

# Chapter E

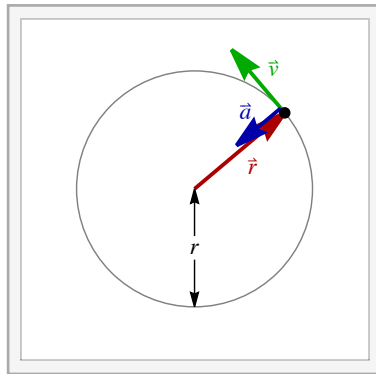
## Circular Motion and More Applications

Blinn College - Physics 2325 - Terry Honan

This chapter discusses the kinematics and dynamics of circular motion, and considers the problem of motion in an accelerated frame of reference. More applications on the second law will also be presented.

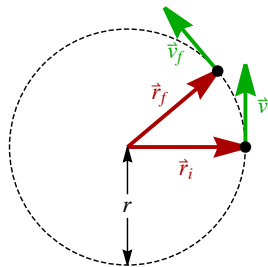
### E.1 - Kinematics of Circular Motion

#### Uniform Circular Motion - Centripetal Acceleration



Interactive Figure

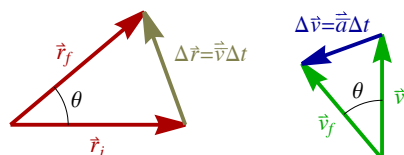
We will first consider the case of uniform circular motion, where uniform means that the speed stays constant. Even with a constant speed there is an acceleration, since the direction is changing. To find the acceleration we will first consider a finite time  $\Delta t$  and the average velocity and average acceleration. We will then let the time approach zero; the average values will then approach the instantaneous values.



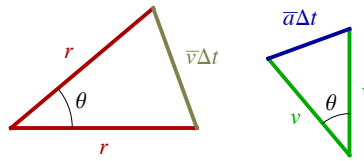
Take  $\vec{r}_i$  and  $\vec{r}_f$  to be the initial and final position vectors at times  $t_i$  and  $t_f$  where  $\Delta t = t_f - t_i$ . The velocities at these times are  $\vec{v}_i$  and  $\vec{v}_f$ ; since the speed is constant these have the same magnitude.

Take the angle between the two position vectors as  $\theta$ . In circular motion the velocity is perpendicular to the position vector; it follows that  $\theta$  is also the angle between the velocities. The sides opposite the angle  $\theta$  in the triangles are

$$\Delta \vec{v} = \vec{v}_f - \vec{v}_i = \vec{a} \Delta t \quad \text{and} \quad \Delta \vec{r} = \vec{r}_f - \vec{r}_i = \vec{v} \Delta t.$$



Looking at just the magnitudes of these vectors we see a pair of isosceles triangles and since the angle is the same for both, we have similar triangles.



Comparing the ratios of similar sides of similar triangles gives

$$\frac{\bar{a}\Delta t}{\bar{v}\Delta t} = \frac{v}{r}.$$

This can be rewritten as

$$\bar{a} = \frac{v\bar{v}}{r}.$$

As  $\Delta t \rightarrow 0$  the average velocity and average acceleration approach their instantaneous values, so the magnitudes of the average quantities approach the instantaneous magnitudes.

$$\bar{v} \rightarrow v \text{ and } \bar{a} \rightarrow a$$

With this we can write the value of the instantaneous acceleration as

$$a_c = \frac{v^2}{r}.$$

As  $\Delta t \rightarrow 0$  it is also true that  $\theta \rightarrow 0$ . It follows that the velocity and acceleration vectors are perpendicular. The direction of the acceleration is toward the center. We will refer to this direction as centripetal and denote the centripetal acceleration as  $a_c$ .

Many modern texts avoid the use of the term centripetal and have adopted the term radial instead. The radial direction could be outward or inward and the term centripetal is unambiguously toward the center. The reason for avoiding the word centripetal is because students confuse it with centrifugal. The centrifugal force is the outward false force associated with the centripetal acceleration.

## Uniform Circular Motion - Period, Speed and Radius

For uniform circular motion we can relate the period, speed and the radius. This will also give us an alternative expression for the centripetal acceleration in terms of the period. The period  $T$  is defined as the time needed for each full revolution.

It is a simple matter to relate the period to the radius and speed. Since the speed is uniform, is the same as the average speed; this is the distance traveled divided by the total time. The distance traveled in one period is one circumference. It follows that

$$v = \frac{2\pi r}{T}.$$

Inserting this into our expression for the centripetal acceleration  $a_c = v^2/r$  gives

$$a_c = \left(\frac{2\pi}{T}\right)^2 r.$$

This will prove to be a useful expression. The student should keep in mind that this *only* applies to uniform circular motion.

## General Circular Motion - Tangential Acceleration

Now consider the more general case of circular motion where we allow the speed to change. Generally, the component of the acceleration perpendicular to the motion is related to the change in direction and the component parallel to the direction of motion is related to the change in speed.

We will choose orthogonal (perpendicular) coordinates in tangential and centripetal directions. The centripetal direction is toward the center and the tangential direction is the direction of motion. If the origin is chosen at the center of the circle then the direction of the position vector  $\vec{r}$  is the unit radial vector  $\hat{r}$ ; this points away from the center, so the unit vector in the centripetal direction  $\hat{u}_c$  is opposite this:

$$\hat{u}_c = -\hat{r}. \quad (\text{unit centripetal vector})$$

The tangential direction is the direction of motion, so it is the direction of the velocity vector  $\vec{v}$ :

$$\hat{u}_t = \hat{v}. \quad (\text{unit tangential vector})$$

The component of the acceleration in the centripetal direction must be the same as in the uniform case

$$a_c = \frac{v^2}{r}.$$

The velocity is purely tangential

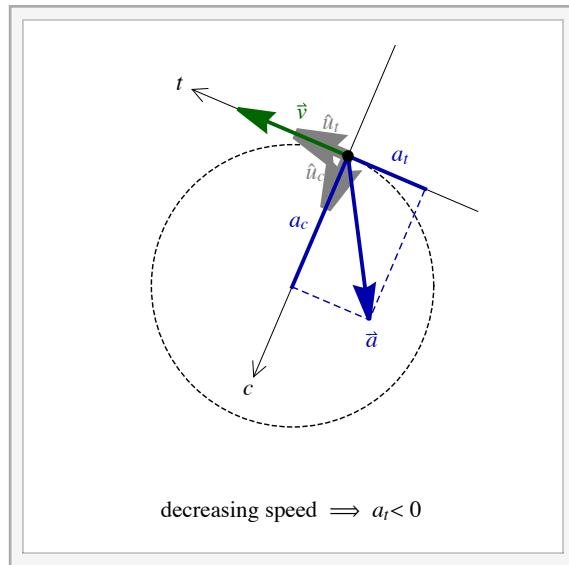
$$\vec{v} = v \hat{u}_t.$$

The component of the acceleration parallel to the direction of motion depends on the change in the speed; it is the time derivative of the speed.

$$a_t = \frac{dv}{dt}$$

Combining these two components we get a general expression for the acceleration in circular motion.

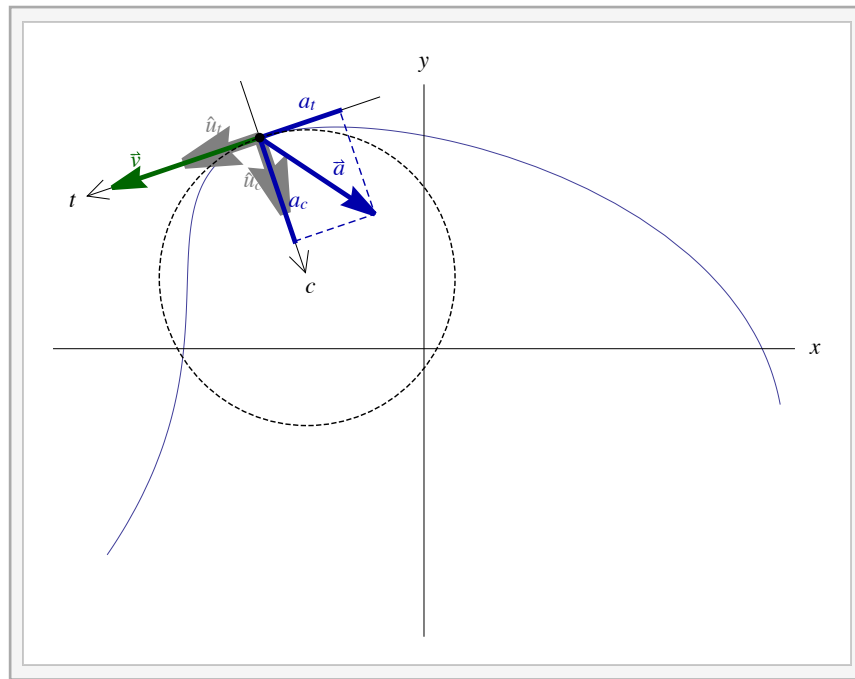
$$\vec{a} = a_c \hat{u}_c + a_t \hat{u}_t = \frac{v^2}{r} \hat{u}_c + \frac{dv}{dt} \hat{u}_t$$



## General Two Dimensional Motion and the Effective Radius

Now we consider the most general problem of motion in a plane. Any trajectory may be approximated by a circle at some position. We refer to the radius of that circle as the effective radius. We may choose centripetal and tangential directions as before and apply the above expression for the acceleration to the general two dimensional problem. If the direction is not changing at some instant, then the effective radius is infinite; the centripetal acceleration then becomes zero.

$$\vec{a} = a_c \hat{u}_c + a_t \hat{u}_t = \frac{v^2}{r} \hat{u}_c + \frac{dv}{dt} \hat{u}_t$$



Interactive Figure

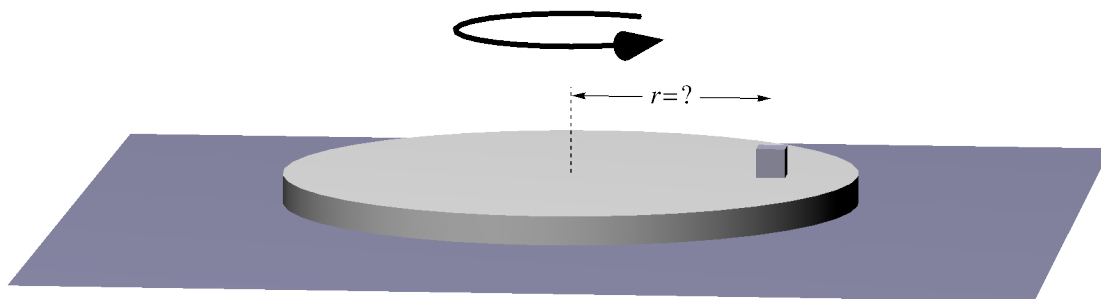
## E.2 - Dynamics of Circular Motion

Newton's Second Law  $\vec{F}_{\text{net}} = m\vec{a}$  is the foundation of classical mechanics and, of course, it must still apply when we have circular motion. We now have new formulas for the acceleration. We will now consider examples with circular motion.

### Examples with Uniform Circular Motion

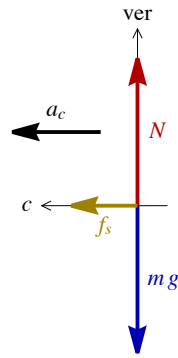
#### Example E.1 - Block on a Rotating Disk

A block sits on a turntable that rotates about its center once every 2.5 s. If the coefficient of static friction is 0.28, then what is the largest distance  $r$  that the block can be from the center without slipping?



#### Solution

We know the period  $T = 2.5$  s and the coefficient of static friction  $\mu_s = 0.28$ . We are not given the mass so it must cancel out in the algebra; call the mass  $m$ .



The only thing touching the block is the surface so we only have the two contact forces of a surface, friction (static here) and the normal force. Now we apply the second law to our perpendicular directions: the centripetal direction is horizontal and parallel to the surface ( $c = \text{hor} = \parallel$ ) and vertical is perpendicular to the surface ( $\text{ver} = \perp$ ).

$$F_{\text{net},c} = m a_c \implies f_s = m a_c$$

$$F_{\text{net,ver}} = m a_{\text{ver}} = 0 \implies N = m g$$

From the period we can find the centripetal acceleration .

$$a_c = \left( \frac{2\pi}{T} \right)^2 r$$

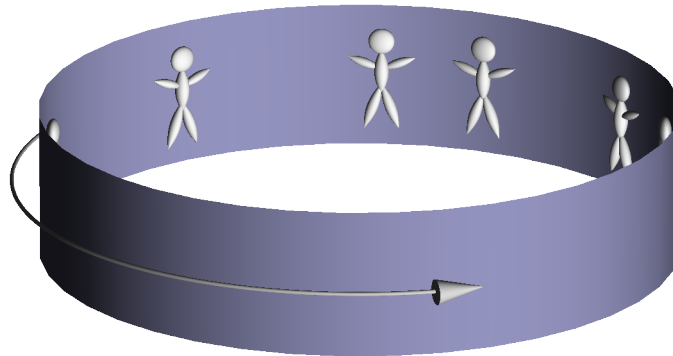
The static friction inequality and the expressions above give us the condition for not sliding.

$$f_s \leq \mu_s N \implies m a_c \leq \mu_s m g \implies a_c \leq \mu_s g \implies \left( \frac{2\pi}{T} \right)^2 r \leq \mu_s g$$

The maximum radius saturates this inequality. Using  $\mu_s = 0.28$ ,  $g = 9.80 \text{ m/s}^2$  and  $T = 2.5 \text{ s}$  gives our result.

$$\left( \frac{2\pi}{T} \right)^2 r_{\text{max}} = \mu_s g \implies r_{\text{max}} = \frac{\mu_s g}{(2\pi/T)^2} = 0.434 \text{ m}$$

### Example E.2 - Rotating Vertical Cylinder in an Amusement Park

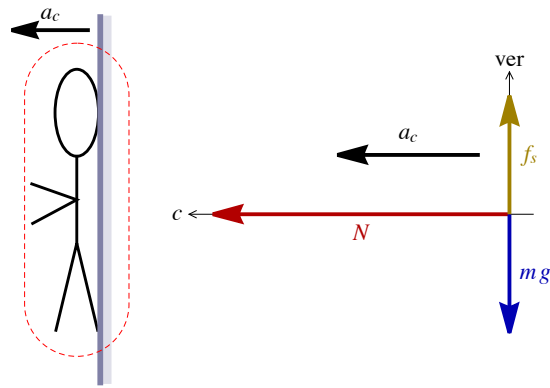


A common amusement park ride consists of vertical cylinder that rotates. Initially, the riders stand on a floor and lean against the inside surface of the cylinder as it starts to rotate. When the rotation is sufficiently rapid, the floor is dropped out and the riders stay pinned against the wall.

To design such a ride you are given the radius of the cylinder  $R$  and an estimate of the smallest coefficient of static friction  $\mu_s$  between a person and the wall. What is the minimum speed of the cylinder's surface for the riders to stay safe and not slide?

#### Solution

As with the previous example, the only contact forces are due to the surface, but since here the surface is vertical the roles of the normal force and friction are reversed.



$$F_{\text{net},c} = ma_c \implies N = ma_c$$

$$F_{\text{net,ver}} = ma_{\text{ver}} = 0 \implies f_s = mg$$

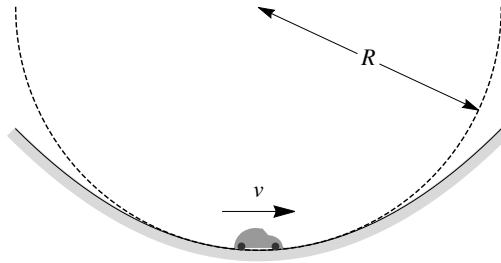
$$f_s \leq \mu_s N \implies mg \leq \mu_s ma_c \implies g \leq \mu_s a_c \implies g \leq \mu_s \frac{v^2}{R}$$

$$v_{\text{min}} = \sqrt{\frac{Rg}{\mu_s}}$$

## Nonuniform Examples

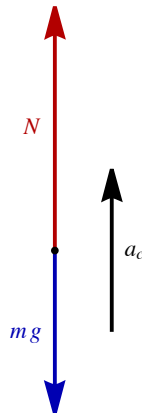
### Example E.3 - Car at the Bottom of a Trough

A car of mass  $m$  drives with speed  $v$  at the bottom of a trough with an effective radius  $R$ . What is the normal force of the road on the car?



#### Solution

The only contact force is the normal force of the road on the car, which acts straight upward. The acceleration is centripetal, which here is upward. Applying the second law in the centripetal direction gives an expression for the normal force.

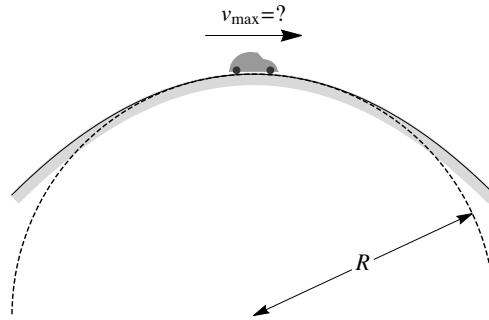


The only contact force is the normal force of the road on the car, which acts straight upward. The acceleration is centripetal, which here is upward. Applying the second law in the centripetal direction gives an expression for the normal force.

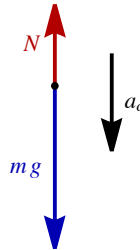
$$F_{\text{net},c} = ma_c \implies N - mg = m \frac{v^2}{R} \implies N = mg + m \frac{v^2}{R}$$

### Example E.4 - Car at the Top of a Hill

A car of mass  $m$  drives at the top of a hill with an effective radius  $R$ . What is the maximum speed the car can have while staying on the road surface?



### Solution



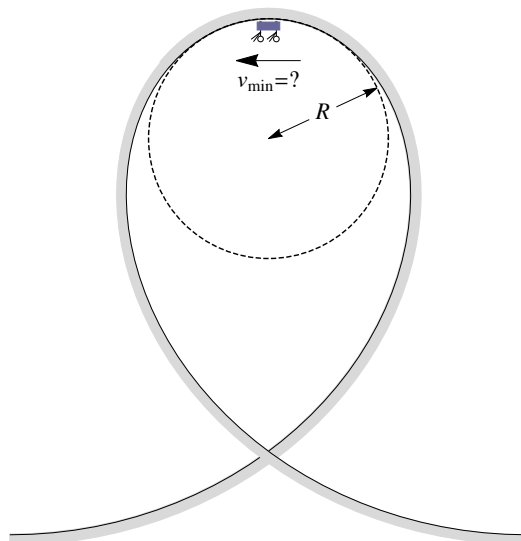
As with the preceding example the only contact force is the upward normal force. The difference is that the centripetal acceleration is now downward. Applying the second law in the centripetal direction to get the normal force.

$$F_{\text{net},c} = ma_c \implies mg - N = m \frac{v^2}{R} \implies N = mg - m \frac{v^2}{R}$$

The normal force constraint inequality  $N \geq 0$  gives an inequality for the speed. Saturating these inequalities gives the maximum speed. At a higher speed the car flies off the road surface.

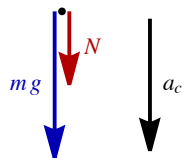
$$N \geq 0 \implies v \leq \sqrt{Rg} \implies v_{\text{max}} = \sqrt{Rg}$$

### Example E.5 - Roller-Coaster Loop-to-Loop



What is the minimum speed of a roller coaster cart at the top of a loop with an effective radius of  $R$ ?

### Solution



The only contact force is the normal force of the roller coaster track on the cart, which acts straight downward. The acceleration is centripetal, which here is also downward. Applying the second law in the centripetal direction gives an expression for the normal force.

$$F_{\text{net},c} = ma_c \implies mg + N = m \frac{v^2}{R} \implies N = m \frac{v^2}{R} - mg$$

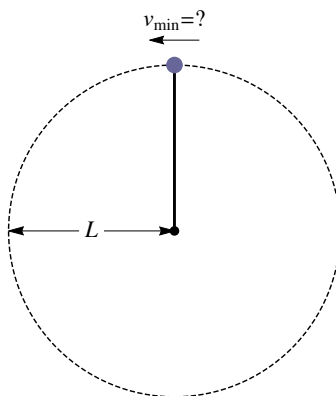
The normal force constraint inequality  $N \geq 0$  gives an inequality for the speed. Saturating these inequalities gives the minimum speed.

$$N \geq 0 \implies v \geq \sqrt{Rg} \implies v_{\text{min}} = \sqrt{Rg}$$

### Example E.6 - Ball Swung in Vertical Circle

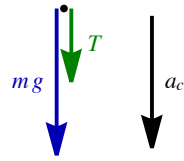
A small ball of mass  $m$  is swung in a vertical circle at the end of a string of length  $L$  with a fixed end.

(a) What minimum speed is needed at the top of the arc for the ball to not fall out of the circle.



### Solution





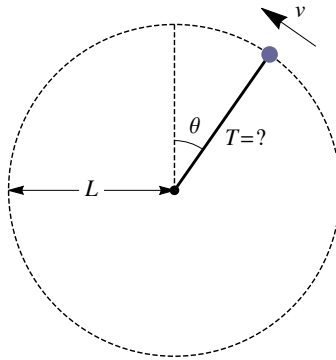
The only contact force is the tension in the string, which acts straight down. Note that this problem is identical to the roller coaster problem above but the tension plays the role of the normal force.

$$F_{\text{net},c} = ma_c \implies mg + T = m \frac{v^2}{L} \implies T = m \frac{v^2}{L} - mg$$

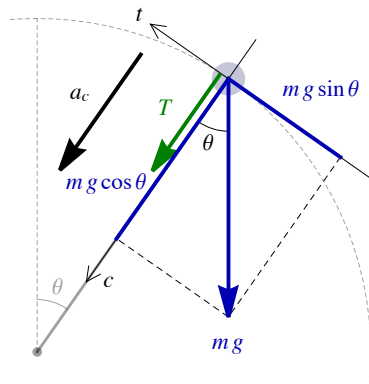
The tension constraint inequality  $T \geq 0$  gives an inequality for the speed. Saturating these inequalities gives the minimum speed.

$$T \geq 0 \implies v \geq \sqrt{Lg} \implies v_{\text{min}} = \sqrt{Lg}$$

(b) If the ball has a speed  $v$  at the instant the string makes an angle of  $\theta$ , measured from the top of the arc, then what is the tension in the string at that instant.



### Solution



The only contact force is due to the tension, which is centripetal. We need to resolve the weight  $mg$  into components, but since we are looking for the tension we only need to consider the centripetal component of the second law. This quickly gives the tension.

$$F_{\text{net},c} = ma_c \implies mg \cos \theta + T = m \frac{v^2}{L} \implies T = m \frac{v^2}{L} - mg \cos \theta$$

## E.3 - Accelerated Frames and False Forces

### False Force and Acceleration

Newton's second law applies to inertial (non-accelerated) frames. In accelerated frames there are false forces. In any case of acceleration there is a false force opposite the acceleration. We will refer to as an artificial gravity.

$$\vec{g}_{\text{art}} = -\vec{a}$$

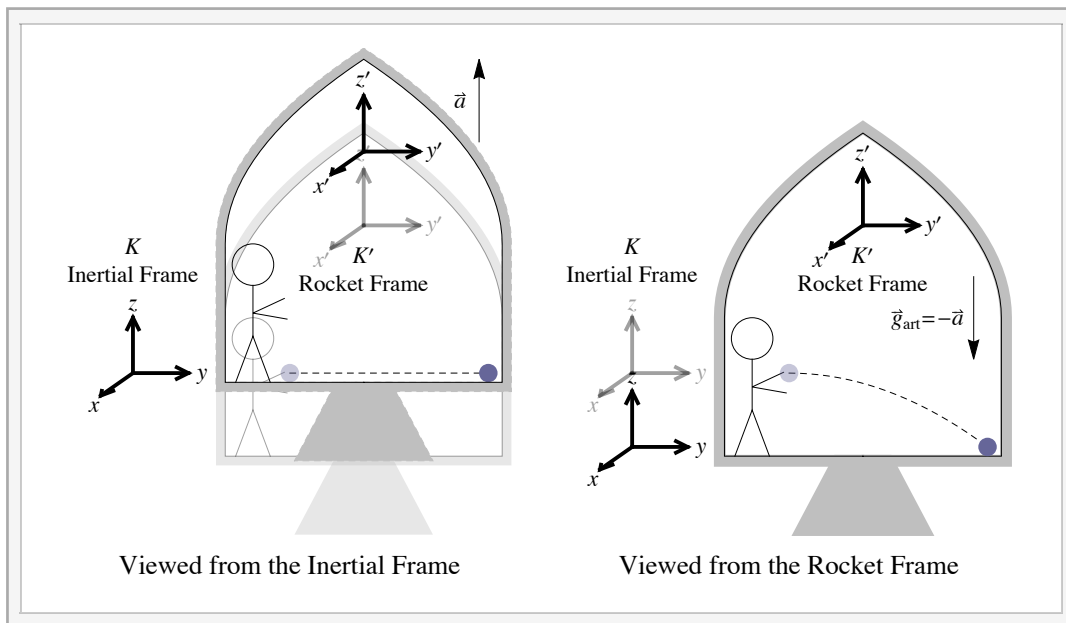
The false force is  $m\vec{g}_{\text{art}}$ . In a braking car the acceleration is backward so the false force is forward. In the case of circular motion the acceleration is centripetal (toward the center) so the false force is outward. The false force associated with a centripetal acceleration is called a *centrifugal force*.

### The Principle of Equivalence

The *principle of equivalence* states that a constant acceleration is equivalent to a uniform gravitational field in the direction opposite the acceleration.

$$\vec{g}_{\text{art}} = -\vec{a}$$

The equivalence principle states that this artificial gravity for a uniformly accelerated frame is *indistinguishable* from a real uniform gravitational field in the opposite direction. A rocket in space accelerating upward at  $9.80 \text{ m/s}^2$  is indistinguishable from standing in a uniform gravity of the same magnitude.



Interactive Figure - A ball is thrown from inside an accelerating rocket.  $K'$  is the accelerated frame of the rocket.  $K$  is the inertial frame, which is moving at a constant velocity and is initially at rest with respect to the rocket at the instant the ball is released.

### Effective Gravity

Where there is acceleration and gravity we can define an effective gravity.

$$\vec{g}_{\text{eff}} = \vec{g} + \vec{g}_{\text{art}}$$

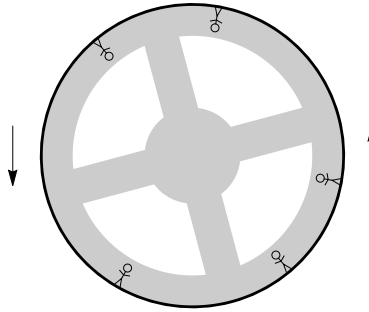
The earth is a rotating frame and there is a centrifugal force. When we drop a plum-bob it does not hang in the direction of gravity but in the direction of the effective gravity. Clearly the centrifugal force is significantly smaller than actual gravity.

### Nonuniform Acceleration and Velocity Dependent Forces

Note that in an accelerated frame with a nonuniform acceleration there are additional velocity-dependent false forces other than the artificial gravity discussed here. For instance, in the case of a rotating frame the artificial gravity force is the outward centrifugal force but there is also the velocity dependent force called the Coriolis force. The Coriolis force is responsible for weather patterns having different senses of rotation

in different hemispheres.

### Example E.7 - Rotating Space Station



A space station with an outside radius of 150 m rotates to create artificial gravity. What period of rotation is needed to create an artificial gravity equal to the earth's gravity?

#### Solution

Because  $\vec{g}_{\text{art}} = -\vec{a}$  we can equate the magnitudes  $g_{\text{art}} = a_c$ . Write  $a_c$  in terms of the period and radius.

$$g_{\text{art}} = a_c = \left(\frac{2\pi}{T}\right)^2 r$$

We are given that  $g_{\text{art}} = g$ . Solving for  $T$  gives our result.

$$g_{\text{art}} = g = 9.80 \frac{\text{m}}{\text{s}^2} \text{ and } r = R = 150 \text{ m} \implies T = 2\pi \sqrt{\frac{R}{g}} = 24.6 \text{ s}$$