Chapter A

Units and Dimension

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A.1 - Introduction

When we give a value to something, like distance, time, mass or velocity, we must specify the appropriate units. Suppose we wanted to add two lengths in different units, for example 2 m + 3 ft. Clearly, since the units are different, we cannot just add the numbers but the expression does make sense; we could first convert ft to m (or m to ft) and then combine the two numbers.

Suppose instead we were asked to add 2 m + 3 s. This is a meaningless expression; we cannot add a length to a time. In the first case we added two quantities with different units but the same dimensions, two lengths, and the expression made sense. The second case consists of adding quantities of different units and different dimensions, a length and a time; this is meaningless and the expression is considered incorrect.

A.2 - Dimensional Analysis

The three fundamental dimensions are length L, mass M and time T. To give the dimension of a variable we use square brackets. For example, if x is some distance then it is dimensionally a length, and we write: $[x] = \mathrm{L}$. A velocity is length per time, so if v is a velocity we would write $[v] = \mathrm{L}/\mathrm{T}$.

If two quantities are equal they must be dimensionally equal. Moreover, as illustrated previously, if we add two quantities they must be dimensionally the same. If $\alpha + \beta = \gamma$ is a correct expression then it must follow that the three variables have the same dimension.

$$\alpha + \beta = \gamma \implies [\alpha] = [\beta] = [\gamma]$$

It is also the case that the dimension of the product of two variables is the product of their dimensions:

$$[\alpha\,\beta] = [\alpha]\,[\beta].$$

A dimensionless quantity is some number, like π, 2 or $\sqrt{3}$, that would be expressed without units. If κ is dimensionless we write $[\kappa] = 1$, since multiplying by a dimensionless quantity doesn't change the dimension of a quantity.

The condition that our expressions must be dimensionally correct puts a constraint on the final form of our expressions. In fact, in many cases dimensional analysis will uniquely determine our expressions up to multiplicative dimensionless constants. Suppose one is trying to recall an expression for the surface area of a sphere in terms of its radius. An area A is dimensionally a length squared and a radius r is a length. It follows that the expression for the area in terms of the radius must have the general form:

$$[A] = \mathrm{L}^2 \text{ and } [r] = \mathrm{L} \implies A = \kappa\, r^2 \text{ where } [\kappa] = 1.$$

In the case the dimensionless constant is $\kappa = 4\,\pi$.

Example A.1 - Dimensional Analysis

As an example of this suppose we want to find an expression for a distance $[x] = \mathrm{L}$ in terms of an acceleration $[a] = \mathrm{L}/\mathrm{T}^2$ and a speed $[v] = \mathrm{L}/\mathrm{T}$. We can choose a general form of this expression as:

$$x = \kappa\, a^m\, v^n \text{ where } [\kappa] = 1.$$

Solution

We can solve for m and n using dimensional analysis by insisting the expression is dimensionally correct.

$$[x] = [\kappa]\,[a]^m\,[v]^n \implies \mathrm{L} = 1\cdot\left(\mathrm{L}\cdot\mathrm{T}^{-2}\right)^m\left(\mathrm{L}\cdot\mathrm{T}^{-1}\right)^n = \mathrm{L}^m\,\mathrm{T}^{-2\,m}\,\mathrm{L}^n\,\mathrm{T}^{-n}$$
$$\implies \mathrm{L}^1\,\mathrm{T}^0 = \mathrm{L}^{m+n}\cdot\mathrm{T}^{-2\,m-n}$$

For the expression to be dimensionally correct each side of the equation must have the same number of length dimensions and the same number of time dimensions. This gives us two linear equations for our two unknowns m and n.

$$1 = m + n \text{ and } 0 = -2\,m - n$$

Solving gives $m = -1$ and $n = 2$ so we can conclude that:

$$x = \kappa \frac{v^2}{a} \text{ where } [\kappa] = 1.$$

A.3 - Systems of Units

Two quantities with the same dimension can have different units. For instance, we can measure length in ft or in m. We will soon discuss generally how to convert between different units. To avoid excessive conversions we use systems of units. Within a system if two quantities have the same dimension they will have the same units. By using a system we can avoid carrying through units while performing difficult calculations.

The metric system was introduced at the time of the French revolution. The intent was to base units on powers of ten and not on obscure factors like 1 ft = 12 in. Within the metric system there are two systems we will consider. For the most part this semester we will use the SI (Systeme International) system, which is also known as the MKS system. Here lengths are measured in m = meter, masses are in kg = kilogram and time is measured in s = second. In the CGS system we use cm = centimeter, g = gram and s. The British Engineering system (BE) uses for length, mass and time dimensions: ft, slugs and s. Note that lb = pound are units of force (or weight) and not a unit of mass.

The following table shows different quantities, their dimensions and units, in each of our three systems. The three fundamental dimensions are Length, Mass and Time, and within a system the corresponding units are the fundamental units. In addition to the fundamental units there are derived units.

Quantity	Dimension	SI Unit	CGS Unit	BE Unit
Length	L	m	cm	ft
Mass	M	kg	g	slug
Time	T	s	s	s
Velocity	L/T	m/s	cm/s	ft/s
Acceleration	L/T^2	m/s^2	cm/s^2	ft/s^2
Force ($F = m\,a$)	$M\,L/T^2$	N $= kg \cdot m/s^2$ (newton)	dyne $= g \cdot cm/s^2$	lb $=$ slug$\cdot ft/s^2$ (pound)
Work or Energy ($W = F\,d$)	$M\,L^2/T^2$	J = N·m (joule)	erg = dyne·cm	ft·lb
Power ($\mathcal{P} = W/\text{time}$)	$M\,L^2/T^3$	W = J/s (watt)	erg/s	ft·lb/s
Pressure ($P = F/\text{Area}$)	$M/(L \cdot T^2)$	Pa $= N/m^2$ (pascal)	dyne$/cm^2$	lb/ft^2

When using a system of units we only need to consider units at the beginning and end of every problem. At the start of a problem we convert all relevant quantities into our system. The calculation can then be performed without considering units; if an equations is correct it must be dimensionally correct and the units will automatically work out. At the end of a problem we then need to restore the appropriate units for the quantity that was calculated.

A.4 - Conversion of Units

Conversion of units is typically a straightforward matter. There are more complicated cases where a systematic approach to conversions is needed. The idea here is simple; to convert we multiply by one, in some form. If $a = b$ is some conversion factor then $a/b = 1$.

Example A.2 - Write *g* in terms of miles and hours

The acceleration due to gravity is $9.80\ m/s^2$. Convert this to mi/hr^2.

Solution

We need to convert from m to mi and from s to hr. For the length conversion we can combine conversions from m to km and then from km to mi.

$$1\ \text{km} = 1000\ \text{m} \text{ and } 1\ \text{mi} = 1.609\ \text{km} \implies 1\ \text{mi} = 1609\ \text{m}$$

We can similarly combine two conversions to get a s to hr conversion.

$$1\ \text{hr} = 60\ \text{min and } 1\ \text{min} = 60\ \text{s} \implies 1\ \text{hr} = 3600\ \text{s}$$

Since in $9.80\ \text{m}/\text{s}^2$, m is in the numerator we must multiply by a form of one where m is in the denominator. Similarly, since s is in the denominator and squared, we must multiply by a form of one with s in the numerator and since it is squared, we must square the conversion.

$$9.80\ \frac{\text{m}}{\text{s}^2} \times \frac{1\ \text{mi}}{1609\ \text{m}} \times \left(\frac{3600\ \text{s}}{1\ \text{hr}}\right)^2 = 78\,900\ \frac{\text{mi}}{\text{hr}^2}$$

A.5 - The SI System and Fundamental Constants

The second, meter and kilogram

The SI base units we will use most are seconds, meters and kilograms. Over the years these definitions evolved to allow for more precise measurements.

The original definition of a second was in terms of a day, $60 \times 60 \times 24$ seconds is one day and a day is the average time for each rotation of the earth relative to the sun. This was not appropriate for accurate measurements and the second was redefined in terms of the current most accurate way of measuring time, atomic clocks. The most common atomic clock is based on a radiation frequency of a "hyper-fine structure" transition in the common isotope of Cesium, ^{133}Cs. Since 1967, a second has been defined as exactly 9192631 cycles of this radiation frequency.

Originally, the meter was defined in terms of the dimensions of the earth; the distance from the north pole to the equator (along a meridian through Paris) was 10 000 kilometers. This was, of course, difficult to reproduce, so the standard meter was converted to the distance between two marks on a metal bar at some fixed temperature. As science progressed, and scientific precision increased, a better standard was needed. Early in the twentieth century it was redefined in terms of a fixed number of wavelengths of a specific emission line of a krypton atom. Since 1983, the modern definition is now in terms of the speed of light. We define the speed of light to be some exact value, given below, and that then defines the meter in terms of the atomic clock definition of a second.

The original definition of the gram and kilogram was in terms of lengths and properties of water; a cubic centimeter of water was one gram. For more precision, a standard kilogram was created; some artifact that defined the kilogram. This artifact changed over time but the use of some artifact as a standard kilogram persisted until 2019.

The 2019 Redefinition of the SI System

In May 2019 the SI system of units was redefined. By choosing an exact value for the speed of light we were able to define a meter. Now all SI base units are defined entirely in terms of fundamental constants whose values are chosen as exact. Planck's constant, which is now referred to officially as the Planck constant, is the fundamental constant of quantum physics. The table below shows its units involve meters, seconds and kilograms, so by choosing its value to be exact we can now define the kilogram. Although, theoretically, this could have been done when Planck first introduced his constant in 1900, as a practical matter there would have been no way to use this definition to accurate calibrate a scale to kilograms. Experimental advances late in the twentieth century made it possible to create such a scale to accurately measure kilograms; this scale is a very complicated apparatus referred to as a Kibble balance, named after its inventor.

Fundamental Constant Name	Symbol Constant	Exact Numerical Value of Constant	SI Units of Constant and Base Units	Defined Base Unit
HFS freq. ^{133}Cs	Δf_{Cs}	9 192 631	$\text{Hz} = \text{s}^{-1}$	s = second
Speed of Light	c	299 792 458	ms^{-1}	m = meter
Planck Constant	h	$6.62607015 \times 10^{-34}$	$\text{J} \cdot \text{s} = \text{kg} \cdot \text{m}^2\text{s}^{-1}$	kg = kilogram
Boltzmann Constant	k_{B}	1.380649×10^{-23}	$\text{J}/\text{K} = \text{kg} \cdot \text{m}^2\text{s}^{-2}\text{K}^{-1}$	K = kelvin
Avogadro Constant	N_{A}	$6.02214076 \times 10^{23}$	mol^{-1}	mol = mole
Elementary Charge	e	$1.60217634 \times 10^{-19}$	$\text{C} = \text{A} \cdot \text{s}$	A = ampere
Luminous Efficacy	K_{cd}	683	$\text{cd}/\text{W} = \text{cd} \cdot \text{kg}^{-1}\text{m}^{-2}\text{s}^3$	cd = candela

In our discussion of thermodynamics at the end of the semester we will introduce two more SI base units: the kelvin, which is an absolute temperature scale, and the mole which is a very basic unit in chemistry. Now the kelvin is defined by choosing the Boltzmann constant to have an exact value and the mole is defined by setting the value of the Avogadro constant (the new name for Avogadro's number) to an exact value.

The other two base SI units are the ampere and the candela. The ampere is the SI unit of electric current and will be central to the Physics II discussion of electromagnetism; it is now defined by choosing the elementary charge e to an exact value. (Note that the electron's charge is $-e$ and the protons charge is $+e$.) The candela is unimportant for our elementary physics courses; it is a measure of the total light output of a source and its constant, the luminous efficacy, relates the total brightness to the power output of radiant energy.

Some Other Units

The atomic mass unit u is related to the Avogadro constant, $N_A\,\mathrm{u} = 1\ \mathrm{g/mol}$. Previously, before 2019, u was defined using the carbon-12 standard, where the mass of $^{12}\mathrm{C}$ was defined as exactly 12 u and the Avogadro constant was then defined in terms of that. Now the Avogadro constant is exact and carbon-12 is now approximately 12 u.

The Celsius scale is now defined in terms of the Kelvin scale, where temperature differences in the two scalse are the same, one celsius-degree equals one kelvin, and the zero of the scale (approximately the freezing point of water at one atmosphere) is defined as exactly 273.15 K.

Name	Constant	Numerical Value	SI Unit	Definition
Atomic Mass Unit	u	$10^{-26}/6.02214076$ (exact)	kg	u
Zero Celsius	0 °C	273.15 (exact)	K	celsius

Other 2019 Redefinitions

At some specific pressure, lower than atmospheric pressure, there is an exact temperature where all three phases of water, solid, liquid and gas, can coexist in equilibrium. This is known as the triple point of water. Previously, this triple point was defined as exactly 273.16 K = 0.01 °C, and this previously defined the kelvin. Now this 273.16 value is approximate and the kelvin is defined in terms of the Boltzmann constant.

The ampere was previously defined by choosing an electromagnetic constant that appears in the equations describing magnetism; the vacuum permeability μ_0 was simply defined as exactly $4\,\pi \times 10^{-7}\ \mathrm{N/A^2}$. Although conceptually this was a very clean way to define the ampere, it turns out that this was impractical for making very precise current measurement. Now the value of μ_0 is approximate and the ampere is defined in terms of the elementary charge. There is another electromagnetic constant ε_0, the vacuum permittivity, that can be written in terms of μ_0 and the speed of light; since μ_0 is no longer exact, neither is ε_0.

Name	Constant	Numerical Value	SI Unit
Triple Point of Water	$T_{\mathrm{tp,H_2O}}$	273.16 (approximate)	K
Vacuum Permeability	μ_0	$4\pi \times 10^{-7}$(approximate)	$\mathrm{N/A^2 = kg \cdot m^2 s^{-2}}$
Vacuum Permittivity	ε_0	$1/(\mu_0\,c^2)$	–