Chapter C

Vectors and Two Dimensional Kinematics

Blinn College - Physics 2325 - Terry Honan

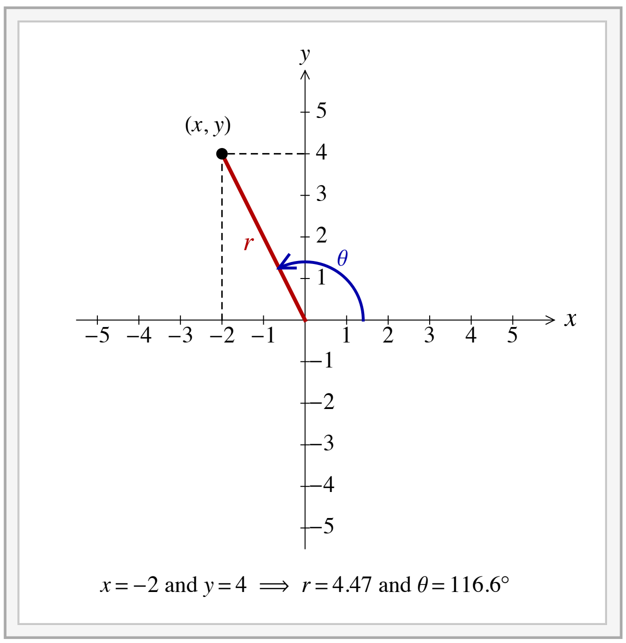
C.1 - Vector Algebra - I

Polar Coordinates

(x, y) are the Cartesian (or rectangular) coordinates of some point on a plane. r and θ are the polar coordinates; $r = \sqrt{x^2 + y^2}$ is the distance from the origin to the point and θ is the angle measured counterclockwise from the positive x axis to the point. The definitions of the trig functions, for general angles, are given by

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r} \quad \text{and} \quad \tan\theta = \frac{y}{x}.$$

Using the above definitions it is a straightforward matter to find the formulas for converting between polar and rectangular coordinates.



Interactive Figure

r and θ $\Longrightarrow$ x and y

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \tag{C.1}$$

x and y $\Longrightarrow$ r and θ

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) \text{ when } & x > 0 \\ 180\,° + \tan^{-1}\left(\frac{y}{x}\right) \text{ when } & x < 0 \end{cases} \tag{C.2}$$

The subtlety in solving for the angle follows from the identity: $\tan\theta = \tan(180\,° + \theta)$. The range of the $\tan^{-1}$ (also written arctan) function is $-90\,° < \theta < 90\,°$, which corresponds to the quadrants I and IV; this is equivalent to the condition that $x > 0$. The case of quadrants II and III, when $x < 0$, requires shifting the result of the inverse tangent by $180\,°$.

Example C.1 - Polar Coordinates

Consider the point $(x, y) = (-2, 4)$ in the Cartesian plane. Find the polar coordinates r and θ of this point.

Solution

This is a straightforward application of the formulas above, where $x = -2$ and $y = 4$.

$$r = \sqrt{x^2 + y^2} = \sqrt{20} = 4.47$$

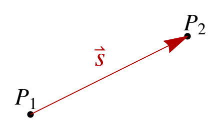
$$\theta = 180^\circ + \tan^{-1}\left(\frac{y}{x}\right) = 116.6^\circ$$

Vector Basics

A vector is a quantity with both a magnitude and a direction. A scalar has only a magnitude; it is just a real number. The magnitude of a vector is a non-negative (positive or zero) scalar. Velocity is a vector quantity and speed is its magnitude. Acceleration, force and momentum are also vectors. Time, temperature, mass and pressure are examples of scalars.

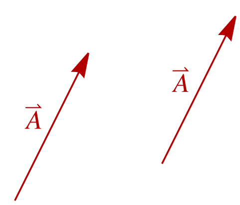
We will write a vector variable by a symbol with an "⇀" over it. The magnitude of a vector is given by the symbol without the arrow or by applying the "|| ||" brackets to the vector. If $\vec{A}$ is some vector then $A = \|\vec{A}\|$ is its magnitude.

We can represent vectors by arrows. Suppose someone walks from a starting point P_1 to a stopping point P_2. A displacement vector $\vec{s}$ (or $\Delta\vec{r}$) may be viewed as an arrow with its tail at P_1 and its tip at P_2.



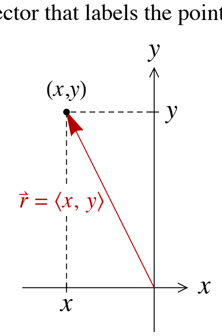
A position vector $\vec{r}$ is a displacement vector with its tail at the origin. For more general vectors we represent them by arrows pointing in the direction of the vectors and the length of the arrow is proportional to the magnitude of the vector. For instance, if one velocity vector has a magnitude of 60 mi/hr and another has 30 mi/hr then the arrow representing the first should have twice the length of the second.

A vector has no fixed position. If a vector arrow is moved keeping its length and direction fixed then it still is the same vector.



Component Definition - Position and Displacement Vectors

A position vector is a way to label a position in the Cartesian plane; it has its tail at the origin and its head at the position it labels. We will use an angled bracket notation for vectors. The position vector that labels the point (x, y) will be written as $\vec{r} = \langle x, y \rangle$.

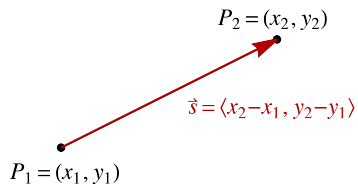


We will define the magnitude and direction of a vector so that the polar coordinates, r and θ, are the magnitude and direction of the vector.

In the Cartesian plane we will denote vector from (x_1, y_1) to (x_2, y_2) by:

$$\vec{s} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

This is called a displacement vector. A position vector is clearly a displacement vector with its tail at the origin.

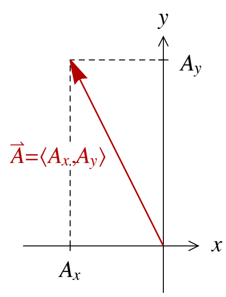


Component Definition - General Vectors

We will write a 2-dimensional vector $\vec{A}$ as a pair of real numbers A_x and A_y called components. (A 3D vector is a triple.) We will use the "angled-bracket" notation for vectors.

$$\vec{A} = \langle A_x, A_y \rangle$$

The component A_x has the interpretation as the amount the vector $\vec{A}$ is in the x-direction and A_y the y-direction.



Magnitude and Direction Angle

We define A_x and A_y as the components of the vector $\vec{A}$. A_x is the part of $\vec{A}$ in the x direction and similarly A_y is the y part. The Cartesian coordinates x and y are the components of a position vector $\vec{r}$. For a two dimensional vector we can represent the direction with an angle, measured as in the polar coordinates. To convert between the magnitude and direction angle and the components of a two dimensional vector we have analogous expressions to the ones for polar coordinates.

$$\boldsymbol{A \text{ and } \theta \implies A_x \text{ and } A_y}$$

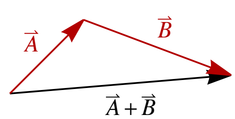
$$A_x = A\cos\theta \quad \text{and} \quad A_y = A\sin\theta \tag{C.3}$$

$$\boldsymbol{A_x \text{ and } A_y \implies A \text{ and } \theta}$$

$$A = \sqrt{{A_x}^2 + {A_y}^2} \quad \text{and} \quad \theta = \begin{cases} \tan^{-1}\left(\frac{A_y}{A_x}\right) & \text{when } A_x > 0 \\ 180^\circ + \tan^{-1}\left(\frac{A_y}{A_x}\right) & \text{when } A_x < 0 \end{cases} \tag{C.4}$$

Vector Addition

Suppose a displacement vector $\vec{s}_1$ corresponds to someone walking from P_1 to P_2. Suppose that then the person walks from P_2 to P_3; call this displacement $\vec{s}_2$. The net displacement is the vector from P_1 to P_3; this is what we will define as the sum of the two displacements $\vec{s}_1 + \vec{s}_2$. To generalize this to any vectors, we will define the sum of general vectors $\vec{A}$ and $\vec{B}$. Draw the vectors as shown, with the tail of $\vec{B}$ at the tip of $\vec{A}$. The sum the vectors $\vec{A} + \vec{B}$ is the vector drawn from the tail of $\vec{A}$ to the tip of $\vec{B}$.



With our component definition vector addition takes the very simple form:

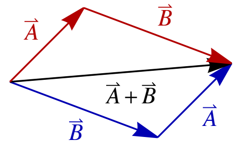
$$\vec{A} + \vec{B} = \langle A_x, A_y \rangle + \langle B_x, B_y \rangle = \langle A_x + B_x, A_y + B_y \rangle. \tag{C.5}$$

Commutative Property

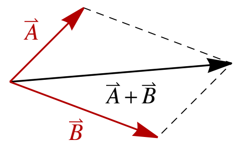
An algebraic operation is commutative when changing the order of the items doesn't affect the result. For vector addition this takes the form.

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

The commutative property of the addition of reals implies this for vectors.



Because of the commutative property there are two more ways of adding vectors we can consider. In addition to placing the tail of $\vec{B}$ at the tip of $\vec{A}$, we can place the tail of $\vec{A}$ at the tip of $\vec{B}$. Also there is the parallelogram rule: Draw the two vector together tail to tail and complete the parallelogram; the sum it the vector from the common tail of the vectors to the opposite corner.

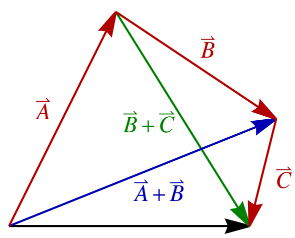


Associative Property

From the definition of vector addition it is clear it satisfies.

$$\left(\vec{A} + \vec{B}\right) + \vec{C} = \vec{A} + \left(\vec{B} + \vec{C}\right)$$

This property is called associativity. For an associative operation it is not necessary to use brackets, since the order of the operation is unimportant.



$(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$

Identity

The identity vector $\vec{0}$ is the vector that leaves any other vector $\vec{A}$ unchanged under addition.

$$\vec{A} + \vec{0} = \vec{A}$$

It is clear that the zero vector has zeros as components.

$$\vec{0} = \langle 0, 0 \rangle$$

The magnitude of the identity vector is 0, $0 = \|\vec{0}\|$. Note that the direction of $\vec{0}$ is undefined; in fact, it is the only vector with an undefined direction.

Additive Inverse.

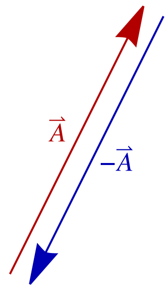
For any vector $\vec{A}$ there is an additive inverse vector $-\vec{A}$ with the property:

$$\vec{A} + \left(-\vec{A}\right) = \vec{0}$$

Clearly, this has the value

$$-\vec{A} = \langle -A_x, -A_y \rangle$$

and has the same magnitude and is in the opposite direction.



Vector Subtraction

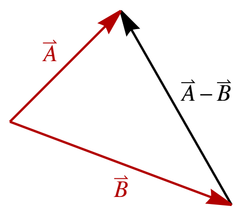
We define vector subtraction by adding the inverse.

$$\vec{A} - \vec{B} = \vec{A} + \left(-\vec{B}\right)$$

In terms of components we have

$$\vec{A} - \vec{B} = \langle A_x, A_y \rangle - \langle B_x, B_y \rangle = \langle A_x - B_x, A_y - B_y \rangle.$$

The simplest way to view the vector $\vec{A} - \vec{B}$ is as the vector that when added to $\vec{B}$ gives $\vec{A}$; if the vectors are drawn tail to tail then it is the vector from the tip of $\vec{B}$ to the tip of $\vec{A}$



Scalar Multiplication

If $\vec{A}$ is a vector and c is a scalar then we can define their product $c\,\vec{A}$ as a vector.

$$c\,\vec{A} = \langle c\,A_x, c\,A_y \rangle \tag{C.6}$$

It is clear that its magnitude is given by

$$\left\| c\,\vec{A} \right\| = |c|\;||\vec{A}||,$$

where |c| is the absolute value of the scalar. The direction of $c\,\vec{A}$ is the same as $\vec{A}$ when $c > 0$ and opposite to $\vec{A}$ when $c < 0$. When $c = 0$ we get $0\,\vec{A} = \vec{0}$. Note also that $1\,\vec{A} = \vec{A}$ and $(-1)\,\vec{A} = -\vec{A}$.

The scalar multiplication operation has the associative and distributive properties.

Associative Property

$$(c\,d)\,\vec{A} = c\left(d\,\vec{A}\right)$$

Distributive Properties

$$(c + d)\,\vec{A} = c\,\vec{A} + d\,\vec{A} \text{ and } c\left(\vec{A} + \vec{B}\right) = c\,\vec{A} + c\,\vec{B}$$

Unit Vectors and Notation

A unit vector is a vector of magnitude one. We denote unit vectors with a " ^ " over its top. For any vector $\vec{A}$ we can simply find the unit vector in its direction $\hat{A}$ by

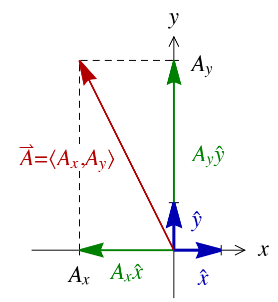
$$\hat{A} = \frac{\vec{A}}{||\vec{A}||}.$$

Basis unit vectors are unit vectors along the coordinate axes. We will use $\hat{x}$ and $\hat{y}$ for the unit vectors in the x and y directions. The more traditional notation for these unit vectors is to write them as $\hat{\imath}$ and $\hat{\jmath}$.

$$\hat{x} = \hat{i} = \langle 1, 0 \rangle \text{ and } \hat{y} = \hat{j} = \langle 0, 1 \rangle$$

Any vector can then be written in terms of these basis unit vectors

$$\vec{A} = \langle A_x, A_y \rangle = A_x \hat{x} + A_y \hat{y}$$



3D Vectors

It is straightforward to generalize our two dimensional discussion to a three dimensional one. We need to add $\hat{z} = \hat{k}$ for the unit vector in the z direction.

$$\hat{x} = \hat{i} = \langle 1, 0, 0 \rangle\,, \quad \hat{y} = \hat{j} = \langle 0, 1, 0 \rangle \text{ and } \hat{z} = \hat{k} = \langle 0, 0, 1 \rangle$$

We can similarly write any vector in terms or its components and unit vectors

$$\vec{A} = \langle A_x, A_y, A_z \rangle = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}.$$

The magnitude of a vector is

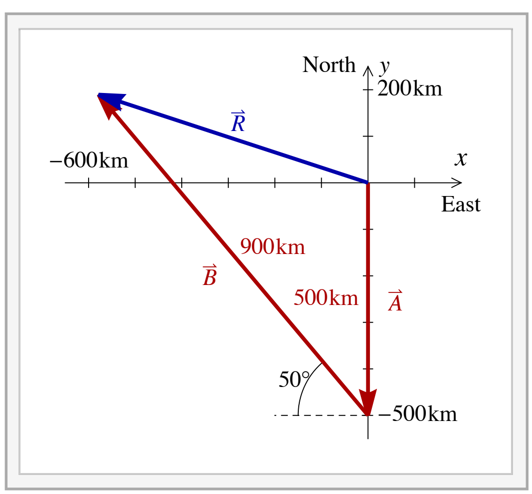
$$A = ||\vec{A}|| = \sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2}\,.$$

The only subtlety with three dimensions is specifying directions as angles. It takes two angles to specify a direction is 3D; these are the θ and ϕ of spherical coordinates. To avoid these issues the best way to represent a direction for a three dimensional vector is to just write a unit vector in the direction of that vector

$$\hat{A} = \frac{\vec{A}}{||\vec{A}||}.$$

Example C.2 - Vector Addition

A plane flies 500 km to the south and then 900 km in the direction 50° north of west. What is the plane's net displacement? What are the magnitude and direction angle of the plane's net displacement?



Interactive Figure

Solution

Here we are adding two vectors, starting with magnitude and direction information. To find the components using equation

(C.3) we must identify the direction angles which are measured counterclockwise from the positive x direction, which we take to be East. Vector $\vec{A}$ is to the South so it has direction angle $-90\,°$ and it has magnitude 500 km. Although we could use (C.3) find A_x and A_y it is easier to see that since it is purely in the negative y direction:

$$\vec{A} = \langle A_x,\ A_y\rangle = \langle 0,\ -500\ \text{km}\rangle\ .$$

The direction angle for vector $\vec{B}$ is $\theta_B = 180\,° - 50\,° = 130\,°$. (See the interactive diagram above.) The magnitude of $\vec{B}$ is $B = 900$ km so using (C.3) we get

$$\vec{B} = \langle B_x,\ B_y\rangle = \langle B\cos\theta_B,\ B\sin\theta_B\rangle = \langle -578.51,\ 689.44\rangle\ \text{km}\ .$$

The net displacement is sum of these two vectors, the resultant vector $\vec{R} = \vec{A} + \vec{B}$.

$$\begin{aligned}\vec{R} = \vec{A} + \vec{B} &= \langle R_x,\ R_y\rangle\\ &= \langle 0,\ -500\rangle\ \text{km} + \langle -578.51,\ 689.44\rangle\ \text{km}\\ &= \langle -578.51,\ 189.44\rangle\ \text{km}\end{aligned}$$

Using (C.4) we can find the Magnitude and direction angle of the net displacement.

$$R = \sqrt{R_x^2 + R_y^2} = 609\ \text{km} \qquad \text{(Magnitude)}$$

$$\theta = 180\,° + \tan^{-1}\frac{R_y}{R_x} = 161.9\,° \qquad \text{(Direction Angle)}$$

The net displacement is just $\vec{R}$.

$$\vec{R} = \langle -579,\ 189\rangle\ \text{km} \qquad \text{(Net Displacement)}$$

C.2 - Kinematics in 2D and 3D

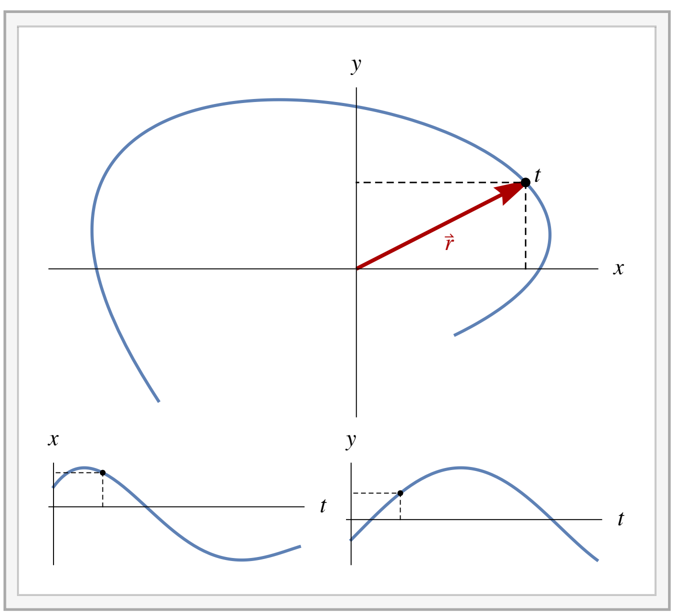
The General Problem

Position as a Function of Time

There are two equivalent ways of describing position in two dimensions. One is by labeling the coordinates $(x,\ y)$. The other is by giving the position vector $\vec{r}$ of the position. The two are related; the coordinates are the components of the position vector. To label position as a function of time we can consider x and y as separate functions of time or as $\vec{r}$ as a function of time.

$$x(t)\ \text{and}\ y(t) \iff \vec{r}(t) = \langle x(t),\ y(t)\rangle$$

The actual path followed by the body is called the trajectory. It isrepresented by a plot of a path in the xy plane.



Interactive Figure

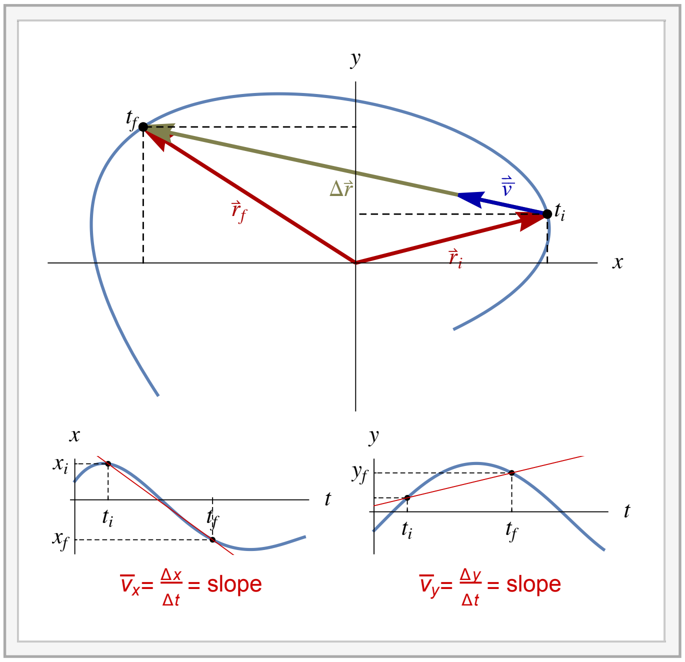
Average Velocity

The average velocity, as we saw in the one dimensional case, refers to two times. At t_i the position vector is $\vec{r}_i = \vec{r}(t_i)$ and at t_f it is $\vec{r}_f = \vec{r}(t_f)$. The displacement is the difference of these two positions

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i = \langle x_f - x_i,\, y_f - y_i\rangle.$$

The average velocity vector is then defined as

$$\overline{\vec{v}} = \frac{\Delta\vec{r}}{\Delta t} = \left\langle \frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}\right\rangle = \langle \overline{v}_x, \overline{v}_y\rangle. \tag{C.7}$$



Interactive Figure

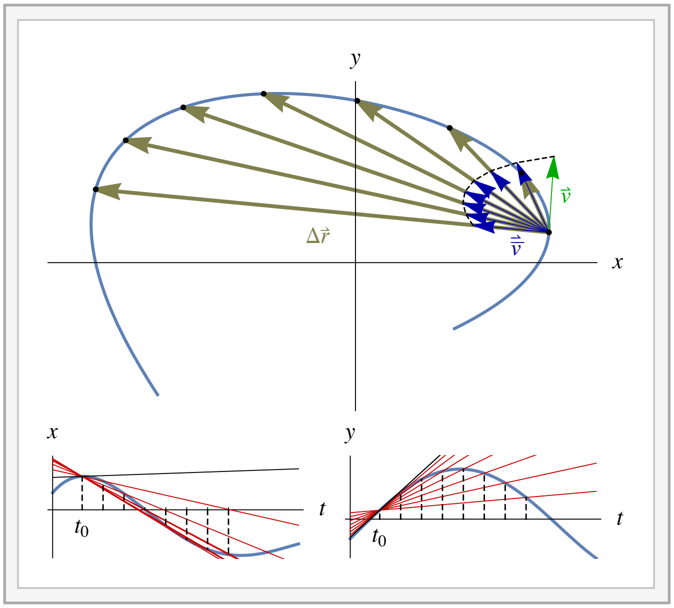
Instantaneous Velocity

The instantaneous velocity is defined as the limit of the average velocity as Δt approaches zero; it is the time derivative of the position vector.

$$\vec{v} = \frac{d\vec{r}}{dt} = \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle = \langle v_x, v_y\rangle. \tag{C.8}$$

The magnitude of the velocity is called the speed.

$$v = \|\vec{v}\| = \sqrt{v_x^2 + v_y^2} = \text{speed} \tag{C.9}$$



Interactive Figure

Average Acceleration

As with the one dimensional case acceleration is to velocity as the velocity is to position.

$$\vec{\bar{a}} = \frac{\Delta \vec{v}}{\Delta t} = \left\langle \frac{\Delta v_x}{\Delta t}, \frac{\Delta v_y}{\Delta t} \right\rangle = \langle \bar{a}_x, \bar{a}_y \rangle. \tag{C.10}$$

Instantaneous Acceleration

The instantaneous acceleration is similarly defined as a limit of the average acceleration or simply as the time derivative of the velocity.

$$\vec{a} = \frac{d\vec{v}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \left\langle \frac{d v_x}{dt}, \frac{d v_y}{dt} \right\rangle = \langle a_x, a_y \rangle. \tag{C.11}$$

Acceleration with Constant Speed

The acceleration is the time derivative of the velocity. Constant velocity implies zero acceleration. Speed is the magnitude of the acceleration so there can be an acceleration if the speed is constant, because the direction is changing. If the speed is constant then the acceleration must be perpendicular to the direction of motion, which is the direction of the velocity.

$$\text{constant speed} \iff \vec{a} \perp \vec{v}$$

If a car is driving at a constant speed and turning left then the acceleration is toward the left.

Parallel and Perpendicular Components of Acceleration

If the speed is changing then there is a forward or back component of the acceleration. If the speed is increasing there is a forward component of the acceleration and if the speed is decreasing there is a backward component. If the speed and direction are changing then you have both components parallel and perpendicular to the direction of motion. Suppose a car is braking while turning left. There are then two components of the acceleration, the parallel component points backward and the perpendicular component points to the right.

Constant Velocity and Acceleration

The cases of constant velocity and acceleration follows the one dimensional example. Since we are now dealing with vectors the arbitrary constants introduced in antidifferentiation become vector quantities.

$$\text{const}\,\vec{v} \implies \vec{a} = \vec{0} \text{ and } \vec{r}(t) = \vec{r}_0 + \vec{v}\,t$$

$$\text{const}\,\vec{a} \implies \vec{v}(t) = \vec{v}_0 + \vec{a}\,t \text{ and } \vec{r}(t) = \vec{r}_0 + \vec{v}_0\,t + \frac{1}{2}\vec{a}\,t^2$$

Projectile Motion

An important case of motion with constant acceleration is that of projectile motion. Projectile motion is two dimensional motion of body under the influence of only gravity. Insisting on only gravity means that we ignore air resistance. Spin effects like a curving baseball or a slicing golf ball are associated with air resistance and will also be ignored.

For 2D motion with constant acceleration we can modify the four constant acceleration equations into two sets of four equations.

x Equations	y Equations
$v_x = v_{0x} + a_x\, t$	$v_y = v_{0y} + a_y\, t$
$\Delta x = \frac{1}{2}(v_x + v_{0x})\, t$	$\Delta y = \frac{1}{2}(v_y + v_{0y})\, t$
$\Delta x = v_{0x}\, t + \frac{1}{2}\, a_x\, t^2$	$\Delta y = v_{0y}\, t + \frac{1}{2}\, a_y\, t^2$
$v_x^2 = v_{0x}^2 + 2\, a_x\, \Delta x$	$v_y^2 = v_{0y}^2 + 2\, a_y\, \Delta y.$

For projectile motion we will take x as the horizontal direction and y as the vertical direction. The acceleration due to gravity is downward and of magnitude g. Writing this as a vector gives

$$\vec{a} = -g\,\hat{y} = \langle 0, -g\rangle \quad \text{or} \quad a_x = 0 \text{ and } a_y = -g$$

Inserting these components into the two sets of equations above gives:

Horizontal Equations	Vertical Equations
$v_x = v_{0x}$	$v_y = v_{0y} - g\, t$
$\Delta x = v_{0x}\, t$	$\Delta y = \frac{1}{2}(v_y + v_{0y})\, t$
	$\Delta y = v_{0y}\, t - \frac{1}{2}\, g\, t^2$
	$v_y^2 = v_{0y}^2 - 2\, g\, \Delta y.$

(C.12)

The horizontal motion is simple. Since there is no horizontal (the x direction) acceleration, the x component of the velocity is constant. The vertical part (the y direction) of the motion is equivalent to free fall. The key to solving projectile motion problems is keeping the two parts separate.

If a projectile is launched at an initial angle of θ with an initial speed v_0 then the components of the initial velocity are given by

$$v_{0x} = v_0 \cos\theta \text{ and } v_{0y} = v_0 \sin\theta.$$

Example C.3 - An Initial Horizontal Velocity

A spring gun shoots a small ball with an initial horizontal velocity from a height of 1.35 m at the same time an identical ball is dropped from the same height. The ball that was shot lands a horizontal distance of 2.42 m from its initial position.

(a) Which ball hits the ground first?

Solution

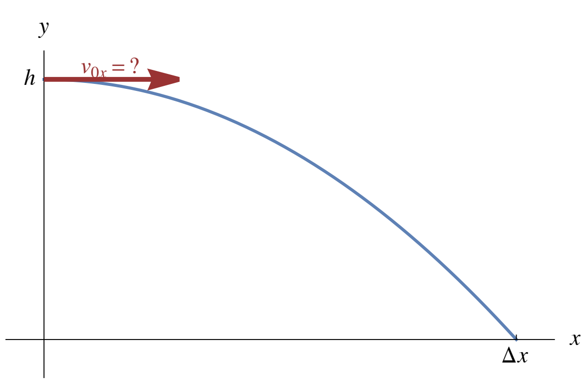
For both balls, the y-component of the initial velocity is zero. The shot ball's velocity has an initial x-component and the dropped ball doesn't. The time of fall is determined only by the vertical equations and the vertical motion of the two is identical. It follows that the two balls hit the floor at the same time.

(b) What is the initial speed of the shot ball?

Solution

First define our variables in terms of the given information.

$$h = 1.35\text{ m}, \quad \Delta x = 2.42\text{ m and } g = 9.80\text{ m}/\text{s}^2$$



The initial speed is also the x-component of the initial velocity.

$$v_0 = \sqrt{v_{0x}^2 + v_{0y}^2} = \sqrt{v_{0x}^2 + 0^2} = v_{0x}$$

First we solve for its time of fall using the vertical equations using $\Delta y = -h$.

$$\Delta y = v_{0y}\, t - \frac{1}{2} g\, t^2 \implies -h = 0 - \frac{1}{2} g\, t^2 \implies h = \frac{1}{2} g\, t^2 \implies t = \sqrt{\frac{2\,h}{g}} = 0.52489\ \text{s}$$

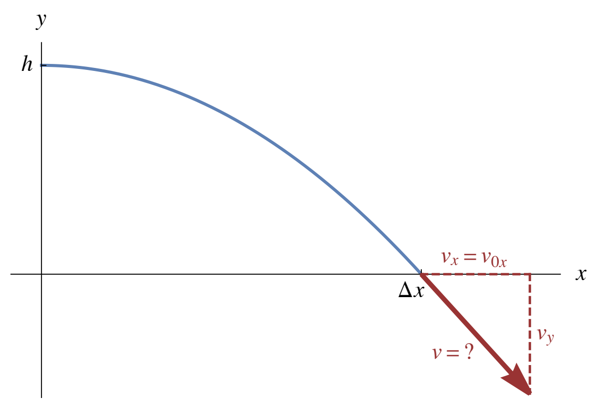
From this we can use the horizontal equation to get v_{0x}.

$$\Delta x = v_{0x}\, t \implies v_{0x} = \Delta x / t = 4.67\ \text{m/s}$$

This is a very standard physics problem. In this class of projectile problems with an initial horizontal velocity, there are three variables: h, Δx and v_{0x}. You are two and asked for the third.

(c) What was the speed of the ball when it hit the floor?

Solution



The x-component of the balls velocity stays constant so $v_x = v_{0x}$. The speed is the magnitude of the velocity.

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{v_{0x}^2 + v_y^2}$$

Using the time and the vertical equations we can find the y-component of the velocity.

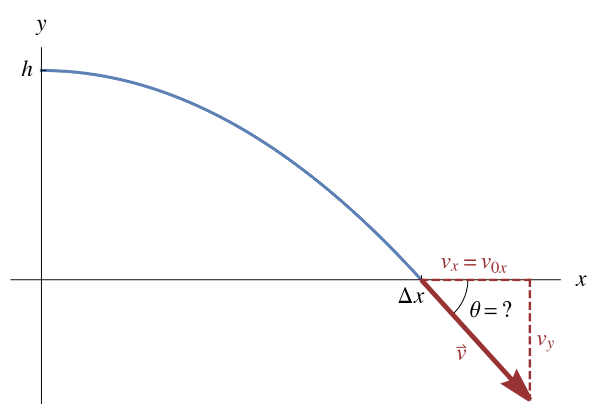
$$v_y = v_{0y} - g\ t = -g\, t = -5.1438\ \text{m/s}$$

and from this we find the speed.

$$v = \sqrt{v_{0x}^2 + v_y^2} = 6.95\ \text{m/s}$$

(d) What was the ball's direction of motion when it hit the floor? Give the angle.

Solution

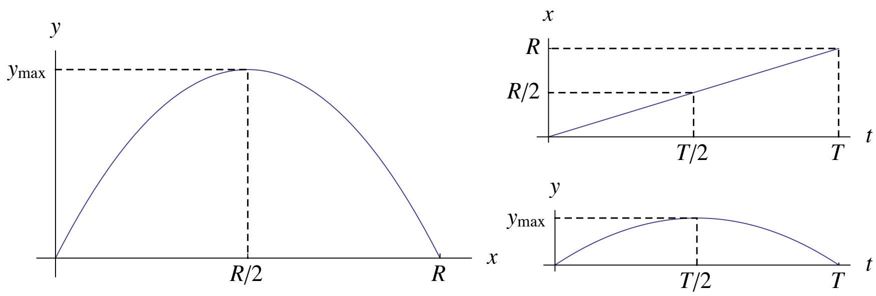


In part (c) we found the magnitude of the velocity. Here we want the direction angle of the same velocity. Recall that the direction of motion is the direction of the velocity vector.

$$\theta = \tan^{-1} \frac{v_y}{v_{0x}} = -47.7\,°$$

Example C.4 - Punted Football

A punter kicks a football from ground level with an initial speed of 27 m/s at an initial angle of 63 °. (Neglect all air resistance.)



(a) What is the maximum height reached by the football?

Solution

Begin with identifying what is given. We know the initial speed and initial angle.

$$v_0 = 27\ \mathrm{m/s} \ \text{ and } \ \theta = 63\,°$$

We can calculate the components of the initial velocity.

$$v_{0x} = v_0 \cos\theta = 12.258\ \mathrm{m/s} \ \text{ and } \ v_{0y} = v_0 \sin\theta = 24.057\ \mathrm{m/s}$$

The maximum height depends only on v_{0y}. At the highest point the velocity is purely horizontal, so $v_y = 0$. We are looking for $y_{\max}$ in the graphs above. $y_{\max} = \Delta y$ when $v_y = 0$. We need the fourth of the vertical projectile equations (C.12).

$$v_y^2 = v_{0y}^2 - 2\,g\,\Delta y \implies 0 = v_{0y}^2 - 2\,g\,y_{\max} \implies y_{\max} = \frac{v_{0y}^2}{2\,g} = 29.5\ \mathrm{m}$$

(b) What is the "hang-time" of the punt? (Note that hang-time is the total time the football is in the air, from the ground to the ground.)

Solution

Since the football ends at the same elevation where it began, so $\Delta y = 0$. We need to solve for time so the relevant formula is the third vertical projectile equation (C.12).

$$\Delta y = v_{0y}\,t - \frac{1}{2}\,g\,t^2 \implies 0 = v_{0y}\,t - \frac{1}{2}\,g\,t^2 = t\left(v_{0y} - \frac{1}{2}\,g\,t\right)$$

This gives two solutions but $t = 0$ is trivially true. We want the other. We will call this T as shown in the graphs above.

$$0 = v_{0y} - \frac{1}{2}\,g\,T \implies T = \frac{2\,v_{0y}}{g} = 4.9096\ \mathrm{s} = 4.91\ \mathrm{s}$$

(c) What is the horizontal range of the football? The range is the total horizontal distance the football travels in the air.

Solution

So far we have not used any horizontal information or equations. The *range* R is Δx when $\Delta y = 0$. Using the time T we just found in the second horizontal projectile equations (C.12).

$$R = \Delta x = v_{0x}\,t = v_{0x}\,T = 60.2\ \mathrm{m}$$

To solve for the trajectory we can choose, for simplicity, that the motion begins at the origin $x_0 = 0 = y_0$ giving $\Delta x = x$ and $\Delta y = y$. Solving the horizontal equation for time gives $t = x/v_{0x}$. Inserting this into the vertical expression $y = v_{0y}\,t - \frac{1}{2}\,g\,t^2$ gives

$$y = \frac{v_{0y}}{v_{0x}}\,x - \frac{g}{2\,v_{0x}^2}\,x^2.$$

It is clear that this is a parabola.

The *range* R of a projectile is the total horizontal distance traveled in the air when it returns to its original level, $y = 0$ in the expression above. We can then factor out an x from the expression and solve for x.

$$x = 0 \text{ and } x = \frac{2\, v_{0x}\, v_{0y}}{g} = \frac{2\, v_0 \cos\theta\, v_0 \sin\theta}{g} = \frac{v_0^2}{g} 2 \cos\theta \sin\theta.$$

The $x = 0$ solution is trivial. Setting the other to R, writing the components in terms of v_0 and θ, and using the identity $\sin 2\theta = 2 \sin\theta \cos\theta$ gives an expression for the range.

$$R = \frac{v_0^2}{g} \sin 2\theta \tag{C.13}$$

Example C.5 - Punted Football (continued)

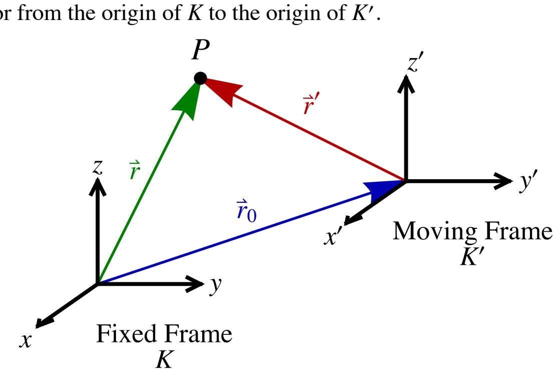
(d) Apply the range formula (C.13) to the football example to get the same answer as in part (c).

Solution

$$R = \frac{v_0^2}{g} \sin 2\theta = 60.2 \text{ m}$$

C.3 - Relative Motion

A frame of reference is some coordinate system used to study motion. Suppose there is a fixed frame K and a moving frame K', that moves with a velocity $\vec{v}_0$ with respect to K. If we take a moving body to be at point P. If $\vec{r}$ is the vector from the origin of K to P, $\vec{r}'$ is the vector from the origin of K' to P, and $\vec{r}_0$ is the vector from the origin of K to the origin of K'.



It follows that the three position vectors are related by

$$\vec{r} = \vec{r}' + \vec{r}_0.$$

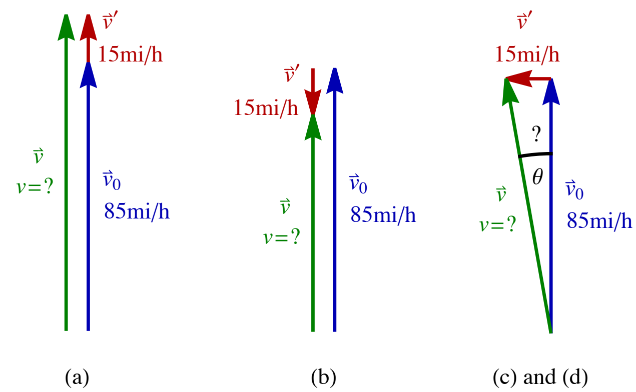
We want to relate the velocities of the moving body with respect to these two frames. The velocities with respect to K and $K'$are $\vec{v}$ and $\vec{v}'$, respectively. We can relate the velocities in these two frames by taking the time derivative of the expression above:

$$\vec{v} = \vec{v}' + \vec{v}_0, \tag{C.14}$$

where $\vec{v}_0$ is the velocity of the moving frame.

Example C.6 - Bubba

Bubba drives his pickup at 85 mi/h while throwing his beer bottle out the window at 15 mi/h, relative to the pickup.



(a) What is the speed of the bottle, relative to the road, if the bottle is thrown forward?

Solution

Take the fixed frame K to be the frame of the road and K' to be the truck's frame. We are studying the motion of the bottle with respect to these two frames. For a one dimensional problem remember that real numbers are one-dimensional vectors where the sign gives the direction. We then have (C.14) without the vector arrows.

$$v = v_0 + v' = 85\ \mathrm{mi/h} + 15\ \mathrm{mi/h} = 100\ \mathrm{mi/h}$$

(b) What is the speed of the bottle, relative to the road, if the bottle is thrown backward?

Solution

Now v' is in the opposite direction, so we will make it negative.

$$v = v_0 + v' = 85\ \mathrm{mi/h} + (-15\ \mathrm{mi/h}) = 70\ \mathrm{mi/h}$$

(c) What is the speed of the bottle, relative to the road, if the bottle is thrown directly to Bubba's left?

Solution

This is now two-dimensional. $\vec{v}_0$ and $\vec{v}'$are perpendicular and form two sides of a right triangle. $\vec{v}$ is the hypotenuse; to find its magnitude use the Pythagorean theorem.

$$v = \sqrt{(85\ \mathrm{mi/h})^2 + (15\ \mathrm{mi/h})^2} = 86.3\ \mathrm{mi/h}$$

(d) Relative to the road, what is the angle the bottle makes from the truck's direction?

Solution

From the right triangle we my find θ using trig.

$$\theta = \arctan \frac{15\ \mathrm{mi/h}}{85\ \mathrm{mi/h}} = 10.0\,^\circ$$

Example C.7 - A Plane in a Cross-wind

A plane flies to a city 800 mi to the east. In still air, the trip takes two hours. Suppose there is a strong $100\ \mathrm{mi/h}$ cross-wind blowing to the south. We will assume that the speed of the plane with respect to the air is fixed.

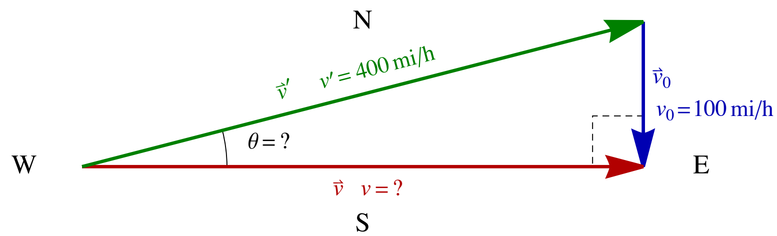
(a) At what angle, north of east, must the plane aim to have a trajectory toward the east?

Solution

In relative motion problems we have $\vec{v} = \vec{v}' + \vec{v}_0$, where $\vec{v}$ is the velocity of some moving object relative to a fixed frame and $\vec{v}'$ is the velocity of the object relative to a moving frame and $\vec{v}_0$ is the velocity of the moving frame. Here the moving object is the plane, the moving frame is the air and the ground is the fixed frame. The wind velocity is then $\vec{v}_0$; choosing east as the x-direction and north as the y-direction we have

$$\vec{v}_0 = \langle 0, -100 \rangle\ \mathrm{mi/h}$$

For $\vec{v}'$ (the plane relative to the air) we can find its magnitude. 800 mi in two hours implies the speed of the plane relative to the air of $v' = 400\ \mathrm{mi/h}$. As for the direction of $\vec{v}'$, it is directed at the unknown angle θ north of east. The velocity of the plane relative to the ground $\vec{v}$ is directed to the east with an unknown magnitude.



Given the right triangle above, we can solve for the angle θ.

$$\theta = \sin^{-1} \frac{100}{400} = 14.5\,^{\circ}$$

(b) How long does the trip take.

Solution

We know that the distance is $d = 800$ mi and $d = v\,t$, where v is the speed of the plane relative to the ground. To find that we use the Pythagorean theorem.

$$v^2 + v_0^2 = v'^2 \implies v = \sqrt{v'^2 - v_0^2} = \sqrt{(400\ \mathrm{mi/h})^2 + (100\ \mathrm{mi/h})^2} = 387.30\ \mathrm{mi/h}$$

From this we can solve for the time.

$$d = v\,t \implies t = \frac{d}{v} = 2.07\ \mathrm{h}$$