# Chapter F

## *AC Circuits*

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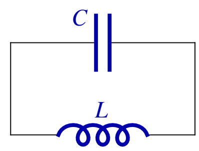
## F.1 - LC Circuits, RLC Circuits and Their Mechanical Equivalents

There is an important analogy between the LC Circuit and the mass-spring system. We will see that in both cases we get oscillatory solutions.

### The LC Circuit




Consider a simple loop circuit containing just a capacitor $C$ and inductor $L$. The loop rule gives

$$0 = L\,\frac{\Delta I}{\Delta t} + \frac{Q}{C} \quad (\text{small } \Delta t)$$

Defining the angular frequency by

$$\omega_0 = \frac{1}{\sqrt{L\,C}} \tag{F.1}$$

gives the following.

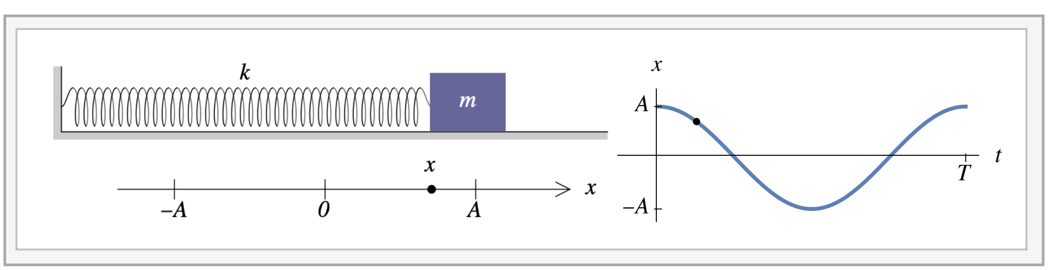
$$\frac{\Delta I}{\Delta t} = -\omega_0^2\,Q \quad \text{where } I = \frac{\Delta Q}{\Delta t} \quad (\text{small } \Delta t) \tag{F.2}$$

The energy is conserved in this system. The energy in the inductor is $U_L = \frac{1}{2}\,L\,I^2$ and the the energy in the capacitor is $U_C = \frac{1}{2\,C}\,Q^2$. The total energy $U$ is then conserved.

$$U = U_L + U_C = \frac{1}{2}\,L\,I^2 + \frac{1}{2\,C}\,Q^2 = \text{constant}$$

The inductor's energy is stored in the magnetic field of the inductor. Similarly, the capacitor's energy is electric.

### The Mass-Spring System




Interactive Figure

The mechanical analog of this is a mass-spring system. The force of a spring is given by Hooke's law $F = -k\,x$. Applying Newton's second law gives a second order ODE

$$F_{\text{net}} = m\,a \quad \Longrightarrow \quad -k\,x = m\,\frac{\Delta v}{\Delta t} \;\; (\text{small}\,\Delta t)$$

Defining the angular frequency by

$$\omega_0 = \sqrt{\frac{k}{m}}\,,$$

gives the following.

$$a = \frac{\Delta v}{\Delta t} = -\omega_0^2\,x \;\; (\text{small}\,\Delta t) \;\; \text{where} \;\; v = \frac{\Delta x}{\Delta t} \;\; (\text{small}\,\Delta t) \tag{F.3}$$

The kinetic energy of the mass $K = \frac{1}{2}\,m\,v^2$ and the potential energy of the spring $U = \frac{1}{2}\,k\,x^2$ also combine to give a conserved energy.

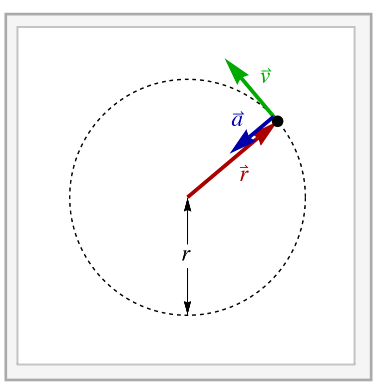
$$E = K + U = \frac{1}{2}\,m\,v^2 + \frac{1}{2}\,k x^2 = \text{constant}$$

## The Analogy without Damping

Summarizing the analogy as stated above: The charge and current are the analogs of the position and velocity. The energy in the inductor $U_L = \frac{1}{2}\,L\,I^2$ and the kinetic energy of the mass $K = \frac{1}{2}\,m\,v^2$ are analogous, as are the energy in the capacitor $U_C = \frac{1}{2\,C}\,Q^2$ and the potential energy of the spring $U = \frac{1}{2}\,k\,x^2$.

| LC Circuit | Mass–Spring |
|---|---|
| $Q$ | $x$ |
| $I = \frac{\Delta Q}{\Delta t}$ (small $\Delta t$) | $v = \frac{\Delta x}{\Delta t}$ (small $\Delta t$) |
| $\frac{\Delta I}{\Delta t}$ (small $\Delta t$) | $a = \frac{\Delta v}{\Delta t}$ (small $\Delta t$) |
| $L$ | $m$ |
| $C$ | $\frac{1}{k}$ |
| $\omega_0 = \frac{1}{\sqrt{L\,C}}$ | $\omega_0 = \sqrt{\frac{k}{m}}$ |
| $U_L = \frac{1}{2}\,L\,I^2$ | $K = \frac{1}{2}\,m\,v^2$ |
| $U_C = \frac{1}{2\,C}\,Q^2$ | $U = \frac{1}{2}\,k\,x^2$ |

## Uniform Circular Motion and the Oscillatory Solutions




Recall from Physics I that uniform circular motion is motion in a circle with a constant speed $v$. The angular velocity $\omega$ is the rate of rotation around the circle; in terms of the speed and radius $r$, the angular velocity is $\omega = v/r$. The acceleration is toward the center and has magnitude

$$a = \frac{v^2}{r} = \omega^2\,r$$

Take the origin to be the center of the circle; the acceleration vector $\vec{a}$ is opposite the position vector $\vec{r}$ and we have

$$\vec{a} = -\omega^2\, \vec{r}$$

If we take one component of this expression, we have:

$$a_y = -\omega^2\, y \tag{F.4}$$

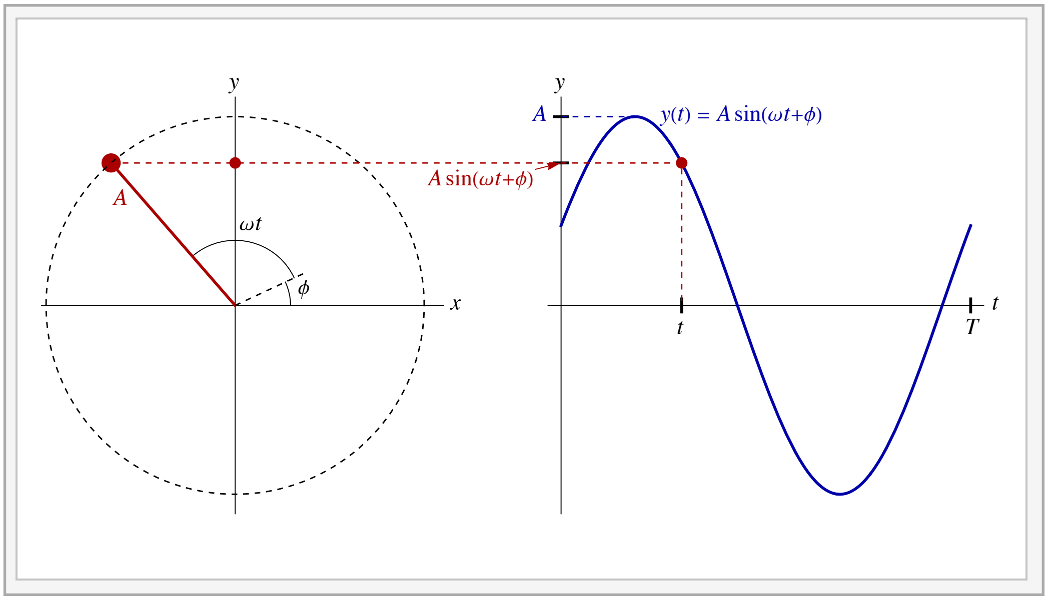
We are considering the $y$-component for reasons that should become clear later.

In polar coordinates we have $y = r\sin\theta$. Replace the radius with $r \to A$. For constant angular velocity we have $\theta = \omega t + \theta_0$ and take the initial angle to be $\theta_0 = \phi$.

$$y = r\sin\theta = A\sin\theta \quad \text{and} \quad \theta = \omega t + \theta_0 = \omega t + \phi$$

This gives our expression for $y$ as a function of $t$.

$$y(t) = A\sin(\omega t + \phi) \tag{F.5}$$




The angular velocity $\omega$ of the circular motion will become the angular frequency $\omega_0$ of the oscillatory solutions. We label these frequencies as $\omega_0$ to distinguish the natural frequency of oscillation of a system, either mechanical or electrical, from some driving frequency; in the electrical case, a sinusoidally varying voltage source will drive the circuit. This is the subject of the rest of the chapter.

The relationship between $a_y$ and $y$ (6.4) is the same as between $a$ and $x$ (6.3) for the mass-spring system and is the same as between $\Delta I / \Delta t$ and $Q$ (6.2) for the $LC$ circuit. We can then obtain those solutions from the $y$-component expression (6.5). The radius of the circular motion, $A$, becomes the amplitude of the mass/spring system, the largest distance from the spring's equilibrium position.

$$x(t) = A\sin(\omega_0 t + \phi) \tag{F.6}$$

For the $LC$ circuit, $A$ becomes $Q_{\text{max}}$ the largest value of the charge on the capacitor during the oscillation.

$$Q(t) = Q_{\text{max}}\sin(\omega_0 t + \phi) \tag{F.7}$$

We will discuss the mathematics of sinusoidal functions in detail in the next section.

In both cases we have oscillatory solutions. In the mass/spring case the energy oscillates between the kinetic energy of the moving mass and the elastic potential energy of the spring; the kinetic energy is its maximum when the mass passes $x = 0$, the equilibrium point of the spring and the potential energy is when $x = \pm A$. In the $LC$ circuit case the energy oscillates between the energy in the capacitor, where $Q = \pm Q_{\text{max}}$ and the energy in the inductor which occurs when $Q = 0$.

## More on the LC Circuit

If we begin with a charge on the capacitor of an LC circuit before connecting it to the inductor, then at time zero the charge is at its maximum $Q_0 = Q_{\text{max}}$ and the phase angle $\phi$ can be chosen to make the sine a cosine.

$$Q(t) = Q_0\cos(\omega_0 t)$$

For this case we will give a formula for the current.

$$I(t) = -\omega_0 Q_0 \sin(\omega_0 t)$$

### Example F.1 - An LC Circuit

A 200-$\mu$F capacitor is given an initial charge of 35 $\mu$C and is connected across an 80-mH inductor.

**(a)** What is the total energy in this circuit?

**Solution**

$$C = 200 \times 10^{-6}\ \text{F}\ , \quad L = 80 \times 10^{-3}\ \text{H} \ \text{ and } \ Q_{\text{max}} = Q_0 = 35 \times 10^{-6}\ \text{F}$$

Since energy is conserved the initial energy is the total energy and initially, there is no current, so that is just the initial energy in the capacitor.

$$U = U_0 = \frac{Q_0^2}{2\,C} = 3.06 \times 10^{-6}\ \text{J}$$

**(b)** What is the maximum current through the inductor?

**Solution**

Where the current is its maximum, then the charge on the capacitor is zero. Using conservation of energy again gives us the maximum current.

$$U = \frac{1}{2}\,L\,I_{\text{max}}^2 \implies I_{\text{max}} = \sqrt{\frac{2\,U}{L}} = 0.00875\ \text{A}$$

**(c)** What are the charge on the capacitor and the current through the inductor 3.2 ms after the circuit is connected?

**Solution**

We use the formulas for the charge and current as functions of time for the LC-circuit. It is important to remember that when calculating such an expression you must have your calculator in the radians mode. Use $t = 3.2 \times 10^{-3}$ s.

$$Q = Q(t) = Q_0 \cos(\omega_0 t) = 2.44 \times 10^{-5}\ \text{C}$$

$$I = I(t) = -\omega Q_0 \sin(\omega_0 t) == 6.28 \times 10^{-3}\ \text{A}$$

**(d)** What are the energy in the capacitor, the energy in the inductor and the total energy at the time in part (c)?

**Solution**

At that instant, we can find the energies. In the inductor we have,

$$U_L = \frac{1}{2}\,L\,I^2 = 1.49 \times 10^{-6}\ \text{J}$$

and for the capacitor:

$$U_C = \frac{Q^2}{2\,C} = 1.58 \times 10^{-6}\ \text{J.}$$

Summing these gives the same tatal energy we found before, as it must.

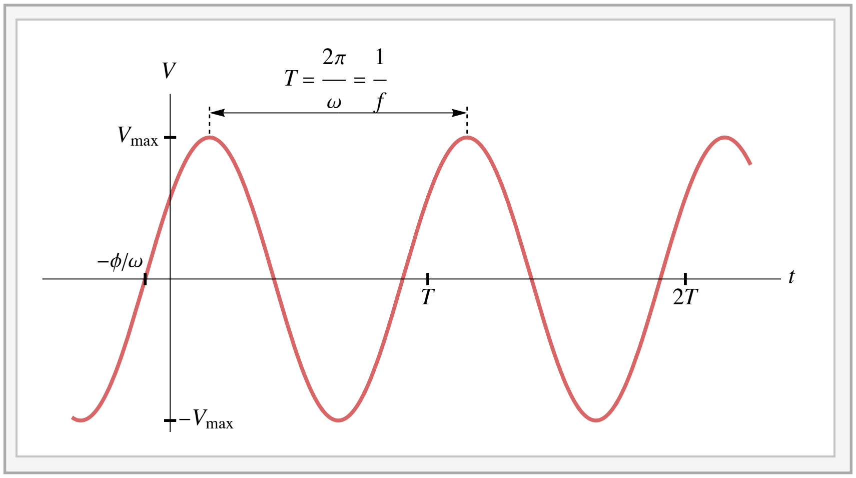
$$U = U_L + U_C = 3.06 \times 10^{-6}\ \text{J.}$$

## F.2 - Sinusoidal Voltage and Current

The general form of a sinusoidal function of voltage is:

$$V(t) = V_{\text{max}} \sin(\omega\,t + \phi)$$

$V_{\text{max}}$ is the *peak voltage*, $\omega$ is the *angular frequency* and $\phi$ is the *phase angle*.




Interactive Figure

## Frequency, Angular Frequency and Period

The rate of oscillation of a sinusoidal function is described by the *angular frequency* $\omega$. The *period* $T$ is the time for each cycle and the *frequency* $f$ is the cycles per time and thus $f = 1/T$. When time changes by one period the argument of the trig function, $\omega t + \phi$, shifts by $2\pi$, which implies that $\omega T = 2\pi$.

$$\omega = \frac{2\pi}{T} = 2\pi f$$

## Phase Angle

Changing the phase shifts the graph along the time axis. (A positive $\phi$ shifts the graph in the negative time direction.) By redefining the zero of time we can always shift the phase angle to zero. Where phase will become important is in the relative phase between two sinusoidal functions, namely voltage and current. Note that a choice of $\phi$ can shift the function from sine to cosine,

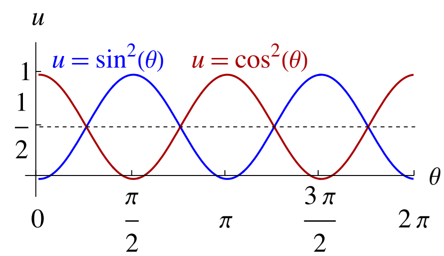
$$\sin\left(\omega t \pm \frac{\pi}{2}\right) = \sin(\omega t \pm 90°) = \pm\cos\omega t.$$

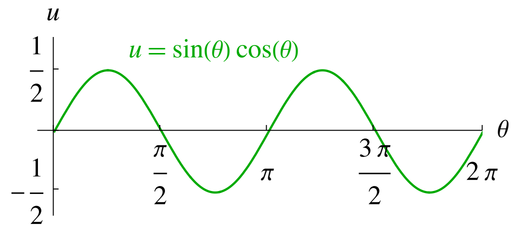
## Averages of Products of Sine and Cosine

The average value of the square of the sine function over one period is 1/2. The average value of $\cos^2$ is also 1/2. Moreover, the average of sine times cosine is zero. Denote the average of a function $f(t)$ over time as $\langle f(t)\rangle_{\text{ave}}$

$$\left\langle\sin^2(\omega t)\right\rangle_{\text{ave}} = \frac{1}{2} = \left\langle\cos^2(\omega t)\right\rangle_{\text{ave}} \quad \text{and} \quad \left\langle\sin(\omega t)\sin\left(\omega t \pm \frac{\pi}{2}\right)\right\rangle_{\text{ave}} = \pm\langle\sin(\omega t)\cos(\omega t)\rangle_{\text{ave}} = 0. \tag{F.8}$$

These average values can be seen by looking at the graphs

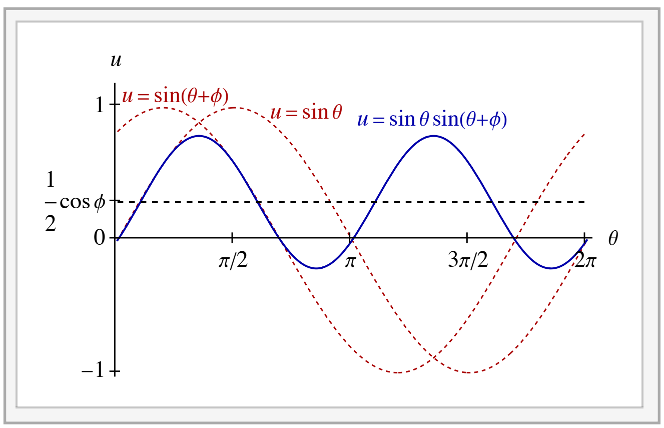



Two sine functions out of phase by $\phi$ average to $\frac{1}{2}\cos\phi$.

$$\langle\sin(\omega\, t)\sin(\omega\, t+\phi)\rangle_{\text{ave}} = \frac{1}{2}\cos\phi. \tag{F.9}$$

Note that the results in (6.8) are the $\phi = 0$ and $\phi = \pi/2$ special cases of this.




## rms and Peak Quantities

It should be clear, from the graph, that the average of a sinusoidal function is zero. To see how much a function that averages to zero deviates from zero, we use the root-mean-square, rms, quantity. This is defined, as its name implies, as the square root (root) of the average (mean) of the square of the function. Using that the average of sine-squared is 1/2 we can find $V_{\text{rms}}$ for a sinusoidal voltage.

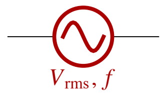
$$V_{\text{rms}} = \sqrt{\langle V^2\rangle_{\text{ave}}} = \sqrt{V_{\max}^2\,\langle\sin^2(\omega t)\rangle_{\text{ave}}} = \frac{1}{\sqrt{2}}\,V_{\max}$$

The above expression applies to any sinusoidal function, so in AC it applies as well to the current.

$$V_{\text{rms}} = \frac{1}{\sqrt{2}}\,V_{\max} \quad\text{and}\quad I_{\text{rms}} = \frac{1}{\sqrt{2}}\,I_{\max}$$

## Circuit Diagrams and the Standard US Outlet

In a circuit diagram, the usual convention for an AC voltage source is to draw use a circle with one period of a sine wave inside, as shown below.




We have to label the source with voltage and frequency, either the rms voltage $V_{rms}$ or peak voltage $V_{max}$, and either the frequency $f$ or angular frequency $\omega$. We will typically use $V_{rms}$ and $f$.

A standard US outlet has

$$V_{rms} = 120\text{ V} \quad \text{and} \quad f = 60\text{ Hz},$$

It follows that $V_{max} = \sqrt{2}\ V_{rms} = 169.7\text{ V}$ and $\omega = 2\pi f = 377\ s^{-1}$. This convention is used in most of the Americas, but some South American countries follow the European convention. Europe uses mostly $V_{rms} = 240\text{ V}$ and $f = 50\text{ Hz}$.

### Example F.2 - European Convention

What is the peak voltage, angular frequency and period of a standard European outlet that has $V_{rms} = 240\text{ V}$ and $f = 50\text{ Hz}$.

#### Solution

$$V_{max} = \sqrt{2}\ V_{rms} = 339\text{ V}, \quad \omega = 2\pi f = 314\text{ Hz} \ \text{ and } \ T = \frac{1}{f} = 0.02\text{ s}$$

## Rate of Change of a Sinusoidal Function

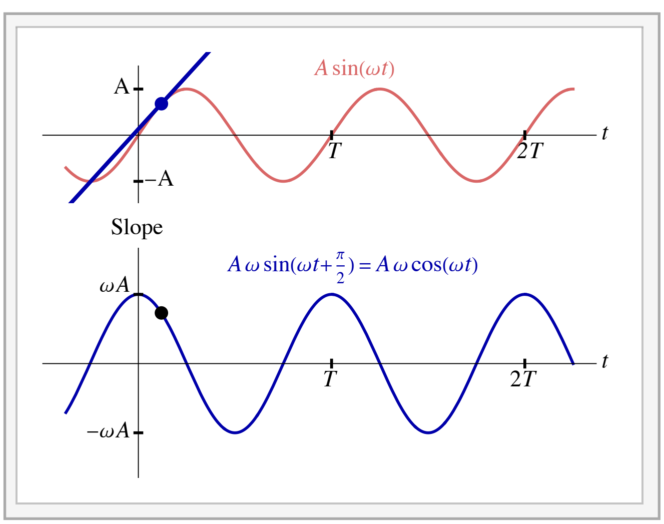
We often use the rate of change of functions of time. The first example of this from Physics I was the instantaneous velocity which is defined as the average velocity for small values of $\Delta t$. We will borrow the notation from calculus and write this as

$$v = \frac{d}{dt} x = \frac{\Delta x}{\Delta t} \text{ (small } \Delta t)$$

For any function of time $f$ we will define its rate of change by

$$\frac{d}{dt} f(t) = \frac{\Delta f}{\Delta t} \text{ (small } \Delta t) \tag{F.10}$$

This has the graphical interpretation as the slope of the line tangent to the curve for the function of time. We will now write, without proof, an expression for the rate of change of a sinusoidal function.




Interactive Figure - The rate of change of $A\sin(\omega t)$ is $A\,\omega\sin\left(\omega t + \frac{\pi}{2}\right)$

The above interactive graphic motivates the result for the rate of change of a sinusoidal function.

$$\frac{d}{dt} A \sin \omega t = A\,\omega \sin\left(\omega t + \frac{\pi}{2}\right) = A\,\omega\cos(\omega t) \tag{F.11}$$

We will use this result shortly.

## F.3 - Relating Voltage and Current

### Impedance and Phase

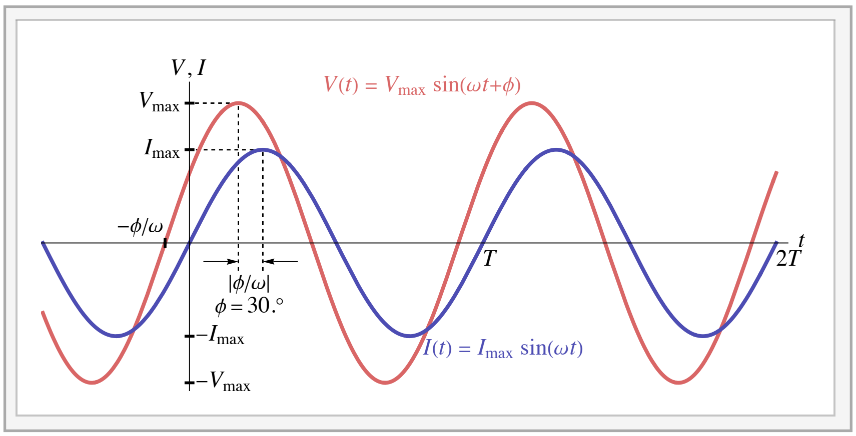
In DC circuits we can relate the voltage and current through a resistor by just one number, the resistance. In AC it takes two numbers, the impedance $Z$ and the phase $\phi$. The peak values, and similarly the rms quantities, are related by the impedance.

$$Z = \frac{V_{\mathrm{max}}}{I_{\mathrm{max}}} = \frac{V_{\mathrm{rms}}}{I_{\mathrm{rms}}}$$

Sinusoidal functions also have a phase angle. The phase angle of some function by itself is unimportant; it just refers to a time shift. What we define as the phase angle $\phi$ is the relative phase between the voltage and current. The voltage is ahead of the current by the phase angle.

$$I(t) = I_{\mathrm{max}} \sin(\omega\, t) \quad \text{and} \quad V(t) = I_{\mathrm{max}}\, Z \sin(\omega\, t + \phi)$$

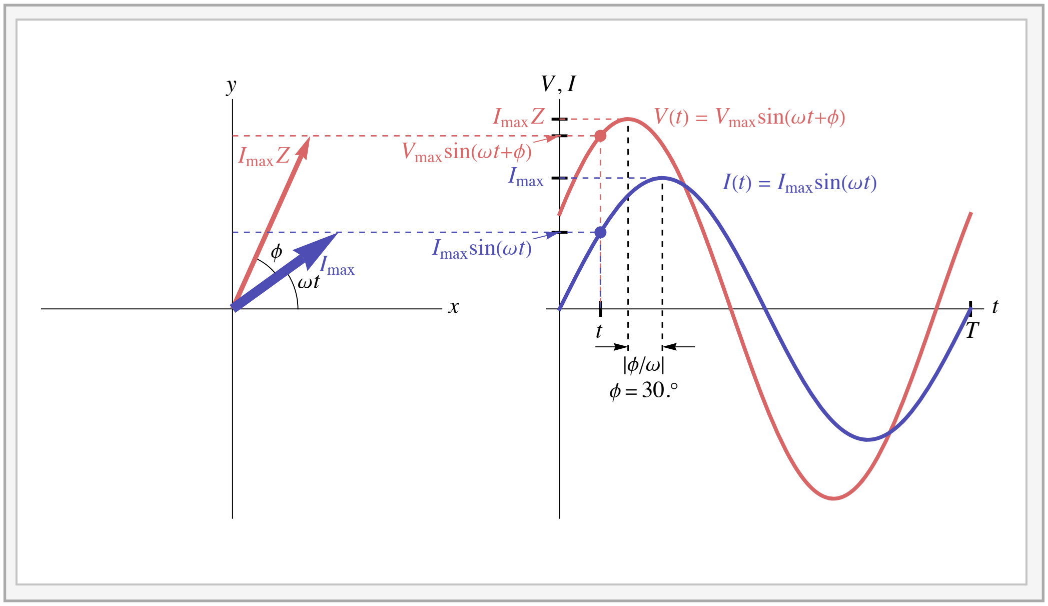
Increasing the phase angle shifts the voltage vs. time graph to the left (ahead in time.)




Interactive Figure - Voltage and Current vs. time

### Phase and Phasors

Following the discussion relating uniform circular motion to sinusoidal functions at the start of the chapter, we can introduce a vector-like notation called phasors to keep track of phase. This is illustrated in the diagram below. A phasor is a rotating vector in the $xy$-plane, as shown in the graph on the left. We project those phasors to the $y$-axis and then to the graphs on the right showing current and voltages as functions of time.




The thick blue (and darker) phasor represents the current. The thinner red (and lighter) phasor represents the voltage; the voltage is ahead of the current by the phase angle $\phi$.

## Power and Average Power

Power is voltage times current, so it follows that as functions of time we have.

$$\mathcal{P}(t) = V(t) I(t)$$

The average power $\mathcal{P}_{ave}$ is then

$$\mathcal{P}_{ave} = \langle \mathcal{P}(t) \rangle_{ave} = \langle V(t) I(t) \rangle_{ave} = V_{max}\, I_{max}\, \langle \sin(\omega\, t) \sin(\omega\, t + \phi) \rangle_{ave} = V_{max}\, I_{max}\, \frac{1}{2} \cos\phi$$

Where we used (6.9) to find the average. Writing the peak values in terms of the rms values we get the result.

$$\mathcal{P}_{ave} = V_{rms}\, I_{rms} \cos\phi \qquad \text{(F.12)}$$

# F.4 - Voltages Across Resistors, Capacitors and Inductors

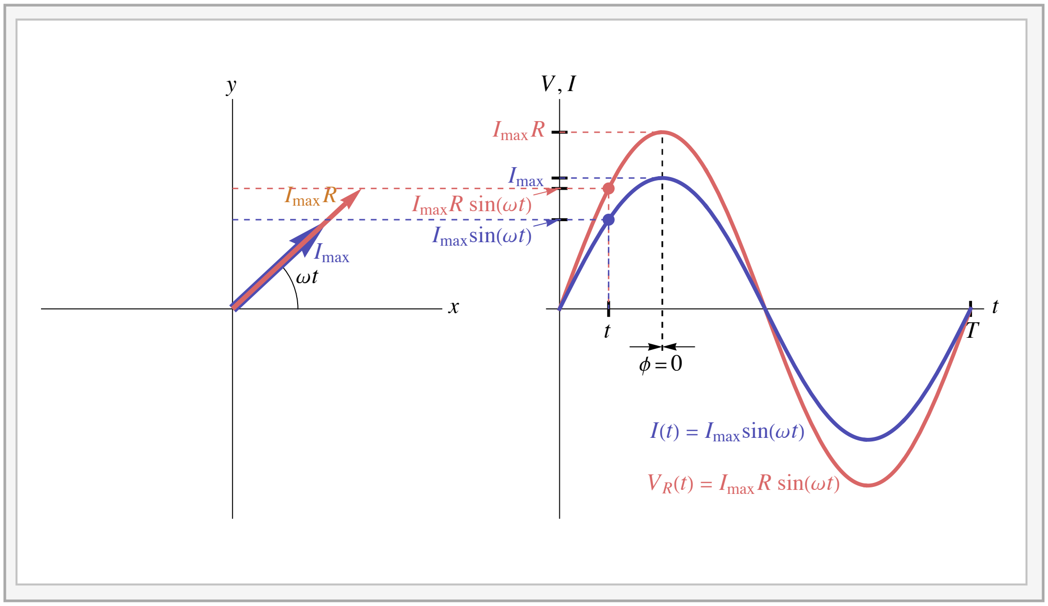
## Voltage Across a Resistor

Ohm's law gives the voltage to current relationship for a resistor, $V(t) = I(t)\, R$. If we have a current of $I(t) = I_{max} \sin(\omega\, t)$ then the voltage drop across the resistor is

$$V_R(t) = I_{max}\, R \sin(\omega\, t)$$

This means that the voltage is in phase with the current, $\phi = 0$. The peak values are related by the resistance $V_{max} = I_{max}\, R$ (and similarly $V_{rms} = I_{rms}\, R$), so $Z = R$.

$$Z = R \text{ and } \phi = 0$$




If we have a purely resistive AC circuit, one with no capacitance or inductance, it behaves like a DC circuit if we use the average power and rms voltages and currents.

$$\text{For DC}: \; V = IR \;\text{ and }\; \mathcal{P} = VI \;\;\text{ and for purely resistive AC}: \; V_{\text{rms}} = I_{\text{rms}}\,R \;\text{ and }\; \mathcal{P}_{\text{ave}} = V_{\text{rms}}\,I_{\text{rms}}$$

#### Example F.3 - Will a Hair Dryer Trip a Circuit Breaker?

An 1800 W hair dryer is connected across a standard US outlet. If your house has 20-A circuit breakers, will the hair dryer trip the breaker? Here, as is typical, voltages and currents are rms values and the power is the average power.

**Solution**

$$\mathcal{P}_{\text{ave}} = V_{\text{rms}}\,I_{\text{rms}}\cos\phi = V_{\text{rms}}\,I_{\text{rms}} \implies I_{\text{rms}} = \frac{\mathcal{P}_{\text{ave}}}{V_{\text{rms}}} = \frac{1800\text{ W}}{120\text{ V}} = 15\text{ A}$$

So the hair dryer will not trip the circuit breaker, but note that two of these hair dryers on the same circuit of a house will surely trip that breaker.

## Voltage Across an Inductor

For an inductor, the voltage to current relationship is

$$V(t) = L\,\frac{d}{dt}\,I$$

where we used the rate of change notation from (6.10). If we have a current of $I(t) = I_{\text{max}}\sin(\omega\,t)$ then using (6.11) we get the voltage drop across the inductor.

$$V_L(t) = L\,\omega\,I_{\text{max}}\,\sin\!\left(\omega\,t + \frac{\pi}{2}\right)$$

Defining the inductive reactance as

$$X_L = \omega L = 2\,\pi\,f\,L$$

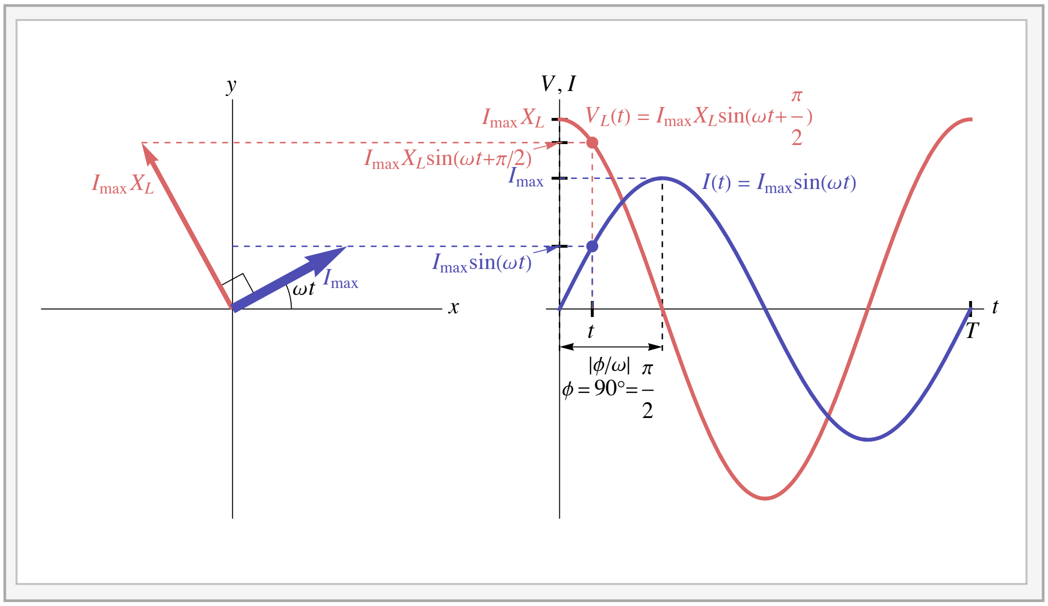
we get

$$V_L(t) = I_{\text{max}}\,X_L\,\sin\!\left(\omega\,t + \frac{\pi}{2}\right).$$

Using the definitions of impedance and phase angle $V(t) = I_{\text{max}}\,Z\sin(\omega\,t + \phi)$ we can write the impedance and phase angle for an inductor.

$$Z = X_L \;\text{ and }\; \phi = \frac{\pi}{2} = 90\,°$$

The phasor diagram below shows the voltage phasor, in red, ahead of the current phasor by 90°.




## Voltage Across a Capacitor

The voltage to current relationship for a capacitor is given by

$$V(t) = \frac{1}{C} Q(t) \text{ where } I(t) = \frac{d}{dt} Q(t)$$

We need to find a sinusoidal function for the charge on the capacitor

$$\frac{d}{dt} Q(t) = I(t) = I_{\max} \sin(\omega t)$$

We can guess the function

$$Q(t) = \frac{1}{\omega} I_{\max} \sin\left(\omega t - \frac{\pi}{2}\right)$$

as a solution, then using (6.11) we see that this guess works; that is, to find the rate of change for our guess we multiply by $\omega$ and then add $\pi/2$ to the phase and that takes us back to the original function for $I(t)$.

Since $V = Q/C$ we get

$$V_C(t) = \frac{1}{\omega C} I_{\max} \sin\left(\omega t - \frac{\pi}{2}\right).$$

If we define the capacitive reactance as

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

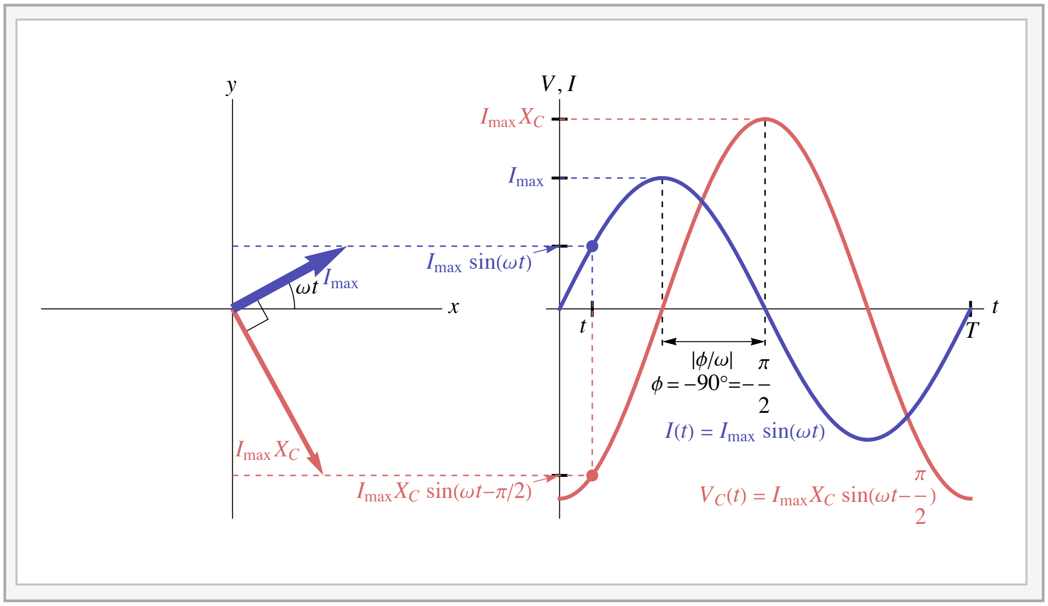
and this gives

$$V_C(t) = I_{\max} X_C \sin\left(\omega t - \frac{\pi}{2}\right).$$

The impedance and phase angle can be read off, as we did for the inductor.

$$Z = X_C \text{ and } \phi = -\frac{\pi}{2} = -90°$$

Now, the phasor diagram shows the voltage phasor, in red, behind of the current phasor by 90°.




## Summary

If we have a current of $I(t) = I_{\max} \sin(\omega\, t)$ passing through just a resistor, just an inductor or just a capacitor, then we get the voltage drops of $V_R$, $V_L$ and $V_C$, respectively. From those expressions we can read off the impedance and phase angle for each circuit element.

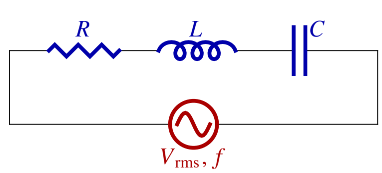
| | $I(t) = I_{\max} \sin \omega\, t$ and $V(t) = I_{\max}\, Z \sin(\omega\, t + \phi)$ | Impedance $Z$ | Phase $\phi$ |
|---|---|---|---|
| $R$ | $V_R(t) = I_{\max}\, R \sin(\omega\, t)$ | $R$ | $0$ |
| $L$ | $V_L(t) = I_{\max}\, X_L \sin\left(\omega\, t + \frac{\pi}{2}\right)$ | $X_L = \omega\, L = 2\,\pi\, f\, L$ | $\frac{\pi}{2} = 90\,°$ |
| $C$ | $V_C(t) = I_{\max}\, X_C \sin\left(\omega\, t - \frac{\pi}{2}\right)$ | $X_C = \frac{1}{\omega\, C} = \frac{1}{2\,\pi\, f\, C}$ | $-\frac{\pi}{2} = -90\,°$ |

# F.5 - The Series RLC Circuit

## The Solution

We saw in Chapter 21 that any combination of resistors that is connected across a DC voltage source behaves as if there is an equivalent resistance across the source. Solving for the circuit consists of finding that equivalent resistance. Analogously for AC circuits, if we have some complicated combination of circuit elements (resistors, inductors and capacitors) connected across an AC voltage source, then there is an equivalent impedance and phase angle that describes the combination of circuit elements. There is, however, one important distinction between the DC and AC cases of this analogy. For any combination of resistors, its equivalent resistance is independent of the DC source. For the AC case, both the impedance and phase angle of the combination depends on the frequency of the AC source.

Now we consider a series combination of a resistor, inductor and capacitor connected across an AC voltage source.




In a series the currents through all circuit elements are the same

$$I(t) = I_R(t) = I_L(t) = I_C(t)$$

and the voltages add.

$$V(t) = V_R(t) + V_L(t) + V_C(t)$$

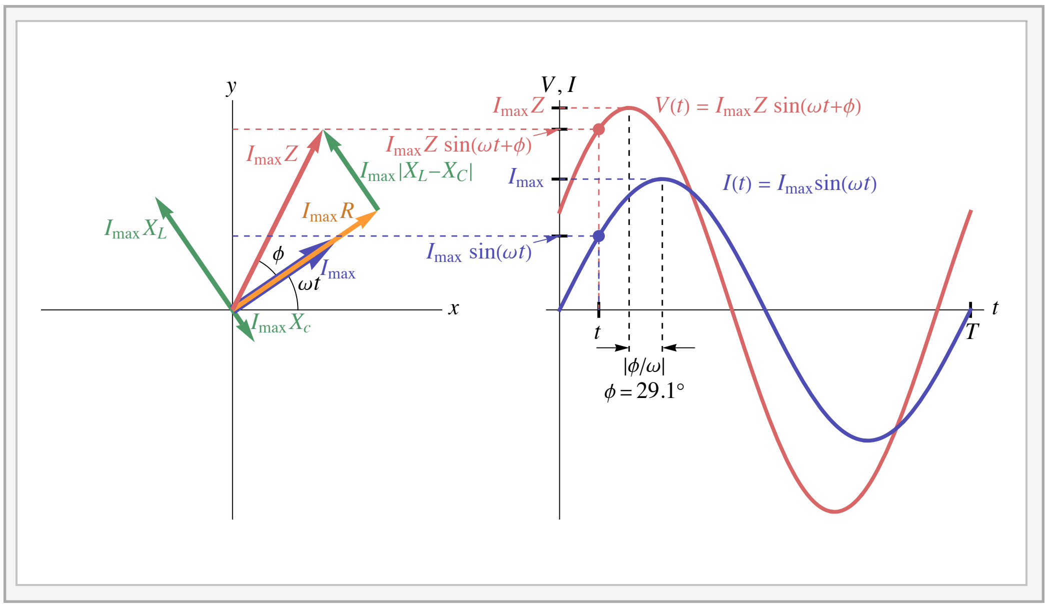
Beginning with the current of

$$I(t) = I_{\max}\sin(\omega t)$$

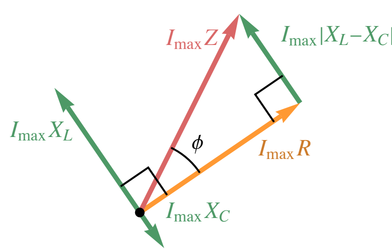
And the voltages across all three circuit elements are found in the table

$$V_R(t) = I_{\max} R \sin(\omega t)$$
$$V_L(t) = I_{\max} X_L \sin\left(\omega t + \frac{\pi}{2}\right)$$
$$V_C(t) = I_{\max} X_C \sin\left(\omega t - \frac{\pi}{2}\right)$$

Because we have a series circuit and the voltages add. If we add the phasors as vectors, then the $y$-component of the sum of these voltage phasors is the $y$-component of the the total voltage phasor, which is the total voltage. This is the payoff of the phasor formalism: for series circuits we can add voltage phasors as vectors to find the total voltage.

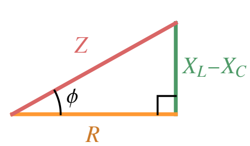



The phasor vectors are labeled by their magnitudes. The diagram below shows the voltage phasors.




The phasor for $V_R$ is orange and is labeled with $I_{\max} R$; it is in the direction of the current phasor. The phasors for the reactances (in green) are perpendicular to the $V_R$ phasor. The phasor for $V_L$ is labeled with $I_{\max} X_L$ and $I_{\max} X_C$ labels the phasor for $V_C$. We sum these three phasor vectors to get the phasor for the total voltage, labeled $I_{\max} Z$. Summing the two reactance phasors gives the phasor for the total reactance which is labeled $I_{\max} |X_L - X_C|$; this is put at the tip of the $V_R$ phasor and the right triangle shows their addition to give the total voltage phasor, labeled with $I_{\max} Z$.

Concentrating on the right triangle of vectors, we can now find the impedance and phase angle. Dividing each side by $I_{\max}$ and rotating gives a similar right triangle.



Using the Pythagorean theorem and the definition of tangent as opposite over adjacent, we get our solution for the impedance and phase angle.

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad \text{and} \quad \tan\phi = \frac{X_L - X_C}{R}.$$

$\phi$ can be negative, as can $X_L - X_C$; with above expression for $\tan\phi$, both have the same sign.

## Power in the Series Circuit

From the right triangle shown above we can also see that $\cos\theta = R/Z$. This allows us to write an expression for power specific to series circuits. Our general expression for power is $\mathcal{P}_{\text{ave}} = I_{\text{rms}}\, V_{\text{rms}} \cos\phi$, (6.12). Using $V_{\text{rms}} = I_{\text{rms}}\, Z$ and the expression for $\cos\theta$ gives.

$$\mathcal{P}_{\text{ave}} = I_{\text{rms}}^2\, R \tag{F.13}$$

This expression only applies to series circuits, but for our purposes it will be more generally applicable, since students will only be responsible for series circuits.

All of the power dissipated in an AC circuit is dissipated in the resistors. Because the phase angle for inductors and capacitors are $\phi = \pm\pi/2$ and $\mathcal{P}_{\text{ave}} = V_{\text{rms}}\, I_{\text{rms}} \cos\phi$, it follows that the power dissipated in each is zero.

$$\mathcal{P}_{\text{ave},R} = \mathcal{P}_{\text{ave}} = I_{\text{rms}}^2\, R \quad \text{and} \quad \mathcal{P}_{\text{ave},L} = 0 = \mathcal{P}_{\text{ave},C}$$

## Other Series Circuits

Any other series AC circuit can be viewed as a special case of this. If there is no inductor, and RC circuit, then we can just set $X_L = 0$. For and RL circuit, without a capacitor, then set $X_C = 0$. If there is no resistor then $R = 0$. The order of the circuit elements doesn't matter. If there are more than one resistor, then combine them in series. The same applies to more than one inductor or capacitor.

### Example F.4 - A Series RLC Circuit

An 80-$\mu$F capacitor, a 50-mH inductor and a 20-$\Omega$ resistor are connected in series across a standard outlet.

**(a)** Complete the table with the rms voltage across, the rms current through and the average power dissipated in each of the three circuit elements.

| | $V_{\text{rms}}$ | $I_{\text{rms}}$ | $\mathcal{P}_{\text{ave}}$ |
|---|---|---|---|
| $R$ | | | |
| $L$ | | | |
| $C$ | | | |

#### Solution

For a standard outlet we have: $V_{\text{rms}} = 120$ V and $f = 60$ Hz. Our circuit elements have the values

$$C = 80 \times 10^{-6}\ \text{F}, \ L = 50 \times 10^{-3}\ \text{H} \ \text{and} \ R = 20\ \Omega$$

First we must calculate the reactances and impedance.

$$X_L = \omega L = 2\pi f L = 18.850\ \Omega$$

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} = 33.157\ \Omega$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = 20.730\ \Omega$$

To begin filling in the table we can find the current, and because it is a series circuit, this will be the same for all three.

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = 5.7889\ \text{A}$$

We then need to find the voltages across each of the circuit elements, using $V_{\text{rms}} = I_{\text{rms}}\, Z$ where Z is the impedances of each, by itself.

$$V_{R,\text{rms}} = I_{\text{rms}}\, R = 86.8\ \text{V}$$

$$V_{L,\text{rms}} = I_{\text{rms}}\, X_L = 109.1\ \text{V}$$

$$V_{C,\text{rms}} = I_{\text{rms}}\, X_C = 191.9\ \text{V}$$

To find the average power lost in each, it is zero for both the inductor and capacitor: $\mathcal{P}_{\mathrm{ave},L} = 0 = \mathcal{P}_{\mathrm{ave},C}$ . For the resistor, all the power dissipated is lost in the resistor.

$$\mathcal{P}_{\mathrm{ave},R} = \mathcal{P}_{\mathrm{ave}} = I_{\mathrm{rms}}^2 R = 503\ \mathrm{W}$$

| | $V_{\mathrm{rms}}$ | $I_{\mathrm{rms}}$ | $\mathcal{P}_{\mathrm{ave}}$ |
|---|---|---|---|
| $R$ | 86.8 V | 5.79 A | 503 W |
| $L$ | 109.1 V | 5.79 A | 0 |
| $C$ | 191.9 V | 5.79 A | 0 |

**(b)** By what angle is the voltage ahead of the current?

**Solution**

This is just asking for the phase angle $\phi$.

$$\phi = \tan^{-1} \frac{X_L - X_C}{R} = -43.6\,° = -0.762\ \mathrm{rad}$$

## F.6 - Resonance

The reactances vary with frequency. The capacitive reactance decreases with frequency and the inductive reactance increases.

| | Low $f$ | High $f$ |
|---|---|---|
| $X_L = 2\,\pi\, f\, L$ | small | large |
| $X_C = \frac{1}{2\,\pi\, f\, C}$ | large | small |
| $X_L - X_C$ | negative | positive |

It is clear that there is some frequency where $X_L = X_C$. This is called the resonant frequency $f_{\mathrm{res}}$. At resonance the condition $X_L = X_C$ implies that the angular frequency is

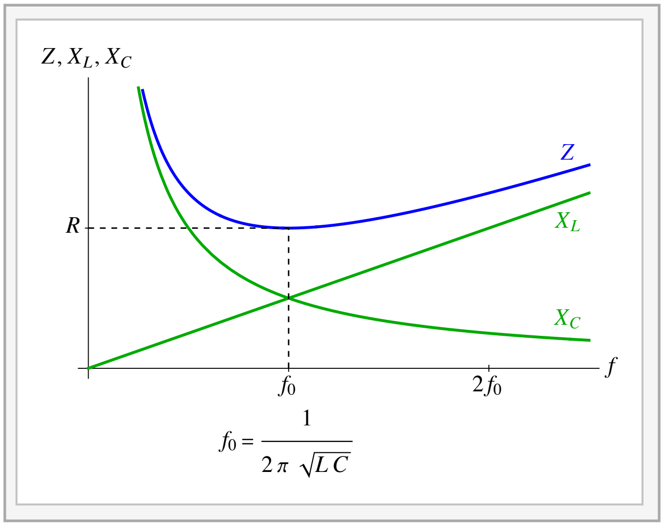
$$X_L = X_C \implies \omega_{\mathrm{res}}\, L = \frac{1}{\omega_{\mathrm{res}}\, C} \implies \omega_{\mathrm{res}} = \frac{1}{\sqrt{L\,C}}$$

and the frequency then becomes

$$f_{\mathrm{res}} = \frac{1}{2\,\pi}\,\omega_{\mathrm{res}} = \frac{1}{2\,\pi\,\sqrt{L\,C}}\,.$$

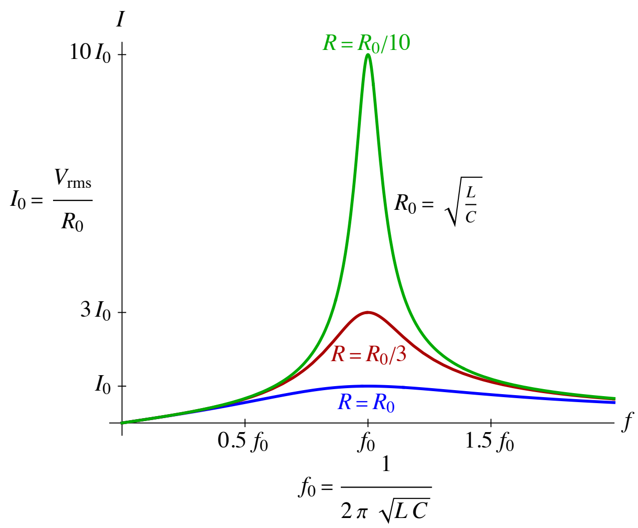
At frequencies below the resonant frequency a circuit is said to be capacitive and above resonance it is inductive.

| | Capacitive | Resonant | Inductive |
|---|---|---|---|
| frequency | $f < f_{\mathrm{res}}$ | $f = f_{\mathrm{res}}$ | $f > f_{\mathrm{res}}$ |
| $X_L$ and $X_C$ | $X_L < X_C$ | $X_L = X_C$ | $X_L > X_C$ |
| $\phi = \tan^{-1} \frac{X_L - X_C}{R}$ | negative | 0 | positive |
| $Z = \sqrt{R^2 + (X_L - X_C)^2}$ | $Z > R$ | $Z = Z_{\mathrm{min}} = R$ | $Z > R$ |




Interactive Figure - Impedance and reactances varying with frequency. $Z_{min} = R$ at $f = f_0 = f_{res}$

If we fix $V_{rms}$ and vary the frequency, the current $I_{rms}$ will peak at the resonant frequency. At resonance the impedance is the resistance, so it follows that the smaller the resistance the more dramatic the peak in the current. The graph shows the behavior of current versus frequency for different resistances.




Note that the resonant frequency is the same as the natural frequency (6.1) for an LC circuit without an AC driving voltage.

$$\omega_{res} = \omega_0 = \frac{1}{\sqrt{LC}}$$

This illustrates a general principal about resonance: when a system, either electrical or mechanical, is driven at its natural frequency, large amplitude oscillations result.

### Example F.5 - A Series RLC Circuit - Continued

**(c)** Using the same 80-$\mu$F capacitor, 50-mH inductor and 20-$\Omega$ resistor as before but now connect them in series across a voltage source with $V_{rms} = 120$ V with a variable frequency. The rms current varies with frequency. At what frequency is the rms current its maximum and what is that maximum value?

#### Solution

Since $I_{rms} = V_{rms}/Z$, the maximum $I_{rms}$ is when the impedance is its minimum and that is at resonance. The frequency is

$$\omega_{res} = \frac{1}{\sqrt{LC}} = 500\ \text{s}^{-1} \implies f_{res} = \frac{1}{2\pi}\,\omega_{res} = \frac{1}{2\pi\sqrt{LC}} = 79.6\ \text{Hz}$$

The minimum impedance is just the resistance of the circuit.

$$\max I_{\text{rms}} = \frac{V_{\text{rms}}}{Z_{\text{min}}} = \frac{V_{\text{rms}}}{R} = \frac{120\ \text{V}}{20\ \Omega} = 6\ \text{A}$$