,\Omega} = I_{12\,\Omega}\,V_{12\,\Omega} = 48\text{ W} \quad\text{and}\quad \mathcal{P}_{24\,\Omega} = I_{24\,\Omega}\,V_{24\,\Omega} = 48\text{ W}$$

	4 Ω	12 Ω	24 Ω
V	12 V	24 V	24 V
I	3 A	2 A	1 A
$\mathcal{P}$	36 W	48 W	24 W

C.5 - Combinations of Capacitors



As we saw for resistors, any network of capacitors can be reduced to an equivalent capacitance. For capacitors its charge plays the role the current played in resistors. (Recall that $I = \Delta Q/\Delta t$.)



The voltage to charge relation for a capacitor is

$$V = \frac{Q}{C}.$$

Series

In the case of series resistors the charge on each capacitor is the same and both are the same as the charge on the equivalent. The voltages add.

$$Q = Q_1 = Q_2 = \ldots \text{ and } V = V_1 + V_2 + \ldots$$

Using the voltage to charge relation gives $Q/C_{\text{eq}} = Q/C_1 + Q/C_2 + \ldots$ which gives the expression for equivalent capacitance

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots.$$

Parallel

For parallel resistors the voltages are equal and the charges add.

$$V = V_1 = V_2 = \ldots \text{ and } Q = Q_1 + Q_2 + \ldots$$

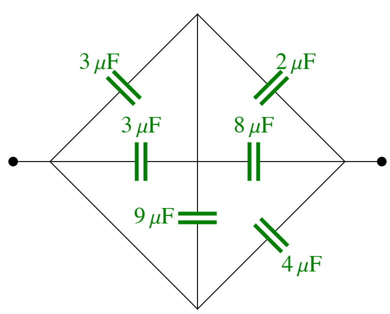
Using $Q = C\,V$ gives $C_{\text{eq}}\,V = C_1\,V + C_2\,V + \ldots$ giving

$$C_{\text{eq}} = C_1 + C_2 + \ldots.$$

Note that the series and parallel formulas for capacitors are reversed relative to their resistor counterparts.

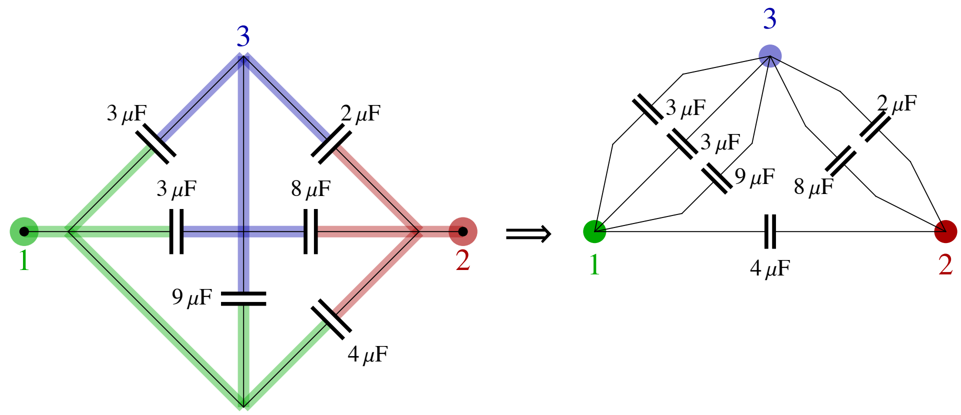
Example C.6 - Equivalent Capacitance

What is the equivalent capacitance of the capacitor network shown?



Solution

First do a nodal analysis on this capacitor network. It is then clear how they are connected.



The two 3 μF and the 9 μF capacitors are in parallel, as are the 2 μF and 8 μF capacitors. We add to find the equivalent capacitances.

$$3\ \mu\text{F} + 3\ \mu\text{F} + 9\ \mu\text{F} = 15\ \mu\text{F} \quad \text{and} \quad 2\ \mu\text{F} + 8\ \mu\text{F} = 10\ \mu\text{F}$$

These two resulting capacitances are in series

$$\left(\frac{1}{15\ \mu\text{F}} + \frac{1}{10\ \mu\text{F}}\right)^{-1} = 6\ \mu\text{F}$$

and this 6 μF is then in parallel with the final 4 μF giving the overall capacitance.

$$C_{\text{eq}} = 6\ \mu\text{F} + 4\ \mu\text{F} = 10\ \mu\text{F}$$

C.6 - Kirchhoff’s Rules

Kirchhoff's rules are used to solve for the currents in the case of a circuit involving many resistors and DC voltage sources. A junction is a point in the circuit where three or more wires meet; if there are just two wires it is just a bend in the wire and not a junction. For every branch in the circuit we can define a current. Kirchhoff's rules gives a set of linear equations in the currents. It is not essential to choose the proper direction for the currents, and in fact one typically doesn't know the current directions until a solution is found. If the chosen current direction is wrong then that current will be negative when the solution is found.

Junction Rule

At every junction in a circuit the total current in is equal to the total current out.

$$\sum I_{\text{in}} = \sum I_{\text{out}}.$$

In every case (at least where the circuit is one connected piece) the junction rule equations will not be independent; there will always be one equation more than is needed. Summing all the equations gives $\sum I = \sum I$ which is equivalent to $0 = 0$. (This is because every current leaves one junction and enters another.) Because of this any one junction rule equation is the negative of the sum of the others. To get an independent set of equations one must delete one (any one) of the equations.

Loop Rule

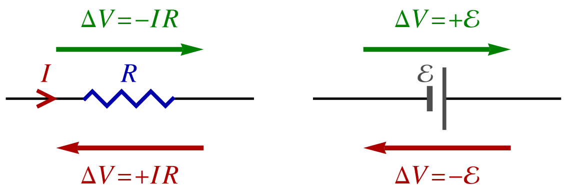
Around every closed loop in a circuit the sum of all the voltage gains is zero.

$$\sum \Delta V = 0$$

The sign conventions are:

When moving through a resistor in the direction of the current: $\Delta V = -I\,R$.
When moving through a resistor opposite the current: $\Delta V = +I\,R$.

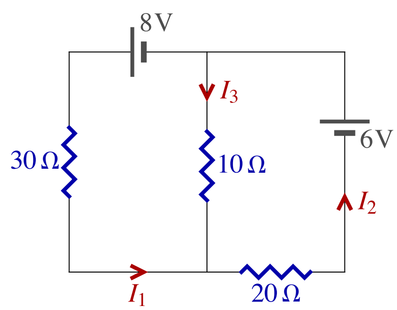
When moving through a DC source from - to + terminals: $\Delta V = +\mathcal{E}$.
When moving through a DC source from + to - terminals: $\Delta V = -\mathcal{E}$.



To avoid non-independent equations consider only the smallest loops.

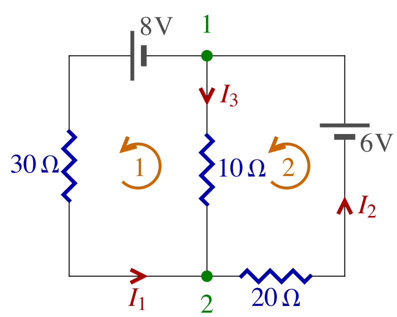
Example C.7 - A Two-loop Circuit

Solve the following 2-loop circuit using Kirchhoff's rules for the three currents shown.



Solution

First, label the junctions and then loops. We have two junctions and two loops.



Using $\sum I_{in} = \sum I_{out}$ for both junctions gives.

$$I_2 = I_1 + I_3 \quad \text{(Junction 1)}$$

$$I_1 + I_3 = I_2 \quad \text{(Junction 2)}$$

Notice that the equation for the second junction is redundant. Just keep the first. Now apply the loop rule $\sum \Delta V = 0$, using the sign conventions, to get the last two equations needed to solve for the three unknown currents. Ignore the units while writing the equations.

$$0 = 8 - 30\,I_1 + 10\,I_3 \quad \text{(Loop 1)}$$

$$0 = 6 - 10\,I_3 - 20\,I_2 \quad \text{(Loop 2)}$$

Use the junction rule equation to eliminate I_2 from the second loop equation

$$0 = 6 - 10\,I_3 - 20\,(I_1 + I_3) \implies 0 = 6 - 20\,I_1 - 30\,I_3 \implies 20\,I_1 + 30\,I_3 = 6$$

Rewrite the first loop equation

$$30\,I_1 - 10\,I_3 = 8$$

To eliminate I_3 multiply this equation by 3 and add it to the other.

$$3 \times (30\,I_1 - 10\,I_3) = 8 \times 3$$
$$20\,I_1 + 30\,I_3 = 6$$

This gives

$$110\, I_1 = 30 \implies I_1 = \frac{30}{110}\text{ A} = 0.27273\text{ A}$$

Solving for I_3 gives

$$30\, I_1 - 10\, I_3 = 8 \implies I_3 = \frac{30\, I_1 - 8}{10} = 0.01818\text{ A}$$

Now solve for I_2.

$$I_2 = I_1 + I_3 = 0.29091\text{ A}$$

The final solution can now be written.

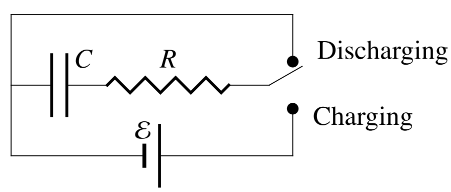
$$I_1 = 0.273\text{ A},\ I_2 = 0.291\text{ A} \text{ and } I_3 = 0.0182\text{ A}$$

C.7 - RC Circuits

When the switch is thrown to the Charging position the current flows from the battery to charge the capacitor. In the Discharging position the charge flows from the capacitor and its energy is dissipated in the resistor. The charge on the capacitor is related to the current in the wire by

$$I = \frac{\Delta Q}{\Delta t} \text{ for small } \Delta t$$

Note that when the capacitor is discharging the charge is decreasing and thus the current is negative.



Discharging

For the discharging case, applying Kirchhoff's loop rule to the circuit gives:

$$0 = I\, R + \frac{Q}{C}.$$

Writing the current in terms of the charge gives.

$$0 = R\, \frac{\Delta Q}{\Delta t} + \frac{1}{C}\, Q \text{ or } \frac{\Delta Q}{\Delta t} = -\frac{Q}{\tau}$$

where τ is defined as the time constant.

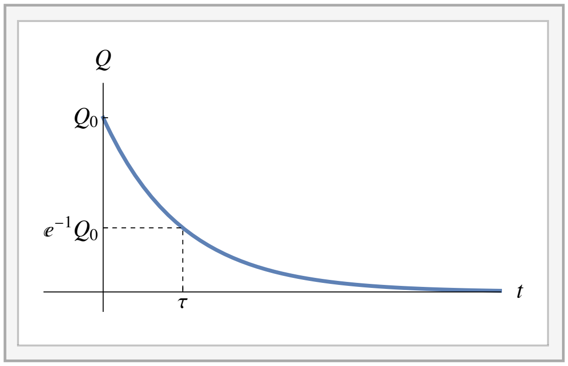
$$\tau = R\, C$$

Note that τ is a time.

Take the initial charge to be Q_0. Solving the equation for t is a calculus problem. We will just write down the solution.

$$Q(t) = Q_0\, e^{-t/\tau}.$$

This is an exponential decay. After on time constant τ the charge drops to $1/e$ of its initial value. A small time constant means a rapid decay and a large time constant means a slow decay.



Interactive Figure - Discharging an RC Circuit

The current is negative, since the charge is decreasing, and is given by

$$I(t) = -\frac{Q_0}{\tau}\, e^{-t/\tau}.$$

Note that with this exponential decay, the charge on a capacitor never goes fully to zero but after several time constants it becomes very small.

Charging

The loop rule for the charging case gives:

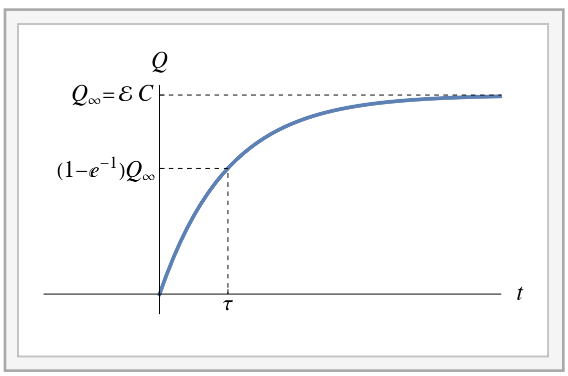
$$0 = \mathcal{E} - I\,R - \frac{Q}{C}$$

and we can rewrite this as:

$$\mathcal{E} = R\,\frac{\Delta Q}{\Delta t} + \frac{1}{C}\,Q$$

Using the same time constant τ we can write down the solution where we take the initial charge, at time zero, to be zero.

$$Q(t) = Q_\infty \left(1 - e^{-t/\tau}\right) \text{ where } Q_\infty = \mathcal{E}\,C.$$



Interactive Figure - Charging an RC Circuit

The current is now positive and is exponentially decaying.

$$I(t) = \frac{\mathcal{E}}{R}\, e^{-t/\tau}$$

Example C.8 - An RC Circuit

Consider a 50 μF capacitor, a 12 V battery and a 300 Ω resistor.

(a) If these three components are connected in a series loop circuit at time zero, then what are the charge on the capacitor and current through the resistor 20 ms after the circuit is connected.

Solution

$$c = 50 \times 10^{-6}\ \text{F}, \quad \mathcal{E} = 12\ \text{V} \ \text{ and } \ R = 300\ \Omega$$

We first need to find the time constant τ.

$$\tau = RC = 0.015\ \text{s}$$

Using the time of $t = 20\ \text{ms} = 0.020\ \text{s}$ we can then use the charging formulas for both charge and current as functions of time to find Q and I.

$$Q(t) = \mathcal{E}\ C\ \left(1 - e^{-t/\tau}\right) \implies Q(20\ \text{ms}) = 4.42 \times 10^{-4}\ \text{C}$$

$$I(t) = \frac{\mathcal{E}}{R}\ e^{-t/\tau} \implies I(20\ \text{ms}) = 0.0105\ \text{A}$$

(b) Suppose the circuit in part **(a)** is allowed to fully charge before the circuit is disconnected and the capacitor is connected across only the resistor. What is the charge on the capacitor and current through the resistor just after the circuit is connected.

Solution

To find the initial charge on the capacitor Q_0, we need to use Q_∞ for a charging capacitor.

$$Q(0) = Q_0 = Q_\infty = \mathcal{E}\, C = 6.0 \times 10^{-4}\ \text{s}$$

The initial current is found using the discharging current formula and that $e^0 = 1$. The time constant τ is the same as in part **(a)**.

$$I(0) = -\frac{Q_0}{\tau}\ e^{-t/\tau} = -\frac{Q_0}{\tau}\ e^0 = 0.040\ \text{A}$$

(c) How long does it take for the charge on the discharging capacitor in part **(b)** to drop to 2% of its initial charge?

Solution

We need to find t when $Q(t) = 0.020\ Q_0$ for a discharging capacitor.

$$0.020\ Q_0 = Q(t) = Q_0\ e^{-t/\tau} \implies 0.020 = e^{-t/\tau}$$

The inverse function for the exponential is the natural log function, ln. If $y = e^x$ then $x = \ln(y) = x$.

$$-t/\tau = \ln(0.020) = -3.91202 \implies t = 3.91202\ \tau = 0.587\ \text{s}$$