Chapter H

Geometric Optics

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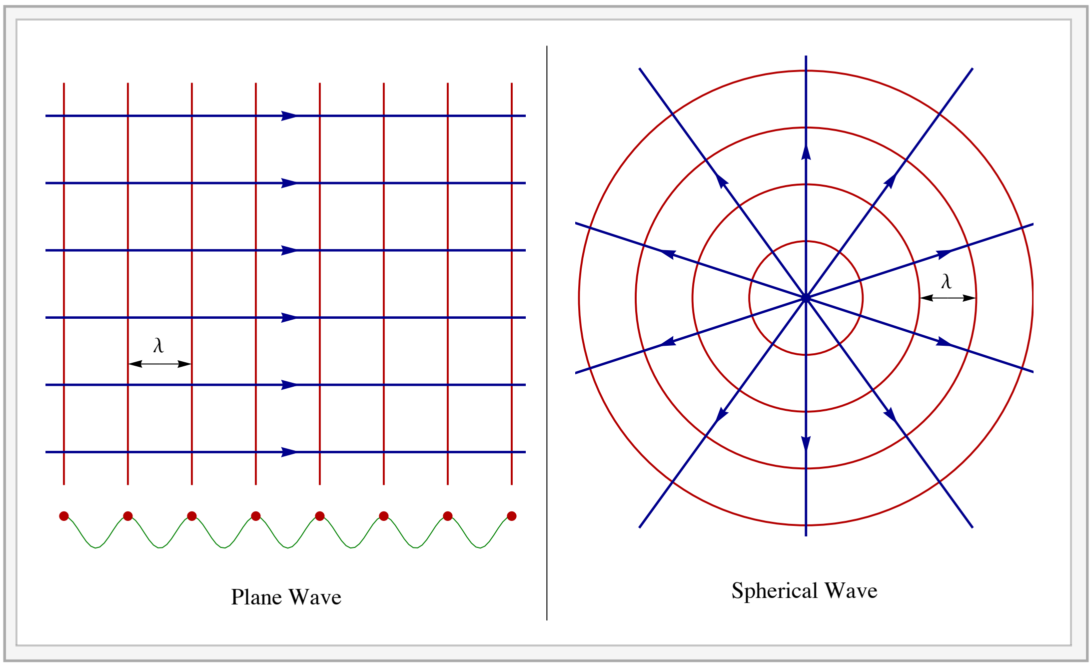
H.1 - Wave Fronts, Rays and Reflection

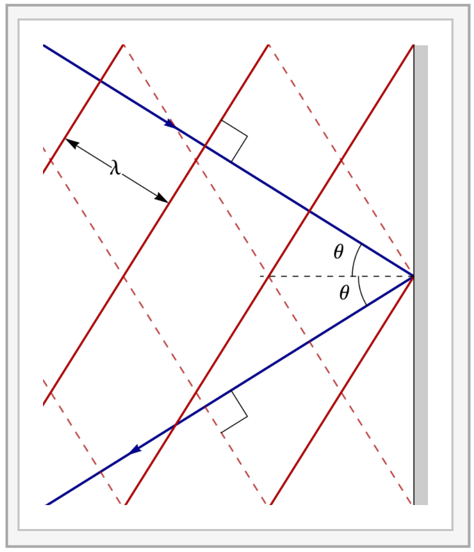
Wave Fronts and Rays

There are two ways we can visualize waves, as wave fronts or as rays. If we mark the crest of each wave then these form the wave fronts. For a plane wave the crest corresponds to parallel planes moving at the wave speed perpendicular to the plane. Light from a point source is a spherical wave. A spherical wave from a distant source approaches a plane wave. In both cases, the distance between wave fronts is one wavelength and the rays are perpendicular to the wave fronts. A light ray is what you expect it to be, it points in the direction of the light propagation.

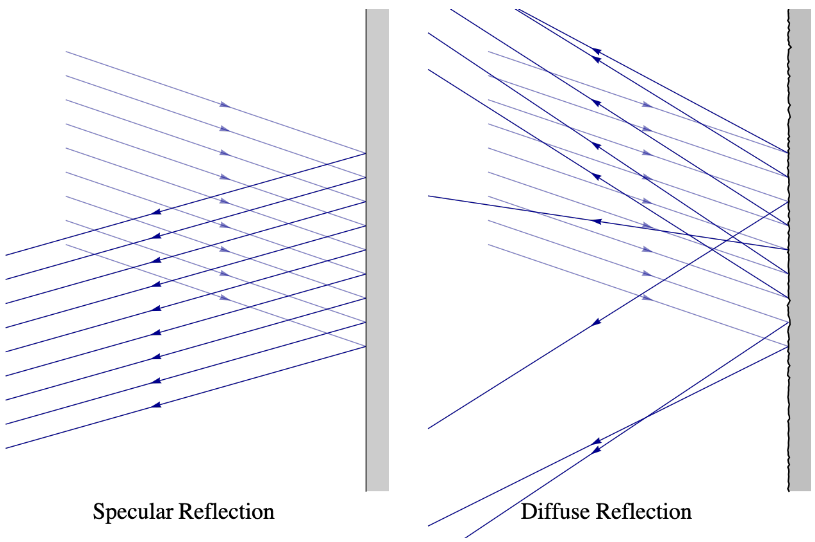


Interactive Figure - Plane Waves and Spherical Waves. Wave fronts are red and rays are blue.

Reflection of Light



When we describe the angle that light makes with a surface, we measure the angle of the ray from the normal to the surface. When light reflects off a smooth shiny surface, it reflects so that the incident angle equals the reflected angle. This is known as the law of reflection. Most of the light we see is reflected light. Reflection off shiny smooth surfaces is called specular reflection. Most surfaces are rough on the order of the wavelengths of light. Because the normal to a rough surface is varying, the light then reflects in random directions. This is called diffuse reflection. Most of the light that we see is from the diffuse reflection off surfaces.



H.2 - Refraction

Light in a Medium

When we refer to c as the speed of light, it is implied that it is the speed in a vacuum. In a medium light slows down. It slows down by a factor called the index of refraction; this is a material dependent constant n defined by

$$v = \frac{c}{n},$$

where v is the speed in the medium. Clearly $n \geq 1$ and the equality applies to a vacuum.

When light hits an interface between indices n_1 and n_2 both the frequency and wavelength cannot remain unchanged, since

$$f\,\lambda = v = \frac{c}{n}.$$

The frequency is the same on both sides of the interface. This is easy to see. At the interface the incoming radiation is varying at some frequency and generally driving a system at some frequency induces oscillations at the same frequency. Given that the frequencies are equal

$$f = f_1 = f_2$$

the wavelengths must change. Since wavelength is proportional to the wave speed we get

$$\frac{\lambda_2}{\lambda_1} = \frac{v_2}{v_1} = \frac{n_1}{n_2} \implies n_1\,\lambda_1 = n_2\,\lambda_2.$$

To view this a different way, if we define the vacuum wavelength as $\lambda_0 = c/f$ then the wavelength in a medium with index n is

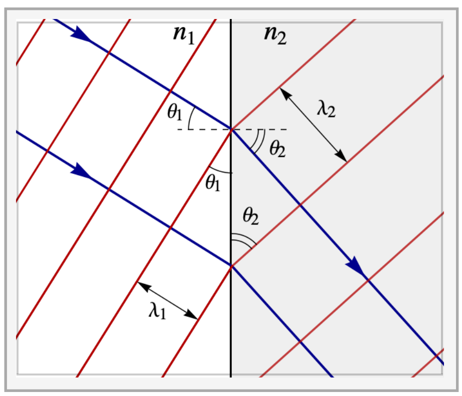
$$\lambda = \frac{\lambda_0}{n}.$$

The index of refraction is a property of a material. Because it is a ratio of a speed to a speed, it is dimensionless; it has units. For a vacuum it is exactly 1. For air it is near 1 and we will usually take it as one, unless directed otherwise.

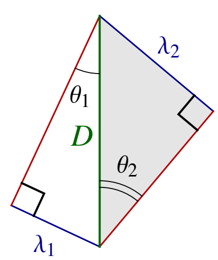
Material	Index of Refraction – n
Vacuum	1 (exact)
Air (0 °C, 1 atm)	1.000277
Water	1.333
Plate Glass	1.52
Diamond	2.417
Polycarbonate	1.60

Snell's Law

As a consequence of the speed of light changing across an interface light will bend as it hits the interface. This is called refraction. Consider light passing from one medium to another through a flat interface. Take it to move from medium 1 with index n_1 to medium 2 with index n_2. We will, as usual, measure the angles of the rays relative to the normal to the surface.



The diagram shows wave fronts and rays on either side of the interface. Since the rays are perpendicular to the wave fronts, the angle between the rays and the normal are the same as the angle between the wave fronts and the interface. Enlarging the triangles at the center



and labeling the common side as D gives

$$\sin\theta_1 = \frac{\lambda_1}{D} \text{ and } \sin\theta_2 = \frac{\lambda_2}{D} \implies \frac{\sin\theta_2}{\sin\theta_1} = \frac{\lambda_2}{\lambda_1} = \frac{n_1}{n_2}.$$

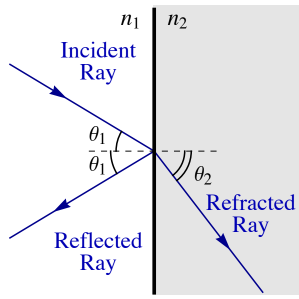
We can now write the law of refraction, called Snell's law, as

$$n_1 \sin\theta_1 = n_2 \sin\theta_2.$$

The usual convention with Snell's law is that the light moves from medium 1 to medium 2.

Incident, Reflected and Refracted Rays

When light in a medium with index n_1 hits a medium with index n_1 some of the light passes through and the angle θ_2 is found by Snell's law. In addition to refraction some of the light is reflected back by the law of reflection. The incoming ray is called the incident ray.



Note that in this example the angle in medium 2 is larger than in medium 1. This implies, by Snell's law, that the index in 2 is smaller than in 1.

$$\theta_1 < \theta_2 \implies \sin\theta_1 < \sin\theta_2 \implies n_1 > n_2$$

Example H.1 - Light from Air to Glass

Laser light of wavelength 632.9 nm shines from air to glass ($n = 1.52$). In the air the beam (ray) makes an angle of 35° from the normal to the surface.

(a) Inside the glass, what is the angle between the light ray and the normal?

Solution

$$n_1 = 1\ ,\quad n_2 = 1.52\ ,\quad \theta_1 = 35\,^\circ \text{ and } \theta_2 = ?$$

This is a straightforward application of Snell's law. Take the index for air to be one.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \implies \theta_2 = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_1\right) = 22.2\,^\circ$$

(b) What is the speed of light inside the glass?

Solution

First we list the relevant information. Call the vacuum wavelength λ_0.

$$\lambda_0 = 632.9\text{ nm} = 632.9\times10^{-9}\text{ m}\ ,\quad n = 1.52 \text{ and } c = 3.00\times10^8\text{ m/s}$$

The speed of light in a medium is

$$v = c/n = 1.97\times10^8\ \frac{\text{m}}{\text{s}}$$

(c) What is the wavelength inside the glass?

Solution

Relate the wavelength to the vacuum wavelength.

$$\lambda = \frac{\lambda_0}{n} = 416\text{ nm} = 4.16\times10^{-7}\text{ m}$$

(d) What is the frequency of the light inside the glass?

Solution

The frequency inside is the same as the frequency outside.

$$f = f_0 = \frac{c}{\lambda_0} = 4.74\times10^{14}\text{ Hz}$$

You would get the same answer if you used $f = v/\lambda$.

Total Internal Reflection

Consider the case where light moves across an interface to a region with lower index.

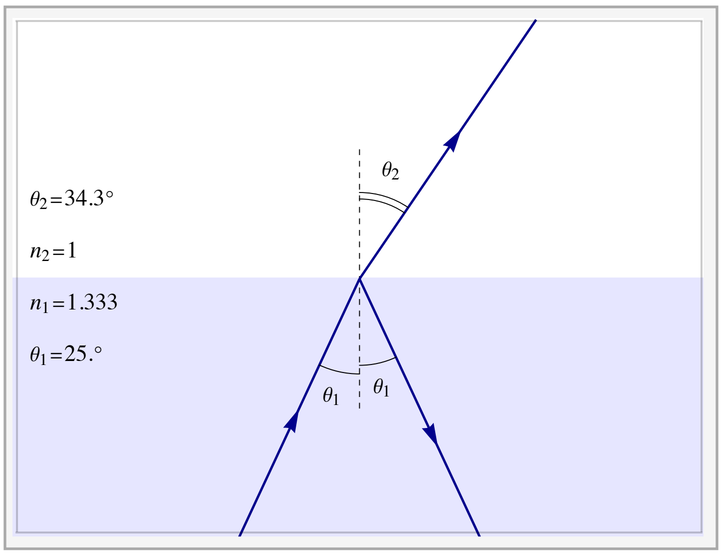
$$n_1 > n_2$$

It follows that the angle from the normal increases ($\theta_1 < \theta_2$). Moreover, there is some angle where there is no refracted ray. In this case there is only the reflected ray. This is what we call total internal reflection. This occurs at incident angles above some critical angle

$$\theta_1 \geq \theta_{\text{crit}},$$

where the critical angle occurs when the refracted angle goes to 90°. That is, $\theta_1 = \theta_{\text{crit}}$ when $\sin\theta_2 = 1$, or

$$\sin\theta_{\text{crit}} = \frac{n_2}{n_1}.$$



A water to air interface. Total internal reflection occurs when $\theta_1 \geq \theta_{\text{crit}} = 48.6°$

Example H.2 - A Water to Air Interface

(a) For what angles of light coming from below the surface of water will there not be a refracted ray?

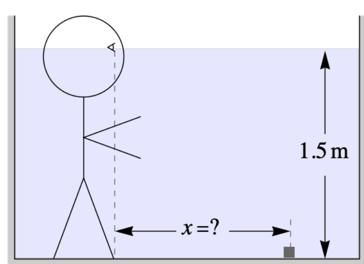
Solution

We first need to find the critical angle for total internal reflection.

$$n_1 = 1.333\,, \quad n_2 = 1, \quad \sin\theta_{\text{crit}} = \frac{n_2}{n_1} \implies \theta_{\text{crit}} = \sin^{-1}\frac{1}{1.333} = 48.6°$$

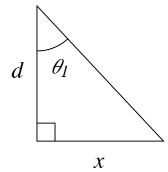
So for all incident angles larger than this $\theta_1 \geq \theta_{\text{crit}}$ there is no refracted ray.

(b) A kid stands in a 1.5-m deep pool with his eye just above the surface of calm water. What is the largest horizontal distance x that a small object can be from his feet for him to be able to see it?



Solution

The light passes from the object to the surface and then refracts out to the eye. The incident angle on the water to air interface is given by



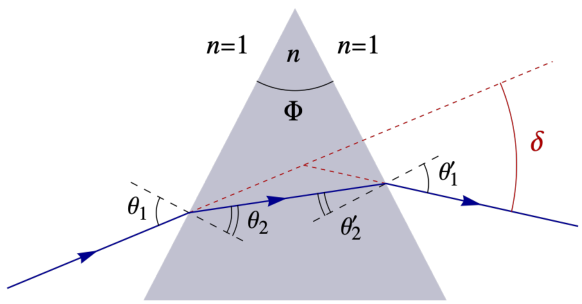
$$\implies \tan\theta_1 = \frac{x}{d} \quad \text{where } d = 1.5\text{ m}$$

The largest x is when $\theta_1 = \theta_{\text{crit}}$.

$$x = d\tan\theta_1 \implies x_{\max} = d\tan\theta_{\text{crit}} = d\tan 48.6\,° = 1.70\text{ m}$$

Prisms and Dispersion

Deflection by a prism



Consider light hitting a prism as shown above. The apex angle Φ is the angle between the two refracting surfaces in the prism. If the ray incident on the prism is at an angle $\theta_1$from the normal it will refract to an angle θ_2, which can be found by Snell's law.

$$\sin\theta_1 = n\sin\theta_2$$

Now consider the triangle formed by the ray inside the prism and the top of the prism. Summing the internal angles of this triangle must give 180 °.

$$(90\,° - \theta_2) + (90\,° - \theta_2') + \Phi = 180\,° \implies \theta_2 + \theta_2' = \Phi$$

The above expression allows us to find θ_2' from θ_2 and Φ. This lets us find θ_1'.

$$\sin\theta_1' = n\sin\theta_2'$$

The total angle of deflection δ is found by summing the bending at both interfaces. At the first the ray bends by $\theta_1 - \theta_2$ and at the second by $\theta_1' - \theta_2'$. It follows that δ is the sum.

$$\delta = (\theta_1 - \theta_2) + (\theta_1' - \theta_2') = \theta_1 + \theta_1' - \Phi$$

Dispersion

A prism splits light into the visible spectrum: red, orange, yellow, green, blue, indigo and violet. We have just seen that using Snell's law and knowing the prism's apex angle Φ and the incident angle θ_1, we can uniquely solve for the position of the ray leaving the prism in terms of the index of refraction of the prism. The fact that different colors refract to different angles implies that the index of refraction varies with color (wavelength). This is known as dispersion.

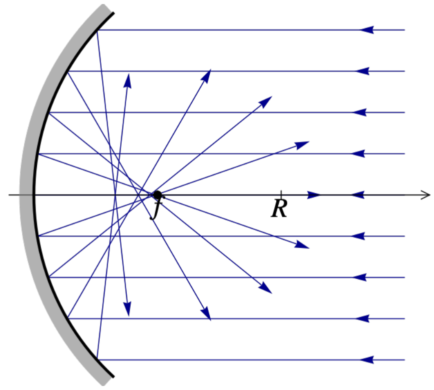
In a prism, red is the color that refracts the least; this means the index of red is the least. Red has the longest visible wavelength. It is generally the case that the index of refraction decreases with wavelength. For instance in the case of crown glass the tabulated index is 1.52; this is a mean value. As the wavelength increases from 400 to 700 nm, the index decreases from about 1.53 to 1.51.

H.3 - Images from Spherical and Parabolic Mirrors

Concave Mirrors

Spherical Mirrors and Spherical Aberration

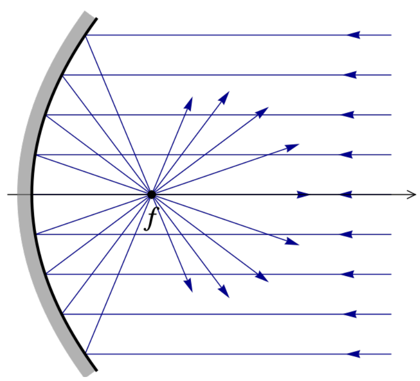
When light from infinity (a plane wave) parallel to the central axis of a concave spherical mirror reflects off the mirror, the rays near the central axis converge toward a point called the focal point, which we will see is at a position half the radius from the mirror.



The rays far from the central axis miss the focal point. This is called spherical aberration.

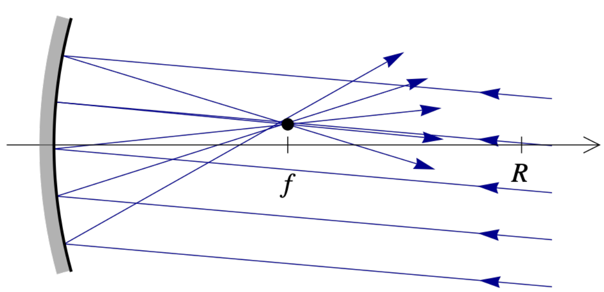
Parabolic Mirrors

The geometrical shape that causes all rays to converge to a single focal point is a paraboloid, which is a parabola under rotation.

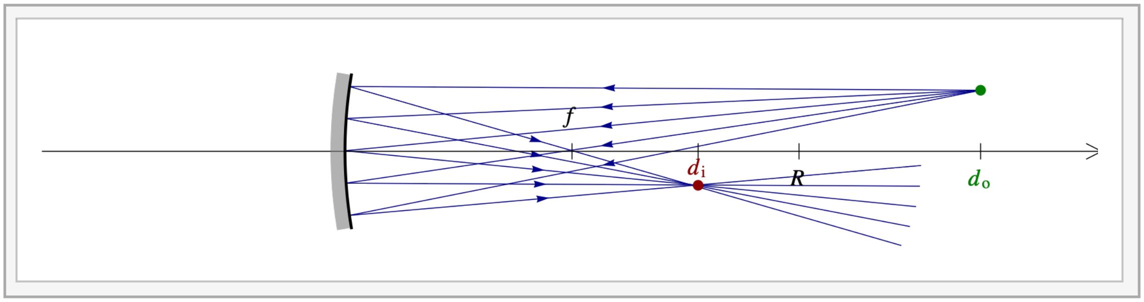


As long as the distance from the central axis is small (compared to the sphere's radius) the sphere approximates a paraboloid well. This is the basic assumption of our analysis of spherical mirrors.

Plane Waves from off the Central Axis

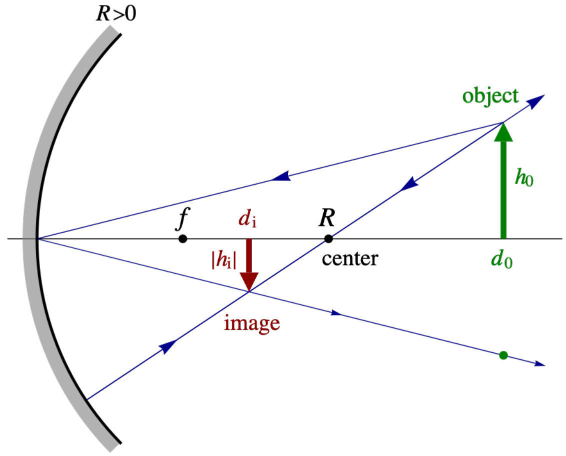


Point Source off the Central Axis

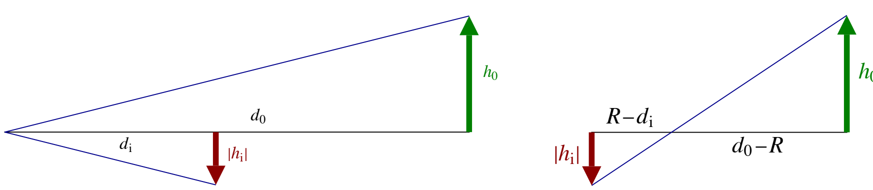


Images from Concave Mirrors

We need to mathematically relate the image position d_i and image height h_i to the object position d_o and object height h_o for the case of a concave spherical mirror. In the preceding diagram it is clear that *every* ray from the tip of the object converges to the tip of the image. It suffices to draw just two rays to find the image position. One ray we will draw is from the tip of the object to the point where the central axis hits the mirror; this will reflect back at an equal angle below the central axis. The other ray we will consider passes through the center of the sphere; this will hit the surface normally and reflect straight backward.



These two rays give two pairs of similar triangles.



These give the expressions

$$\frac{|h_i|}{h_o} = \frac{d_i}{d_o} \text{ and } \frac{|h_i|}{h_o} = \frac{R-d_i}{d_o-R}$$

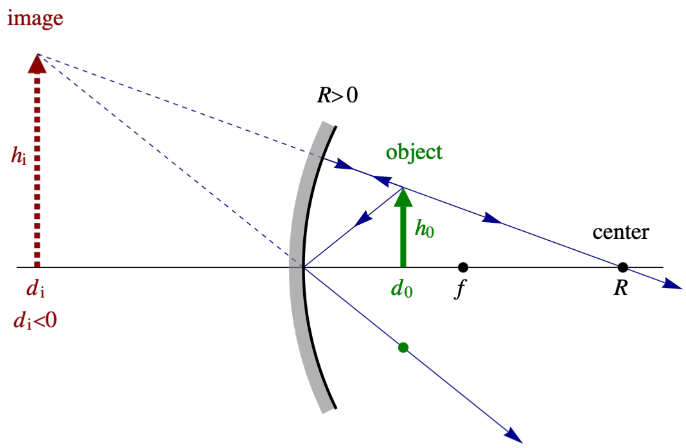
$$\frac{d_i}{d_o} = \frac{R-d_i}{d_o-R} \implies \frac{1}{d_o} + \frac{1}{d_i} = \frac{2}{R}$$

If the object is at infinity ($d_o \to \infty$) the image is at the focal point ($d_i = f$). This gives $f = R/2$. Since the image is inverted we choose, by convention, that $h_i < 0$ and thus $|h_i| = -h_i$. We can rewrite the above expressions as

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \text{ and } m = \frac{h_i}{h_o} = -\frac{d_i}{d_o}.$$

These expressions will apply generally to spherical mirrors, either concave or convex, and to thin lenses, either converging or diverging. m is defined as the magnification; it has the same sign convention as the image height h_i.

The preceding image is called a *real image*. A real image occurs when the light rays converge to a point. A real image is on the side where the light is. If you hold a screen to a real image you have a projected image on the screen. If the object is inside the focal point, it turns out that the reflected rays do not converge. We trace the same rays as before but to find the image we trace the diverging rays backward to see where the reflected light rays appear to originate; this is the image position. Mathematically, when the object is inside the focal point $0 < d_o < f$ the image position is negative $d_i < 0$. There is no light at a virtual image. We will see that a plane mirror, like a bathroom mirror, gives only virtual imagesl that are behind the mirror; ther eis no light there.



Example H.3 - Concave Spherical Mirror

A concave spherical mirror with a 60-cm radius is used with a 10-cm high object.

(a) If the object is 40 cm from the mirror, then where is the image and what is the image height and magnification? Also, sketch the rays.

Solution

We are given the radius, which then gives the focal length, the object height and object distance.

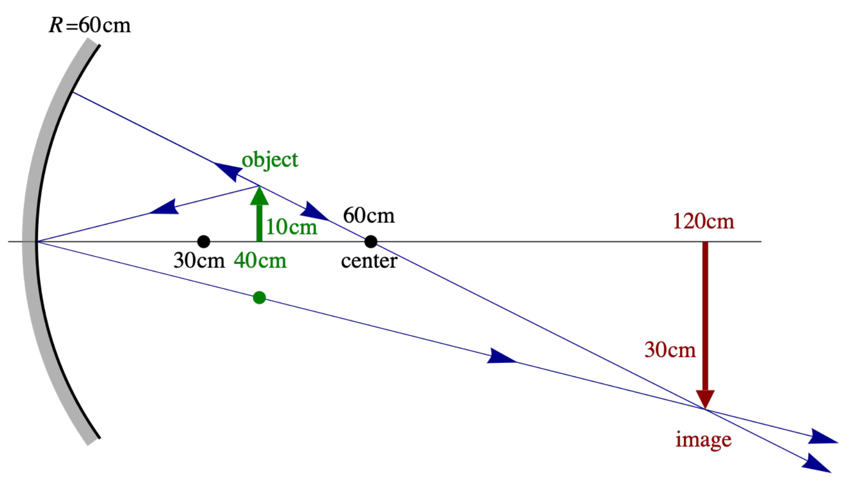
$$R = 60\text{ cm} \implies f = \frac{R}{2} = 30\text{ cm}\,,\quad h_o = 10\text{ cm and } d_o = 40\text{ cm}$$

With this we can solve for the image position.

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \implies d_i = \left(\frac{1}{f} - \frac{1}{d_o}\right)^{-1} = 120\text{ cm}$$

Here we used $1/x = x^{-1}$. We can then solve for the image height and magnification.

$$m = \frac{h_i}{h_o} = -\frac{d_i}{d_o} \implies h_i = -h_o\frac{d_i}{d_o} = -30\text{ cm and } m = -\frac{d_i}{d_o} = -3$$



Some comments:

- With as single mirror (or lens as we will see later) a real image is always inverted and inverted images are always real. Inverted images have negative h_i and m.
- We can use any length unit (cm, m, ft, in, ...) as long as we are consistent.

(b) If the object is 20 cm from the mirror, then where is the image and what is the image height and magnification? Also, sketch the rays.

Solution

We are given the radius, which then gives the focal length, the object height and object distance.

$$R = 60\text{ cm} \implies f = \frac{R}{2} = 30\text{ cm}, \quad h_\text{o} = 10\text{ cm and } d_\text{o} = 20\text{ cm}$$

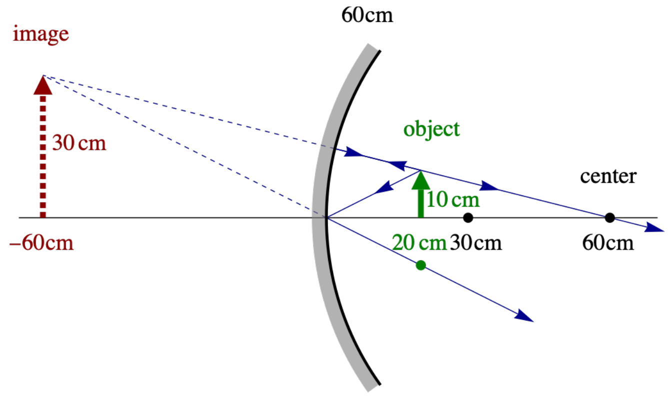
With this we can solve for the image position.

$$\frac{1}{d_\text{o}} + \frac{1}{d_\text{i}} = \frac{1}{f} \implies d_\text{i} = \left(\frac{1}{f} - \frac{1}{d_\text{o}}\right)^{-1} = -60\text{ cm}$$

Here we used $1/x = x^{-1}$. We can then solve for the image height and magnification.

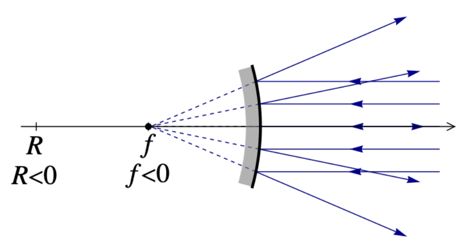
$$m = \frac{h_\text{i}}{h_\text{o}} = -\frac{d_\text{i}}{d_\text{o}} \implies h_\text{i} = -h_\text{o}\frac{d_\text{i}}{d_\text{o}} = +30\text{ cm and } m = -\frac{d_\text{i}}{d_\text{o}} = +3$$

This is an upright virtual image.

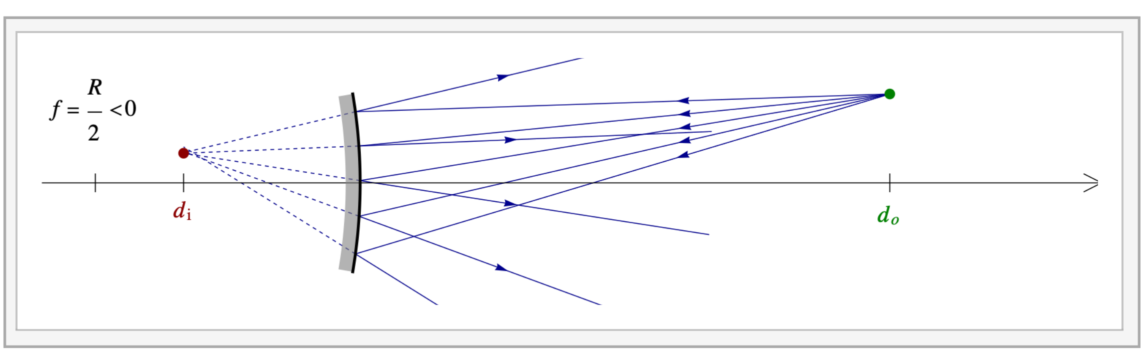


Images from Convex Mirrors

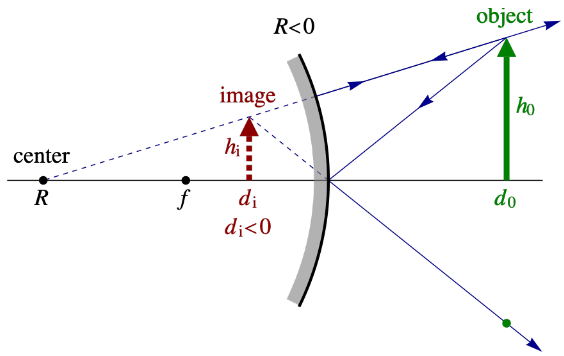
The center of a convex mirror is behind the mirror. We take the radius to be negative. An incident plane wave parallel to the central axis will reflect away from a focal point behind the mirror. When the reflected rays are extrapolated backward they meet at the focal point. We also take the focal length to be negative.



Light from an off-axis point source will reflect away from a point behind the mirror, the image position. This will create a virtual image.



The ray tracing is similar to the concave case except that the center is now behind the mirror.



An analogous analysis with similar triangles yields the same expressions

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \quad \text{and} \quad m = \frac{h_i}{h_o} = -\frac{d_i}{d_o}.$$

It is still true that $f = R/2$ but now both R and f are negative. It follows that since $d_o > 0$ we will always get $d_i < 0$ and thus a virtual image.

Example H.4 - Convex Spherical Mirror

A convex spherical mirror with a 60-cm radius is used with a 10-cm high object. If the object is 20 cm from the mirror, then where is the image and what is the image height and magnification? Also, sketch the rays.

Solution

We are given the magnitude of the radius and we have to add its sign by hand. This then gives the focal length. The object height and object distance are also given.

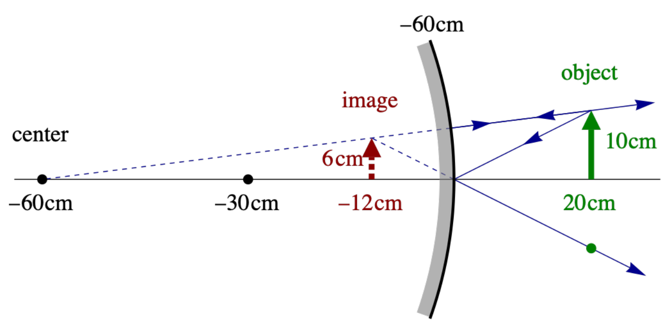
$$R = -60\,\text{cm} \implies f = \frac{R}{2} = -30\,\text{cm}\,, \quad h_o = 10\,\text{cm} \text{ and } d_o = 20\,\text{cm}$$

With this we can solve for the image position.

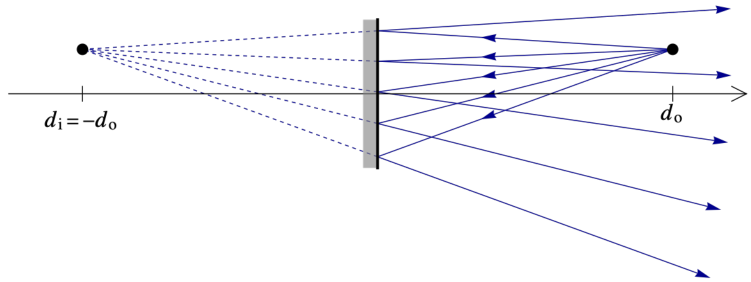
$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \implies d_i = \left(\frac{1}{f} - \frac{1}{d_o}\right)^{-1} = -12\,\text{cm}$$

Here we used $1/x = x^{-1}$. We can then solve for the image height and magnification.

$$m = \frac{h_i}{h_o} = -\frac{d_i}{d_o} \implies h_i = -h_o \frac{d_i}{d_o} = +6\,\text{cm} \text{ and } m = -\frac{d_i}{d_o} = +0.6$$



Flat Mirrors

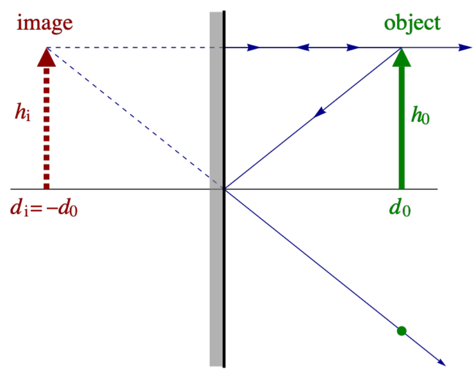


A flat mirror can be viewed as a special case of a spherical mirror with $R \to \pm\infty$. This implies that $f \to \pm\infty$ and $1/f \to 0$

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} = 0 \implies d_i = -d_o$$

$$m = \frac{h_i}{h_o} = -\frac{d_i}{d_o} \implies m = 1 \text{ and } h_i = h_o$$

This is what we would expect: There is an upright virtual image the same size as the object and equal distance behind the mirror.



Example H.5 - A Make-up Mirror

A magnifying mirror, like a make-up mirror, is used to create an enlarged upright image. An upright image must be virtual. For an enlarged virtual image, the mirror must be concave. Suppose we want to design such a mirror so that when the object (the face) is 10-cm in front of the mirror, the image is upright and has a magnification of three. What radius concave mirror is required?

Solution

We are given the magnification and the object position. We are looking for the radius.

$$m = 3, \quad d_0 = 10 \text{ cm}, \quad R = ?$$

We can solve for the image position.

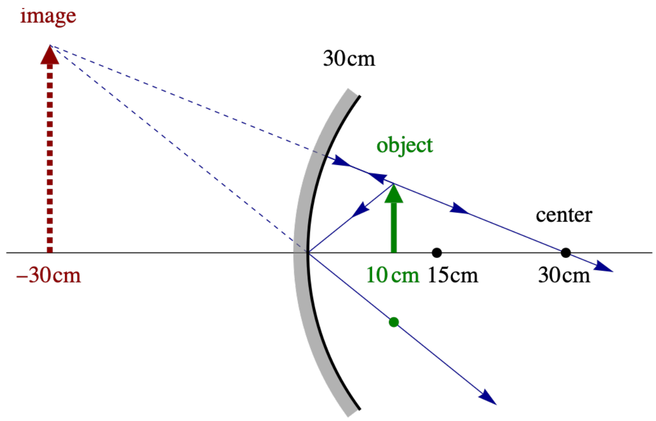
$$m = -\frac{d_i}{d_o} \implies d_i = -m\, d_o = -30 \text{ cm}$$

From this we can find the focal length

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \implies f = \left(\frac{1}{d_o} + \frac{1}{d_i}\right)^{-1} = 15 \text{ cm}$$

and the radius.

$$f = \frac{R}{2} \implies R = 2f = 30 \text{ cm}$$



H.4 - Images from Thin Lenses

Thin Lenses

A lens has two refracting surfaces. Solving for images from two refracting surfaces is a complex thing but when the lens is thin, meaning that the two surfaces are close to each other, the problem becomes simple. To relate the image position and height to the object position and height, we have the same equations we used for mirrors.

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \quad \text{and} \quad m = \frac{h_i}{h_o} = -\frac{d_i}{d_o}.$$

Although we have two surfaces that refract we will treat a thin lens as if all refraction happens at the central plane

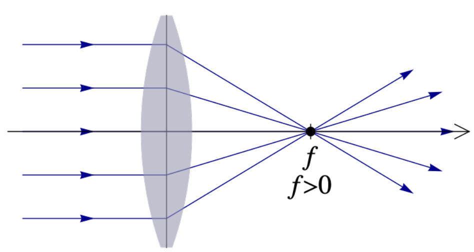
General Sign Conventions in Optics

In the case above where images were formed by mirrors, the sign conventions were relatively simple. d_o is positive when the object is on the front side of the mirror. Similarly, d_i, f and R are positive when the image, focal point and center, respectively, are on the mirror's front side.

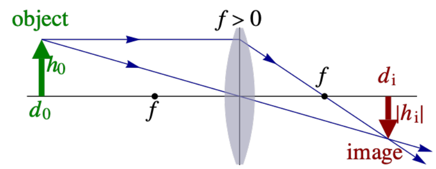
With a lens the light leaves on the opposite side where it enters. Label the ray that approaches a mirror or lens as the entering ray and the ray leaving as the *exiting ray*; this is the reflected ray for mirrors or the refracted ray for a lens.

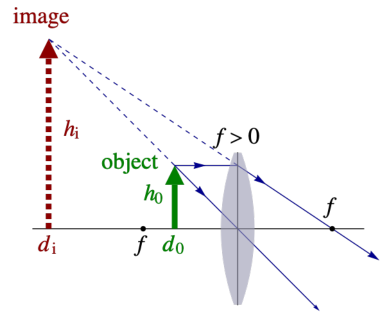
A general sign convention can now be given. d_o is positive when the object is on the side of the entering ray, the side where the light originates. d_i, f and R are positive when the image, focal point and center, respectively, are on the side of the exiting ray, on the side where the light ends up.

Converging Lenses

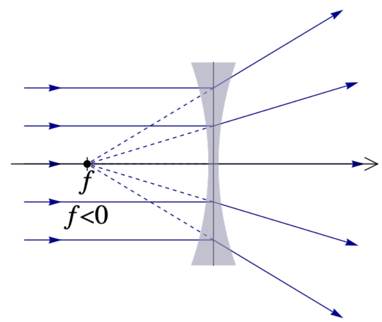


A thin lens with a positive focal length is called a converging lens. This is analogous to a concave mirror mathematically. If the object distance is larger than the focal length $d_o > f$ then there is a real image and when the object distance is less than the focal length $d_o < f$ there is a virtual image. To trace the rays: Draw one ray that passes straight through where the central axis hits the lens. Draw the other ray parallel to the central axis; this will bend through the focal point. The two rays (as all rays do) will converge to the image. Below are examples of ray tracing in both cases.

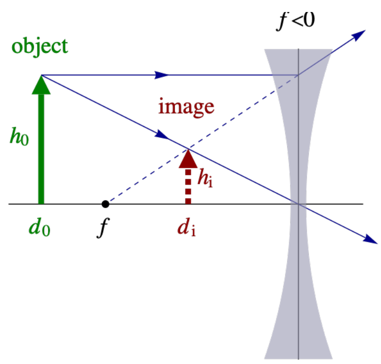




Diverging Lenses



A diverging lens has a negative focal length. The ray tracing is similar but the ray that begins parallel to the central axis will diverge away from the focal point, which is on the same side as the object. Here is the ray tracing for a diverging lens.



Example H.6 - A Slide Projector

A slide projector uses a bright light to shine through a small slide. The slide is the object and it is projected onto a screen using a converging lens. For a focused image, the image is on the screen. Note that this is the same optics as a classroom projector, where there is a small backlit LCD screen that is focused, via a converging lens, to a screen.

A slide projector uses a converging lens with a 12-cm focal length to project a 35-mm wide slide to fill a 2.1-m wide screen. Relative to the lens, where must the slide and screen be placed?

Solution

We are given the focal length $f = 12\,\text{cm}$ and the object and image heights. (They are described as widths, but that is unimportant.) $h_o = 35\,\text{mm} = 3.5\,\text{cm}$ and $|h_i| = 2.1\,\text{m} = 210\,\text{cm}$. The use of the absolute value for h_i is a bit subtle. We will see that the image is inverted but a positive h_i will give us a clear inconsistency.

$$f = 12\,\text{cm}\,,\quad h_o = 3.5\,\text{cm} \text{ and } h_i = \pm 210\,\text{cm}.$$

A projected image must be a real image.

$$\frac{h_i}{h_o} = -\frac{d_i}{d_o} \implies d_i = -\frac{h_i}{h_o}\,d_o$$

For a single lens or mirror, the object position d_o must be positive and a real image must have a positive d_i. If both h_o and h_i are positive then d_i is negative and we have an inconsistency.

$$h_i = -210\text{ cm} \implies d_i = -\frac{h_i}{h_o}\, d_o = 60\, d_o$$

We now have two equations for the two unknowns d_o and d_i

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \implies \frac{1}{d_o} + \frac{1}{60\, d_o} = \frac{1}{12\text{ cm}}$$

Cross-multiplying gives

$$12\text{ cm}\left(1 + \frac{1}{60}\right) = d_o$$

We can then solve for d_o and d_i.

$$d_o = 12.2\text{ cm} \quad \text{and} \quad d_i = 60\, d_o = 732\text{ cm}$$