Chapter J

Relativity

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J.1 - Before 1905

The Principle of Relativity

The principle of relativity is built in to Newtonian mechanics. It is a result of Galileo's observations on motion. This notion of relativity is known as *Galilean Relativity* and should be contrasted with *Special Relativity* which was introduced by Albert Einstein in 1905.

Moving with a constant velocity is equivalent to being at rest.

If someone (preferably a passenger) throws a ball straight upward in a car moving with a constant velocity, it will move in a way that is indistinguishable from free fall. To an observer on the side of the road, the ball would move as a projectile, but to both observers the acceleration would be the same, a downward acceleration of g. If the car is turning, or accelerating in any way, then there will be false forces and the motion will deviate for free fall.

A frame of reference is just some coordinate system we use to study motion. Recall from Physics I, that an inertial frame is a non-accelerated frame. The principle of relativity says that all inertial frames are equivalent; moving with a constant velocity is indistinguishable from being at rest. Suppose some experiment is performed in a van moving at a constant velocity. (We assume the road is as smooth as possible.) The result of that experiment will be give the same result as if the van were at rest. Nature has no absolute rest frame.

Relative Motion

In Physics I, relative motion was discussed. Suppose you have three things labeled 1, 2 and 3. The velocity of 2 relative to 1 is $\vec{v}_{21}$, $\vec{v}_{13}$ is the velocity of 1 relative to 3 and $\vec{v}_{23}$ is the velocity of 2 relative to 3. The relative motion expression from Physics I was just.

$$\vec{v}_{23} = \vec{v}_{21} + \vec{v}_{13}$$

Consider a one-dimensional example, suppose I was walking toward you at 1 m/s and throwing a ball toward you at 2 m/s relative to me, then the ball moves at 3 m/s relative to you.

$$v_{by} = v_{bm} + v_{my} \implies +3\text{ m/s} = 2\text{ m/s} + 1\text{ m/s} \text{ where } m \leftrightarrow \text{me},\ \ y \leftrightarrow \text{you and } b \leftrightarrow \text{ball}$$

That is, the velocity of the ball to you, is the sum of the velocity of the ball to me and the velocity of me to you.

Light and Electromagnetism

Now suppose that I am moving toward you at half the speed of light with a flashlight. The light leaves the flashlight, relative to me, at c, which we write $v_{lm} = c$, where l = light. Clearly we have just replaced the ball with the light.

$$v_{ly} = v_{lm} + v_{my} \implies \frac{3}{2}\, c = c + \frac{c}{2}$$

So the light is now moving toward you at $(3/2)\, c$. This should bother you, because what hits you is light and that has to move at the speed of light. To expand on this: from the equations of electromagnetism, electromagnetic waves travel at c and this can be written in terms of the electromagnetic constants, μ_0 and ε_0.

$$c = \frac{1}{\sqrt{\mu_0\, \varepsilon_0}}.$$

To the nineteenth century physicist, this implied that electromagnetism violated relativity, this age-old principle of physics. It was believed that there was a preferred rest frame, the frame of the “ether”; in only this frame the equations of electromagnetism were valid and light travels at the speed of light. A similar situation occurs with sound waves; sound travels at the speed of sound with respect to the stationary air. There *is* a preferred rest frame, that of the air.

Perhaps the only problem is the velocity addition formula; could we just toss that out? The only assumptions behind the velocity addition formula are absolute space and absolute time, meaning that whether moving or not we all agree on lengths and times.

Space and Time

Space and time are absolute in Galilean relativity. Absolute space means that space is three dimensional Euclidean space. The length of something in a stationary frame is the same as the length in a moving frame.

$$\Delta x^2 + \Delta y^2 + \Delta z^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

In this expression the left-hand side is the squared-length in a stationary frame and the right-hand side is the squared-length relative to a moving frame

Absolute time implies that the time between two events (an event is some position at some time) is the same for all observers.

$$\Delta t = \Delta t'$$

Here Δt and $\Delta t'$ are the times in rest frame and a moving frame, respectively.

Einstein's Special Relativity eliminated the notions of absolute space and time.

In 1905 Einstein realized that electromagnetism was valid in all inertial (non-accelerated) frames and he put relativity back into fundamental physics. To do this however he had to modify the notions of absolute space and time that were implicit in Galilean relativity. Einstein’s first paper on relativity was: “On the Electrodynamics of Moving Bodies”; he was considering electromagnetism and relative motion.

J.2 - Length and Time

The Postulates of Relativity

Einstein’s theory of special relativity is based on two postulates.

The Principle of Relativity

The laws of physics are the same in all inertial frames of reference.

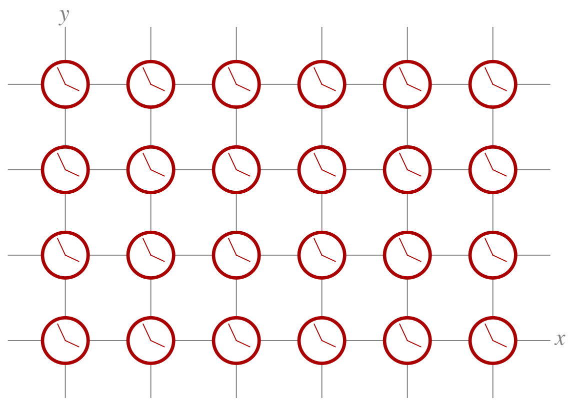
The Constancy of the Speed of Light

The speed of light in a vacuum c, is the same in all inertial frames of reference. This is just applying the first postulate, the principle of relativity, to electromagnetism. As we saw in the previous section, if the laws of electromagnetism are the same in all frames, then light must always travel at the same speed in a vacuum.

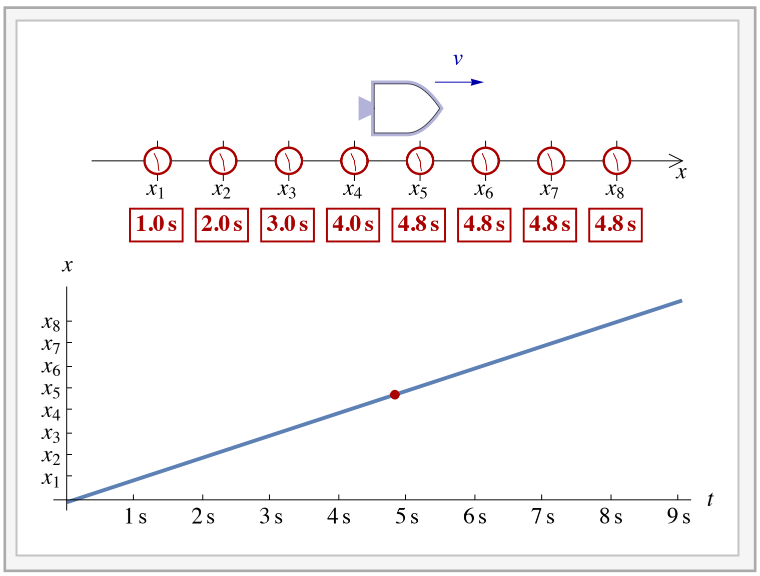
Frames of Reference

One cannot just throw out notions like absolute space and absolute time without replacing them with different and substantial concepts of space and time. For an inertial (non-accelerated) frame we can define lengths and times clearly. We will first consider this and then discuss how lengths and times are different in different inertial frames. If a meter stick is at rest relative to your inertial frame of reference then that defines a meter for you. Lay that meter stick along a line and mark off ticks of one meter. That labels position.

Time is a bit more subtle. When you are at rest with a clock, that clock defines time for you. How do you define time for a frame of reference? If two clocks are at rest with respect to each other we can synchronize them: Suppose you are at position A with a clock and someone at a different position B also has a clock. You send a light signal from your position A to the other position B. At B they have a mirror to reflect the light signal back. The person at B records the time the light signal hits and the person at A then averages the times when the signal was sent and returned. The average of the times for A is then set as simultaneous with the time the person at B recorded. Imagine a frame of reference as an array of synchronized clocks at each position in the frame.



How do we describe motion in a frame of reference? In the diagram below we have an axis with a series of marks labeling positions. At each mark place a clock and synchronize all these clocks following to the procedure described above. If a rocket is zipping by at high speeds, then we record the time as read by a clock as the rocket passes that clock. This is the relevant time for studying motion within a frame of reference. It is not when one person sees something pass a distant point, because there is a time delay for the light to get back. It is the time recorded at the position of the moving rocket that we record. We then can, as we did in Physics I, write a graph of position versus time.



In discussing relativity, we often use the term event. An event is a point in space at an instant in time. To label an event you specify a position and a time, in three dimensions it takes four numbers (x, y, z, t) to label an event. This is in line with the standard usage of the term event; if someone invites you to a party, then to describe the event in an invitation requires saying where it is and when it will be.

The Light Clock and Perpendicular Lengths

We want to relate times in different frames of reference. To do this we will exploit the fact that the speed of light is the same in all frames and invent type of clock, a light clock. If we reflect light between two parallel mirrors separated by a distance d then the time for the light signal to return is

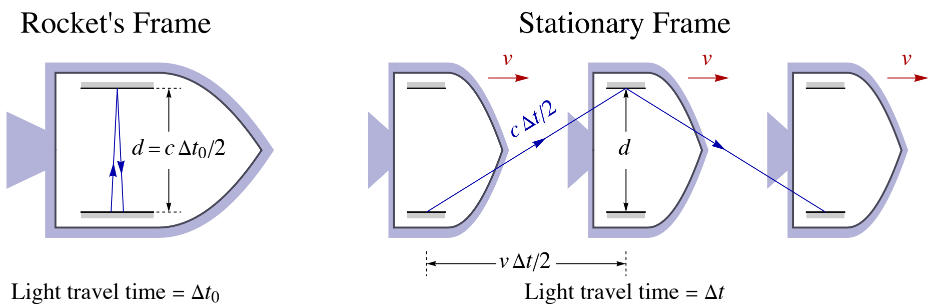
$$\Delta t_0 = \frac{2\,d}{c}$$

since the total distance the light travels is $2\,d$. Think of this as one tick of our light clock. We refer to this time Δt_0 as the proper time. Proper time is the time between two events at the same position; it is what a clock reads in its rest frame.

Now we will put our light clock in a moving rocket and relate time in the rocket's frame to the time relative to a fixed frame. Before we can do this we must establish something about lengths. Lengths that are perpendicular to the direction of motion are the same in all frames. Say you have two meter sticks A and B oriented in the y-direction while the relative motion is in the x-direction. Suppose that in relativity, lengths are such that a moving stick is shorter than the longer one. That means in the frame where A is at rest, B is shorter but in the frame where B is at rest then A would be shorter. If the moving meter stick were longer we would have a similar situation. To make this point more vividly, imagine putting knives on the ends of both meter sticks so as they pass the shorter one would cut the longer one. Then in different frames different meter sticks would be cut and this shows a clear inconsistency. So, lengths perpendicular to the direction of motion are the same in all frames.

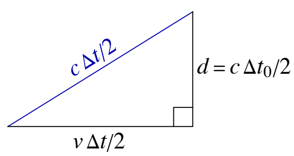
Time Dilation

We now arrange our light clock in a moving rocket to be perpendicular to the rocket's velocity. As mentioned above, in the frame of the rocket, the proper time is $\Delta t_0 = 2\,d/c$. We now look at the light clock as it speeds through a stationary frame and find the time Δt in that frame.



On the left is a light clock in its rest frame. On the right it is moving at high speeds. Note that we will discuss the rocket's contraction in the next section.

Since the light always moves at c, the distance it moves in the stationary frame when moving from one mirror to the other is $c\,\Delta t/2$. Since the rocket has moved by $v\,\Delta t/2$ and since d is the same in both frames we get a right triangle. We then apply the Pythagorean theorem to this.



$$\Longrightarrow \left(\frac{c\,\Delta t}{2}\right)^2 = \left(\frac{v\,\Delta t}{2}\right)^2 + \left(\frac{c\,\Delta t_0}{2}\right)^2 \Longrightarrow \left(c^2 - v^2\right)\Delta t^2 = c^2\,\Delta t_0^2$$

This allows us to solve for Δt in terms of Δt_0.

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} = \gamma\,\Delta t_0$$

where we have defined the *Lorentz factor* γ by

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

γ has the properties that as v approaches zero, γ approaches one and as v approaches c, γ approaches infinity. For routine everyday speeds this Lorentz factor γ is essentially 1. If using your calculator, you try to calculate this with a car's speed for v, you will get exactly one. This is because your calculator does not keep enough digits to see the difference. For very small speeds it is actually more accurate to use an approximation. For speeds much less than c, which we write $v << \text{c}$ we have the approximation

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \simeq 1 + \frac{1}{2}\frac{v^2}{c^2} \quad \text{for } v << \text{c}$$

Solving for the speed in terms of γ gives.

$$v = c\,\sqrt{1 - 1/\gamma^2}$$

Example J.1 - A Trip to Alpha Centauri

Alpha Centauri is a nearby star 4.37 ly away, where a light-year, written ly, is the distance light travels in one year. $1\text{ ly} = c \cdot 1\text{ yr}$. Suppose a rocket travels from the earth to Alpha Centauri at the speed of $0.99\,c$. How long does the trip take in years, relative to the earth and relative to the rocket?

Solution

We are given

$$d = 4.37\,c \cdot 1\text{ yr} \quad \text{and} \quad v/c = 0.99$$

Calculate the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.99^2}} = 7.089$$

Now, solve for the time relative to the earth.

$$d = 4.37\,c\cdot\text{yr}\,,\quad v = 0.99\,c \implies \Delta t = \frac{d}{v} = \frac{4.37\,c\cdot\text{yr}}{0.99\,c} = \frac{4.37}{0.99}\,\text{yr} = 4.41\,\text{yr}$$

Next, find the time in the rocket's frame, the proper time.

$$\Delta t = \gamma\,\Delta t_0 \implies \Delta t_0 = \Delta t/\gamma = 0.623\,\text{yr}$$

This means that in the time when people on the earth age 4.41 yr, people in the rocket would age 0.623 yr.

Example J.2 - Time Dilation in a Car

Suppose you drive in your car at 30 m/s for 2 hours. How much less will you have aged in that time than if you stayed still?

Solution

What we are looking for is the difference between Δt, if you stayed still, and Δt_0 when you are moving.

$$v/c = \frac{30\,\text{m/s}}{3\times 10^8\,\text{m/s}} = 10^{-7}$$

$$\Delta t - \Delta t_0 = (\gamma - 1)\,\Delta t_0$$

If you calculate γ on any calculator or computer with floating point math you will get $\gamma = 1$ and thus get a zero result. Ironically, we have to use the approximate formula to get an accurate result for $\gamma - 1$. The result is very very small but not zero. For $v << c$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \simeq 1 + \frac{1}{2}\frac{v^2}{c^2} \implies \gamma - 1 = \frac{1}{2}\frac{v^2}{c^2} = 5\times 10^{-15}$$

So we get for the time difference

$$\Delta t_0 = 2\,\text{h} = 2\times 3600\,\text{s} \implies \Delta t - \Delta t_0 = (\gamma - 1)\,\Delta t_0 = 3.6\times 10^{-11}\,\text{s}$$

Example J.3 - The Decay of the Muon

When very high energy cosmic rays hit the atmosphere, about 10 km above the earth, they interact with the atmosphere and one of the particles that can be created is a muon. A muon is an elementary particle that behaves like a heavy electron; it is about 207 times more massive than the electron. Unlike the electron it is unstable and decays with a half-life of 2.2×10^{-6} s. What this means is that in the rest frame of a muon it has a 50% probability of decaying in that time. These muons are created with a very relativistic speed of $0.999\,c$.

(a) How far does a muon travel in 2.2×10^{-6} s at this speed?

Solution

This is a simple calculation. We are given v and t and want the distance d.

$$c = 3.00\times 10^8\,\text{m/s}\;\; v = 0.999\,c = 3.00\times 10^8\,\text{m/s}\;\text{and}\; t = 2.2\times 10^{-6}\,\text{s} \implies d = v\,t = 677\,\text{m}$$

This shows that a muon created in the upper atmosphere would never make it to earth if this were its lifetime. However, the half-life refers to its lifetime *in* the muon's rest frame and it lasts longer relative to the atmosphere.

(b) What is the half-life of a muon in the frame of the atmosphere?

Solution

First find the Lorentz factor.

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.999^2}} = 22.4$$

The given half-life is the proper time Δt_0. The time relative to the atmosphere is Δt.

$$\Delta t_0 = 2.2\times 10^{-6}\,\text{s} \implies \Delta t = \gamma\,\Delta t_0 = 5.05\times 10^{-5}\,\text{s}$$

(c) How far does a muon travel in the time found in part (b)?

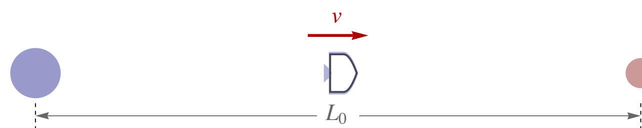
Solution

$$d = v\,\Delta t = 15\,100\text{ m} = 15.1\text{ km}$$

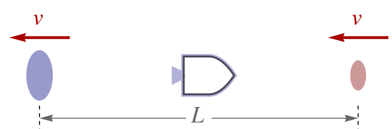
These muons will make it to the ground.

Length Contraction

The proper time Δt_0 is the time measured by a clock in its rest frame. Similarly, we will define the proper length L_0 to be the length of a ruler in its rest frame or, equivalently, the distance between two events in a frame where they are at the same time. Suppose a rocket is moving through the solar system at a high speed v moving past the earth and mars. The distance between the planets in the solar system's frame is L_0.



Solar System's Frame



Rocket's Frame

There are two events, the rocket passing the earth and the rocket passing mars. The time between these two events in the rocket's frame is Δt_0 and the distance is L. In the solar system's frame the time between the events is Δt and the distance is L_0. The speed of the solar system relative to the rocket is the same as the speed of the rocket relative to the solar system.

$$v = \frac{L}{\Delta t_0} = \frac{L_0}{\Delta t} \implies \frac{L}{L_0} = \frac{\Delta t_0}{\Delta t} = \frac{1}{\gamma}$$

We then get the result

$$L = L_0/\gamma = L_0\,\sqrt{1 - v^2/c^2}$$

Since the Lorentz factor γ is greater than one $\gamma \geq 1$, the length of a moving rod or ruler is contracted. Summarizing for moving rods: lengths parallel to the direction of the velocity are contracted and, as we saw previously, lengths perpendicular to the direction of motion are unchanged.

Example J.4 - A Rocket Moving through the Solar System

Suppose a rocket is 55-m long and 20-m wide (in its rest frame.) Take the length to be in the direction of motion and the width to be perpendicular. If this rocket is moving through the solar system at 85 % the speed of light, first moving past the earth and then past mars when the distance between the earth and mars is 2.2×10^{11} m. (This is the speed that was chosen for the diagram above, so the relative lengths in the diagram apply here.)

(a) What is its length and width of the rocket relative to the solar system?

Solution

We are given the proper length L_0 and the speed and we are looking for L, the contracted length.

$$L_0 = 55\text{ m}\,,\quad v = 0.85\,c \text{ and } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.85^2}} = 1.89$$

The length to the solar system is then

$$L = L_0/\gamma = 29.0\text{ m}\,.$$

The width is perpendicular to the direction of motion, so that is the same in both frames.

$$W = W_0 = 20\text{ m}$$

(b) What is the distance between the earth and mars relative to the rocket?

Solution

We are given the proper length L_0. The speed and γ are the same. We want L, the contracted length.

$$L_0 = 2.2\times 10^{11}\ \text{m} \quad \Longrightarrow \quad L = L_0/\gamma = 1.16\times 10^{11}\ \text{m}$$

Aside on Space-time

In 1908, Hermann Minkowski gave a geometrical interpretation to special relativity. Minkowski was a college math professor of Einstein's and, although he was not that impressed with Einstein as a student, changed his impression of his former student after his 1905 relativity papers. Let us begin with our time dilations formula.

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}}$$

Multiply both sides by the denominator and square

$$\Delta t^2 \left(1 - v^2/c^2\right) = \Delta t_0^2$$

The speed v is the distance per time $v = \Delta s/\Delta t$. Insert this for the speed and also multiply by c^2.

$$\Delta t^2 \left(c^2 - \frac{\Delta s^2}{\Delta t^2}\right) = c^2\,\Delta t_0^2 \implies c^2\,\Delta t^2 - \Delta s^2 = c^2\,\Delta t_0^2$$

Δs is the distance between two positions so, $\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$. Multiplying the expression above by -1 and substituting for Δs^2 gives

$$\Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\,\Delta t^2 = -c^2\,\Delta t_0^2$$

The right-hand side of the above expression involves only the proper time, which is the time between two events in the frame where they are at the same position. The left-hand side involves the distance and time between the same two events in any frame. But the left-hand side is the same for all frames. This give Minkowski's result.

$$\Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\,\Delta t^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2 - c^2\,\Delta t'^2$$

Although lengths and times are different in different frames, the length-squared minus time-squared is the same in all frames, where we must multiply time by c to make the units work out.

The reason for the term space-time is that the underlying geometry of special relativity is four-dimensional, where time is the fourth dimension. However, because of the minus sign before the time variable, the time variable is still different than spatial variables. One way of looking at Minkowski's result is that special relativity replaced the concepts of absolute space and time with an absolute space-time.

Relativistic Addition of Velocities

We will only consider the relativistic addition of velocities in the one-dimensional case where all the velocities are in the same direction. The three-dimensional formulas are more complex. The classical (meaning before relativity) one-dimensional formula was $v_{23} = v_{21} + v_{13}$. View these as one-dimensional vectors where a 1D vector is a real number and the sign gives its direction. The relativistic formula is

$$v_{23} = \frac{v_{21} + v_{13}}{1 + v_{21}\,v_{13}/c^2}$$

I am following the textbook's notation; it can be confusing to keep track of the 1, 2, 3 labels. The key thing is to decide how you would combine the velocities classically, and then divide by the appropriate denominator. For the first example, lets consider the example I referred to in the first section.

Example J.5 - Light from a Moving Source

(a) Suppose I am in a rocket approaching you at half the speed of light with a flashlight aimed at you. What is the speed of the light hitting you?

Solution

Hopefully, at this point it is obvious that the answer is c. But show that from the formulas.

$$v_{lm} = c \text{ (vel. of light to me)}, \;\; v_{my} = \frac{c}{2} \text{ (vel. of me to you)} \;\text{ and }\; v_{ly} = \text{? (vel. of light to you)}$$

$$v_{ly} = \frac{v_{lm} + v_{my}}{1 + v_{lm}\, v_{my}/c^2} \implies \frac{c + c/2}{1 + \frac{c \times c/2}{c^2}} = \frac{3\,c/2}{3/2} = c$$

(b) Suppose instead I am moving away from you and and shining the light back at you?

Solution

$$v_{lm} = c \text{ (vel. of light to me)}, \;\; v_{my} = -\frac{c}{2} \text{ (vel. of me to you) and } v_{ly} = ? \text{ (vel. of light to you)}$$

$$v_{ly} = \frac{v_{lm} + v_{my}}{1 + v_{lm}\, v_{my}/c^2} \implies \frac{c - c/2}{1 + \frac{c \times (-c/2)}{c^2}} = \frac{c/2}{1/2} = c$$

So our new velocity addition formula now gives the expected result.

Example J.6 - A Probe from a Rocket

(a) Suppose a rocket moving toward you at 90% the speed of light launches a probe moving toward you at 99% the speed of light relative to the rocket. What is the speed of the probe relative to you?

Solution

$$v_{pr} = 0.99\, c \text{ (vel. of probe to rocket)}, \;\; v_{ry} = 0.90\, c \text{ (vel. of rocket to you) and } v_{py} = ? \text{ (vel. of probe to you)}$$

$$v_{py} = \frac{v_{pr} + v_{ry}}{1 + v_{py}\, v_{ry}/c^2} \implies \frac{0.99\, c + 0.90\, c}{1 + (0.99)\,(0.90)} = 0.9995\, c$$

(b) Suppose instead the rocket is moving away from you at the same speed and launching the probe back at you. What is the speed of the probe relative to you?

Solution

$$v_{pr} = 0.99\, c \text{ (vel. of probe to rocket)}, \;\; v_{ry} = -0.90\, c \text{ (vel. of rocket to you) and } v_{py} = ? \text{ (vel. of probe to you)}$$

$$v_{py} = \frac{v_{pr} + v_{ry}}{1 + v_{py}\, v_{ry}/c^2} \implies \frac{0.99\, c - 0.90\, c}{1 + (0.99)\,(-0.90)} = 0.826\, c$$

J.3 - Momentum, Energy and Mass

Rest Mass and Relativistic Mass

When reading about mass in relativity, it is easy to get confused. There are two notions of mass: The rest mass m_0 is the mass you measure when you set something (at rest, of course) on a scale. When we tabulate masses of particles those are the rest masses. The other notion is the relativistic mass, which we will refer to as m_r. Some books use m for the rest mass and some use it for the relativistic mass. The Walker text is particularly bad here; they write inconsistent formulas jumping back and forth between the two notations. We will avoid writing m at all, and use the subscripts, m_0 and m_r.

The rest mass of an object or a particle is a fixed thing. As a particle increases it's speed, the mass, its *relativistic mass*, increases.

$$m_r = \frac{m_0}{\sqrt{1 - v^2/c^2}} = \gamma\, m_0$$

Here the γ is the same Lorentz factor as before. Because of the properties of the Lorentz factor, as the speed approaches c, the relativistic mass goes to infinity.

Example J.7 - An Electron in a Particle Accelerator

Suppose an electron is accelerated to a speed of $0.999995\, c$ in a particle accelerator.

(a) What is the relativistic mass of the electron?

Solution

From the speed we can find the Lorentz factor γ

$$v = 0.999995\, c \implies \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.999995^2}} = 316$$

Using the listed value of the mass of the electron as the rest mass we get the relativistic mass.

$$m_0 = m_{\text{electron}} = 9.11 \times 10^{-31}\text{ kg} \implies m_r = \gamma\, m_0 = 2.88 \times 10^{-28}\text{ kg}$$

Energy and Kinetic Energy of a Particle

The relativity formula everyone knows is $E = m\, c^2$. The m here is the relativistic mass and not the rest mass. We will first discuss this in the context of a particle or object moving at high speeds. The $E = m\, c^2$ formula applies more generally than just the motion of a particle; this will be discussed in the next section.

For a particle or object with a rest mass m_0 moving with a speed v has a total energy of

$$E = m_r\, c^2 = \gamma\, m_0\, c^2 = \frac{1}{\sqrt{1 - v^2/c^2}}\, m_0\, c^2.$$

But what is its kinetic energy. When the velocity goes to the we get $E_0 = m_0\, c^2$. This is called the rest energy. The kinetic energy, which is the energy of motion, is then

$$K = E - E_0 = m_r\, c^2 - m_0\, c^2 = (\gamma - 1)\, m_0\, c^2 = \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right) m_0\, c^2.$$

One cannot propose a new theory in physics that invalidates all that came before it. What of all those problems you solved using $K = \frac{1}{2}\, m\, v^2$? Are they now wrong? Is all of Newtonian mechanics wrong? Any new theory must reproduce the successes of the previous theory, at least as a special case.

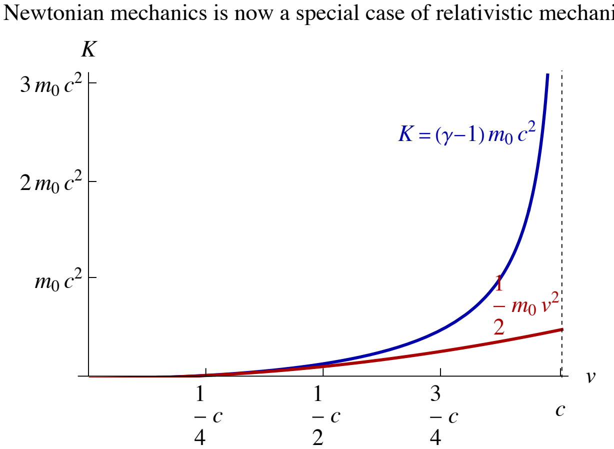
Consider the case of small speeds $v << c$, where small means when compared to c. Recall that for small speeds.

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \simeq 1 + \frac{1}{2}\frac{v^2}{c^2} \quad \text{for } v << c$$

Inserting this into our expression for kinetic energy gives

$$K = (\gamma - 1)\, m_0\, c^2 = \frac{1}{2}\frac{v^2}{c^2}\, m_0\, c^2 = \frac{1}{2}\, m_0\, v^2.$$

So all is good; the new formula reproduces the old formula in the limit of small speeds. So relativity provides a correction to Newtonian mechanics when speeds are large, but Newtonian mechanics is now a special case of relativistic mechanics when speeds are small.



Above is a graph of kinetic energy versus speed and the classical formula $\frac{1}{2}\, m_0\, v^2$ is included for reference. Note that as the speed approaches c, the kinetic energy goes to infinity. This means that it takes more and more energy to get closer and closer to the speed of light and a massive object, anything with a rest mass, cannot be accelerated to c but with enough energy, it can get arbitrarily close to c.

Some particles are referred to as massless particles. This means the particle has no rest mass. There is energy, so there is a relativistic mass $m_r = E/c^2$. However, the energy formula $E = \gamma\, m_0\, c^2$ does not apply, since both γ and m_0 are undefined. Massless particles must always travel at the speed of light. An example of a massless particle is a photon, the particle of light. Since light must always travel at c, then the photon must be massless and always travel at c.

Example J.8 - Grandma's Particle Accelerator

One might think that it requires a particle accelerator to get particles up to relativistic speeds, however old tube televisions, like what your grandmother might have had in her living room, involve sufficiently high voltages to accelerate electrons up to relativistic speeds. Tube-based televisions and computer monitors are called CRTs, where CRT stands for cathode ray tube; a cathode ray is just an archaic term for a beam of electrons. In a CRT, electrons are accelerated across large voltages and then hit a phosphorescent screen to create a dot; electric and magnetic field manipulate the beam to make an image.

The picture tube of a CRT-based television uses a voltage of 25 kV to accelerate electrons.

(a) Calculate the kinetic energy of the electrons after accelerating from rest across this voltage.

Solution

This is essentially a Chapter 20 problem. The conservation of energy allows us to find the kinetic energy. $K = \Delta K = K_f - K_i$ since it starts from rest. The conservation of energy can be written as $0 = \Delta K + \Delta U$, To avoid considering signs take the absolute values.

$$K = \Delta K = |\Delta K| = |\Delta U|$$

The change in potential energy can be written in terms of the potential difference.

$$\Delta U = Q\, \Delta V = -e\, \Delta V \implies |\Delta U| = e\, |\Delta V| = e\, V$$

Where we have used the voltage V for the magnitude (absolute value) of the change in electric potential $V = |\Delta V|$.

The relevant given information and constants are

$$e = 1.60 \times 10^{-19}\ \text{C} \quad \text{and} \quad V = 25\,000\ \text{V}$$

The expressions above give an expression for the kinetic energy

$$K = e\, V = 4.0 \times 10^{-15}\ \text{J}$$

(b) Calculate the speed of the electrons and also give that speed as a fraction of c.

Solution

The constants needed are

$$m_{\text{electron}} = 9.11 \times 10^{-31}\ \text{kg}\ , \quad c = 3.00 \times 10^{8}\ \text{m/s}$$

From the kinetic energy we can find the Lorentz factor γ

$$K = (\gamma - 1)\, m_0\, c^2 \implies \gamma = 1 + \frac{K}{m_0\, c^2} = 1 + \frac{K}{m_{\text{electron}}\, c^2} = 1.0488$$

and from that, using the expression for v from γ given in the last section, find the speed.

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \implies v = c\, \sqrt{1 - 1/\gamma^2} = 9.04 \times 10^{7}\ \text{m/s}$$

Diving this by c gives

$$v/c = 0.301$$

So this old television is accelerating electrons up to 30% the speed of light!

(c) Calculate the speed of the electrons using the incorrect non-relativistic Physics I formula, $K = \frac{1}{2}\, m\, v^2$ and give the percent error in v when using the old formula.

Solution

$$K = \frac{1}{2}\, m_{\text{electron}}\, v_{\text{nr}}^2 \implies v_{\text{nr}} = \sqrt{\frac{2\,K}{m_{\text{electron}}}} = 9.371 \times 10^{7}\ \text{m/s}$$

$$\% \text{ error} = \frac{v_{\text{nr}} - v}{v} \times 100\ \% = 3.62\ \%$$

Example J.9 - An Electron in a Particle Accelerator (continued)

Suppose an electron is accelerated to a speed of 0.999995 c in a particle accelerator.

(b) What is the kinetic energy of the electron?

Solution

As we did before, from the speed we can find the Lorentz factor γ

$$v = 0.999995\, c \implies \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.9995^2}} = 316$$

But now we find the kinetic energy using the rest mass and c.

$$m_0 = m_{\text{electron}} = 9.11 \times 10^{-31}\ \text{kg} \ \text{ and } \ c = 3.00 \times 10^8\ \text{m/s} \implies K = (\gamma - 1)\, m_0\, c^2 = 2.58 \times 10^{-11}\ \text{J}$$

Mass-Energy Equivalence

In Einstein's first relativity paper in 1905 discussed the kinetic energy of a particle, but his most famous result $E = m\,c^2$ was introduced in a different paper later the same year: "Does the Inertia of a Body Depend upon Its Energy-Content?" Inertia refers to the inertial mass, which is the mass associated with Newton's second law. The m refers to what we are labeling as the relativistic mass.

$$E = m_r\, c^2$$

Although we had this result already for a particle, it is a more general thing. Suppose you have some nuclear or chemical reaction. If the reaction gives off energy then the masses of the byproducts of the reaction are less than the constituents before the reaction. In chemistry class you are taught the conservation of mass. Fundamentally, mass is not conserved. In a chemical reaction, where you just move around electrons to change bonds, the energy differences ΔE are small, meaning that mass difference $\Delta m = \Delta E / c^2$ is negligibly small compared to the original masses and can be ignored. Fundamentally, mass is not conserved, but its conservation is close enough to being exact for purposes of chemistry. In a nuclear reaction, where the structure of the atomic nucleus changes, the energy difference are still small but measurable and not negligible.

In particle physics if a collision has sufficient energy, by $E = m\,c^2$ to create a new particle then new particles can be formed. These new particles must be consistent with conservation laws and the constraints of fundamental interactions. There are a huge number of new particles discovered this way. For every particle there is an antiparticle which has the same mass and the opposite charge. The antiparticle of an electron is called a positron. The antiparticles of protons and neutrons are antiprotons and antineutrons. A photon is its own antiparticle. If all of the energy stored in the mass of something could be released the resulting energy release would be huge; however this energy is not typically available. An example where all the available energy is releases is when a particle collides with its antiparticle. The resulting annihilation releases all of the $2\,m\,c^2$ of available energy. Antimatter is matter made from antiparticles. A bound state of a positron and antiproton is an anti-hydrogen atom. When matter and antimatter collide, essentially all of its available rest energy is released. Antimatter has been created in only small quantities and is still mostly the stuff of science fiction.

Example J.10 - A Hydrogen Atom

It takes 2.18×10^{-18} J of energy to break up a hydrogen in its ground state to a separate proton and electron. Which has more mass the atom or the separate proton and electron and by how much? Calculate the percentage of the total mass this it. It should be a very small amount.

Solution

Since it takes energy to break up the atom into its constituent parts, the atom has less energy and thus less mass. The given values and needed constants are.

$$\Delta E = 2.18 \times 10^{-19}\ \text{J}\,, \quad c = 3.00 \times 10^8\ \text{m/s}\,, \quad m_{\text{proton}} = 1.67 \times 10^{-27}\ \text{kg} \ \text{ and } \ m_{\text{electron}} = 9.11 \times 10^{-31}\ \text{kg}$$

The mass change Δm is then found

$$\Delta m = \frac{\Delta E}{c^2} = 2.42 \times 10^{-35}\ \text{kg}$$

The fractional change can then be found

$$\frac{\Delta m}{m_{\text{total}}} = \frac{\Delta m}{m_{\text{proton}} + m_{\text{electron}}} = 1.45 \times 10^{-8}$$

Example J.11 - An Electron-Positron Pair

What is the smallest amount of energy needed to create an electron-positron pair?

Solution

The minimum amount of energy is to create both with their minimum mass, their rest mass. For both that is the mass of an electron, since the positron has the same mass as an electron.

$$c = 3.00 \times 10^8 \text{ m/s} \quad \text{and} \quad m_{\text{electron}} = 9.11 \times 10^{-31} \text{ kg}$$

The minimum energy is then given by

$$E = 2\, m_{\text{electron}}\, c^2 = 1.64 \times 10^{-13} \text{ J}$$

Relativistic Momentum

Momentum is still $m\, v$ but which m? It depends on the relativistic mass. For a particle or body with a rest mass, we have.

$$p = m_r\, v = \gamma\, m_0\, v = \frac{1}{\sqrt{1 - v^2/c^2}}\, m_0\, v$$

This cannot apply to a massless particle. For massless particles we can write $v = c$ and $m_r = E/c^2$ and get:

$$p = m_r\, c = \frac{E}{c^2}\, c = E/c$$

Recall in the chapter on electromagnetic radiation we had for radiation $p = U/c$, where U is the energy. It should be expected that a photon, a particle of light should satisfy the same.

Example J.12 - An Electron in a Particle Accelerator (continued again)

Suppose an electron is accelerated to a speed of 0.999995 c in a particle accelerator.

(c) What is the momentum of the electron?

Solution

The speed is the same as before so Lorentz factor γ must also be the same

$$v = 0.999995\, c \implies \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.9995^2}} = 316$$

But now we find the momentum using the rest mass and c.

$$m_0 = m_{\text{electron}} = 9.11 \times 10^{-31} \text{ kg} \quad \text{and} \quad c = 3.00 \times 10^8 \text{ m/s} \implies p = \gamma\, m_0\, v = 8.64 \times 10^{-20} \text{ kg} \cdot \text{m/s}$$

Example J.13 - Momentum of a Photon, a Massless Particle

A photon of 550-nm (visible) light has an energy of 3.61×10^{-19} J. What is its momentum?

Solution

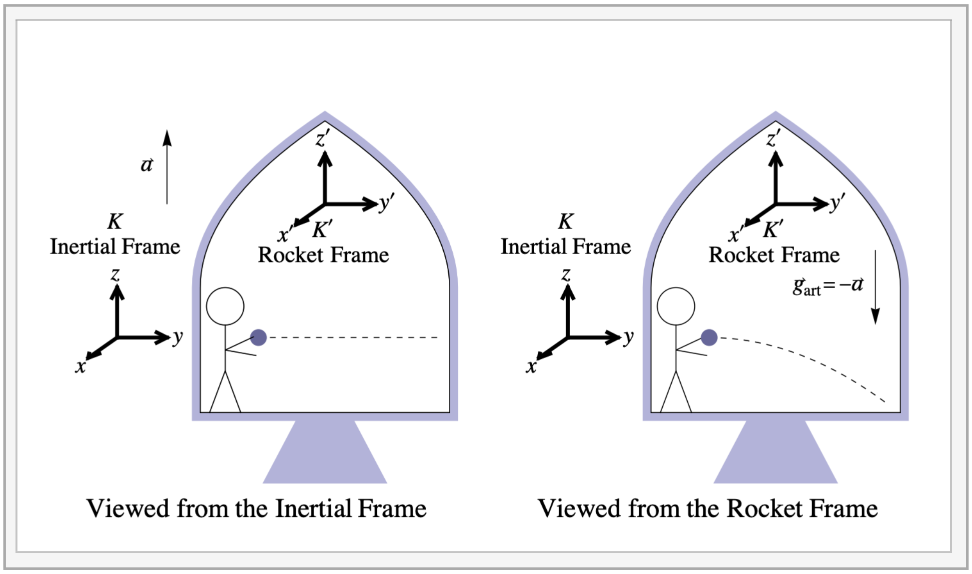
$$E = 3.61 \times 10^{-19} \text{ J} \implies p = E/c = 1.20 \times 10^{-27} \text{ kg} \cdot \text{m/s}$$

J.4 - General Relativity and Cosmology (Aside)

The Principle of Equivalence

It bothered Einstein that his theory of relativity applied only to inertial (or non-accelerated) frames. He set out to generalize his theory. What he ended up with was a theory of gravity. While pondering his generalized theory he had what he referred to as “the most beautiful idea of my life” while riding an elevator. He had the sudden realization that gravity and acceleration were one in the same. In an elevator accelerating upward you feel heavier and when accelerating downward you feel lighter. In a free-fall elevator you would feel weightless.

This idea became what he called the principle of equivalence. Suppose you are in a rocket far from any source of gravity and the rocket is accelerating with a upward steady acceleration of g. The principle of equivalence says that you cannot distinguish being in this rocket from being inside a room with a steady and uniform gravity g acting downward.



Interactive Figure - A ball is thrown from inside an accelerating rocket. K' is the accelerated frame of the rocket. K is the inertial frame, which is moving at a constant velocity and is initially at rest with respect to the rocket at the instant the ball is released.

Einstein published his principle of equivalence in 1907, but it took him nine more years to complete his general theory. The mathematics needed for his theory of gravity was quite difficult and when he started he was actually not much of a mathematician, but by the time he completed his theory he was. In the nineteenth century mathematicians had generalized Euclidean geometry to describe curved spaces. For example, the surface of a sphere is a curved two dimensional surface and when viewed on the very small scale it looks like flat plane, Euclidean space. To complete his theory he had to find a curved four-dimensional theory that, on the small scale, looks like Minkowski’s geometry. Throw a ball and it moves in a straight line, relative to Einstein’s curved geometry. The straight lines of general relativity are the projectile trajectories.

The Bending of Light

Suppose in the rocket example the astronaut had a flashlight. The floor would still accelerate up to the light. This meant to Einstein that light must bend under gravity. In 1919 the astronomer Arthur Eddington set to measure this bending during a solar eclipse. When he reported his results proving Einstein’s prediction, Einstein went from a well-known figure in the scientific community to one of the most famous people in the world.

This bending of light has become an important tool in astronomy. It is known as gravitational lensing and is observed with light bending around galaxies. Multiple images of the same more distant galaxy can be seen placed around a galaxy.

Black Holes

The equations of general relativity are notoriously difficult to solve but many solutions in cases of symmetry do exist. The first significant such solution was due to Schwarzschild. To study stars with general relativity he found a solution with spherical symmetry. The solution had a anomalous result; at some small radius, the solution went haywire. Although this baffled people at first, it eventually was accepted as physically significant and it became what is now known as a black hole. A black hole is where gravity is so strong that not even light can escape. In Schwarzschild’s solution the mass of a star collapses to a point, a singularity, which is infinitely dense. At some finite distance from the singularity there is the point beyond which nothing, even light can escape. This is called the Schwarzschild radius.

It turns out that we can, essentially by luck, get the correct expression for the Schwarzschild radius by solving for the point with Newtonian gravity where the escape speed is c. From Physics I, the escape speed from the surface of a planet of mass M and radius R is

$$v_{\text{escape}} = \sqrt{\frac{2\,G\,M}{R}}$$

Setting $v_{\text{escape}} = c$ and solving for R gives the Schwarzschild radius.

$$c = v_{\text{escape}} = \sqrt{\frac{2\,G\,M}{R}} \implies R = \frac{2\,G\,M}{c^2}$$

Any gravitating body that collapses beyond this point becomes a black hole.

Example J.14 - The Earth or Sun as a Black Hole

(a) If an object the mass of the earth were to collapse to a black hole, what would be its Schwarzschild radius?

Solution

$$G = 6.673 \times 10^{-11} \frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}, \; M_{\text{earth}} = 5.97 \times 10^{24}\ \text{kg} \;\; \text{and} \;\; c = 3.00 \times 10^8\ \text{m/s}$$

This gives a very small result.

$$R = \frac{2\,G\,M}{c^2} = \frac{2\,G\,M_{\text{earth}}}{c^2} = 8.85\ \text{mm}$$

(b) Repeat this with the mass of the sun.

Solution

$$M_{\text{sun}} = 1.99 \times 10^{30}\ \text{kg}$$

This gives a realistic result for a black hole because most black holes are formed by the gravitational collapse of stars in a supernova, however stars do need to be a bit more massive than the sun for such a full collapse

$$R = \frac{2\,G\,M}{c^2} = \frac{2\,G\,M_{\text{sun}}}{c^2} = 2950\ \text{m}$$

Cosmology

It is quite audacious to mathematically solve for a universe. However, with general relativity we can find solutions describing the large-scale structure of the universe. There are many open questions in astronomy but general relativity is an essential tool for studying them. When Einstein first tried to apply general relativity to the universe as a whole, he had a significant problem with his solution. He could not model a static universe with it. He was so troubled by this that he added an extra fudge factor to general relativity, known as the cosmological constant, to allow for static solutions. Several years later, the astronomer Edwin Hubble discovered the universe was expanding. Einstein referred to the introduction of the cosmological constant as “his biggest mistake”; he could have predicted the expanding universe and the big bang but instead lost confidence in his own beautiful equations.

Gravity Waves

Just as the equations of electromagnetism give rise to solutions describing electromagnetic waves, the equations of general relativity give solutions of gravity waves. Although electromagnetic waves are easy to both create and detect, gravity waves are difficult to both create and detect. Creating significant gravity waves involves cataclysmic astrophysical events, like colliding black holes. Detecting gravity waves is even more difficult but gravity waves were first detected in 2016 and that merited the 2017 Physics Nobel prize. What is remarkable about this gravity wave detection is that it is not merely seeing some small effect. It has now become a new and important tool for studying astrophysics.