Chapter F

Work and Energy

Blinn College - Physics 2325 - Terry Honan

F.1 - Introduction to Work

Mechanical Advantage

In Chapter D we considered the example of a pulley system lifting a weight. We saw that using multiple pulleys one can lift a heavy object with a smaller force. In the example, a weight W could be lifted by a tension $T = W/5$, but the smaller force must act over a larger distance. To lift the weight by Δy one must pull on the rope by $\Delta x = 5\,\Delta y$. This suggests that force times distance is an important quantity; we will define it as the work.

One Dimensional Work by a Constant Force

If in one dimension we move something by a displacement Δx with a constant force F. We will define the work done in this case by

$$W = F\ \Delta x.$$

Note that even in one dimension force and displacement are vector quantities; a one dimensional vector is a real number and the sign gives its direction. It follows that if the force and displacement are in the same direction then the work is positive and if opposite it is negative. The work is a scalar quantity.

> **Units:** The SI unit for work and energy is: J = Joule = N m

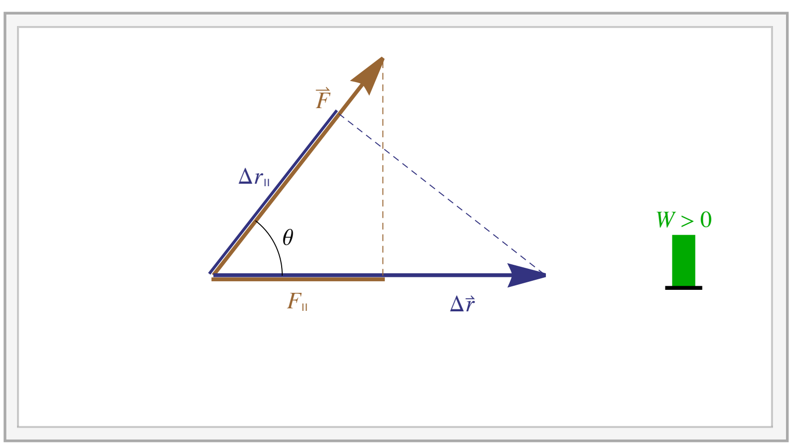
Two and Three Dimensional Work by a Constant Force

We now need to generalize this to the case of a constant force $\vec{F}$ and a straight-line path in two or three dimensions. Here we have a displacement $\Delta\vec{r}$ and we will define θ to be the angle between the two vectors $\vec{F}$ and $\Delta\vec{r}$. What is important is the component of the force along the direction of the displacement, which will be written $F_{\parallel}$ and is given by $F_{\parallel} = F\cos\theta$.

The definition of work becomes

$$W = F_{\parallel}\ \Delta r = F\,\Delta r\cos\theta = F\ \Delta r_{\parallel}$$

where $\Delta r_{\parallel}$ is the component of $\Delta\vec{r}$ parallel to the force $\vec{F}$.



Interactive Figure

Since $\vec{F}$ and $\Delta\vec{r}$ are both vectors and work W is a scalar, the above expressions may be viewed as a product of two vectors giving a scalar. We will next define the dot or scalar product of two vectors so that the work is written as

$$W = \vec{F}\cdot\Delta\vec{r}.$$

F.2 - Vector Algebra II - The Dot Product

The Definition in Terms of Magnitudes and Directions

Given two vectors $\vec{A}$ and $\vec{B}$ we define their Dot or Scalar product to be the scalar quantity given by

$$\vec{A}\cdot\vec{B} = A\,B\cos\theta, \tag{F.1}$$

where θ is the angle between the two vectors.

Properties of the Dot Product

Symmetry

Changing the order of the two vectors doesn't affect the angle between them. It is clear then that we have the symmetry property :

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}.$$

Associative Property of Scalar Product with Scalar Multiplication

It doesn't make sense to consider the dot product of more than two vectors, since the dot product of two is no longer a vector. Associativity is a meaningful thing when combining scalar multiplication with the dot product. It takes the form:

$$\left(c\,\vec{A}\right)\cdot\vec{B} = c\left(\vec{A}\cdot\vec{B}\right) = \vec{A}\cdot\left(c\,\vec{B}\right).$$

Distributive Property

When we have both an addition and multiplication operation defined we can ask if the operations are distributive. Here, there are two ways that the dot product could be distributive and it is distributive both ways

$$\left(\vec{A}+\vec{B}\right)\cdot\vec{C} = \vec{A}\cdot\vec{C}+\vec{B}\cdot\vec{C} \text{ and } \vec{A}\cdot\left(\vec{B}+\vec{C}\right) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}.$$

Note that mathematically the second expression follows from the first and from the symmetry property. Combining the distributive and associative properties mentioned shows that the dot product is a linear operation.

The Square of a Vector

Since the angle between a vector and itself is zero and $\cos 0 = 1$ it follows that for any $\vec{A}$

$$\vec{A}\cdot\vec{A} = A^2 = \|\vec{A}\|^2.$$

Perpendicular Vectors

Since $\cos 90° = 0$ it follows that

$$\vec{A} \perp \vec{B} \iff \vec{A}\cdot\vec{B} = 0.$$

Note that mathematically "$\Longrightarrow$" means implies and "$\iff$" means that either side implies the other side, or that the two sides are mathematically equivalent. Here we are defining the zero vector to be perpendicular to everything.

The Dot Product and Components

Using the two results above we can evaluate the dot products of the basis unit vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$.

$$1 = \hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} \text{ and } 0 = \hat{x}\cdot\hat{y} = \hat{x}\cdot\hat{z} = \hat{y}\cdot\hat{z}$$

Using this and the linearity of the dot product we can derive an expression for the dot product in terms of components.

$$\begin{aligned}\vec{A}\cdot\vec{B} &= (A_x\,\hat{x} + A_y\,\hat{y} + A_z\,\hat{z})\cdot(B_x\,\hat{x} + B_y\,\hat{y} + B_z\,\hat{z}) \\ &= A_x\,B_x\,\hat{x}\cdot\hat{x} + A_x\,B_y\,\hat{x}\cdot\hat{y} + A_x\,B_z\,\hat{x}\cdot\hat{z} \\ &\quad + A_y\,B_x\,\hat{y}\cdot\hat{x} + A_y\,B_y\,\hat{y}\cdot\hat{y} + A_y\,B_z\,\hat{y}\cdot\hat{z}\end{aligned}$$

$$+A_z B_x\,\hat{z}\cdot\hat{x} + A_z B_y\,\hat{z}\cdot\hat{y} + A_z B_z\,\hat{z}\cdot\hat{z}$$

Using the expressions above for the dot products of unit vectors we get all the off-diagonal terms going to zero and the dot products in the diagonal terms giving one. We end up with the result

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z. \tag{F.2}$$

This will be referred to as the *component form* of the dot product and should be viewed as an alternative definition.

Example F.1 - Angle Between Two Vectors

Find the angle between $\vec{A}$ and $\vec{B}$.

$$\vec{A} = \langle 3,\ 5,\ -4\rangle \quad \text{and} \quad \vec{B} = \langle -3,\ 2,\ 1\rangle$$

Solution

Here we will equate the component form of the definition (F.2) of the dot product with the original definition (F.1). Find the dot product using (F.2).

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z = 3\,(-3) + 5\cdot 2 + (-4)\,1 = -3$$

We also need to calculate the magnitudes of both vectors.

$$A = \|\vec{A}\| = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{3^2 + 5^2 + (-4)^2} = \sqrt{50} \quad \text{and}$$

$$B = \|\vec{B}\| = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{14}$$

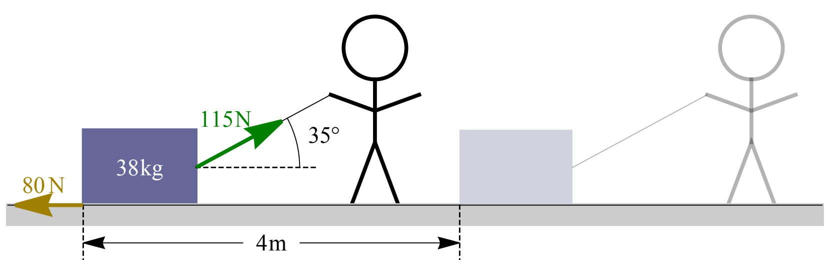
The angle then follows from the original definition (F.1).

$$\vec{A}\cdot\vec{B} = A\,B\cos\theta \implies \theta = \cos^{-1}\left(\frac{\vec{A}\cdot\vec{B}}{A\,B}\right) = 96.5\,°$$

Because the dot product was negative, the angle had to be obtuse.

Example F.2 - Dragging a Crate

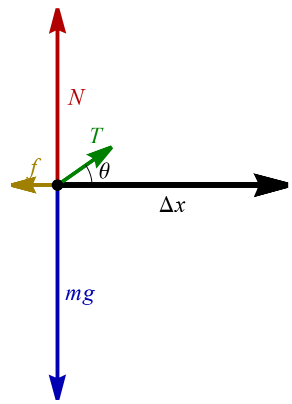
A 38-kg crate initially at rest is dragged by a rope a distance of 4m along a horizontal floor. The rope has a tension of 115 N and makes an angle of 35° from horizontal. There is a backward friction force of 80 N acting on the crate. There are four forces acting on the crate: tension, friction, the normal force and gravity.



(a) What is the work done by each force?

Solution

The free-body diagram for the crate shows the four forces and their directions. The displacement $\Delta\vec{r}$ is shown for reference, where $\|\Delta\vec{r}\| = \Delta x$, but $\Delta\vec{r}$ is not a force and not part of the free-body diagram.



$$T = 115\text{N},\ f = 80\text{N},\ m = 38\text{kg},\ \theta = 35\,°\ \text{and}\ \Delta x = 4\text{m}$$

Since we have constant forces with a straight-line path and using $\|\Delta\vec{r}\| = \Delta x$, we can write the work for each force as:

$$W = \vec{F}\cdot\Delta\,\vec{r} = F\,\Delta x\cos\theta.$$

For the tension we have

$$W_T = T\,\Delta x\cos\theta = 377.\,\text{J}$$

For the friction force the angle is $\theta = 180\,°$.

$$W_f = f\,\Delta x\cos 180\,° = -f\,\Delta x = -320\,\text{J}$$

Both the normal force and gravity (the weight) are perpendicular to the displacement. Since $\cos 90\,° = 0$ both forces give zero work.

(b) Find the acceleration of the crate and its speed after moving 4m.

Solution

To find the acceleration, which is horizontal, we only need to consider the horizontal components of forces. The horizontal component of the tension is $T\cos\theta$ and friction is backward and negative.

$$F_{\text{net,hor}} = T\cos\theta - f = m\,a \implies a = \frac{1}{m}(T\cos\theta - f) = 0.374\,\frac{\text{m}}{\text{s}^2}$$

Using constant acceleration kinematics and that the initial velocity is zero, $v_0 = 0$, allows us to find the final speed.

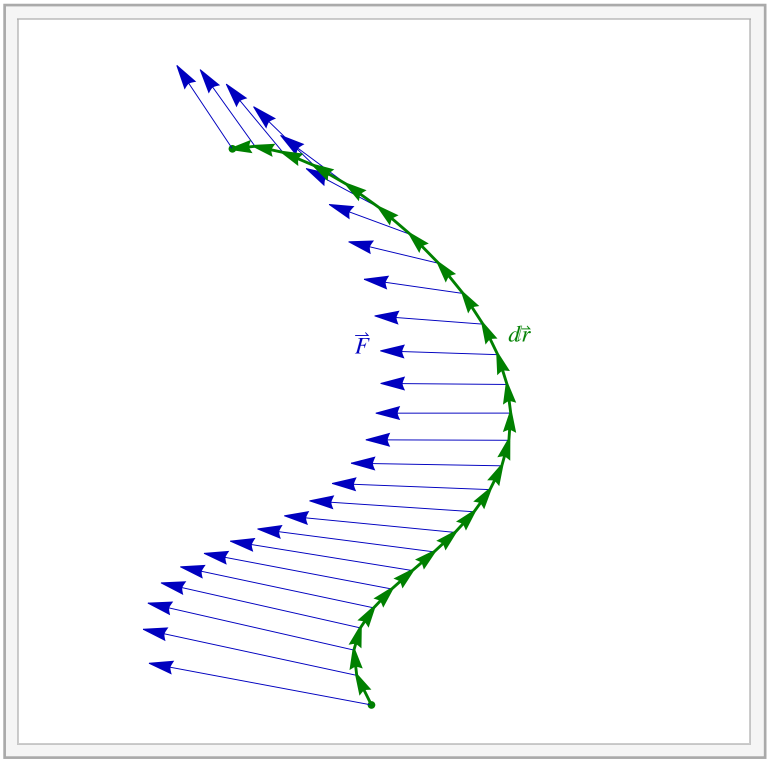
$$v^2 = v_0^2 + 2\,a\,\Delta x \implies v = \sqrt{2\,a\,\Delta x} = 1.73\,\text{m/s}$$

F.3 - Work in General

So far we have the definition of work for a constant force and a straight line path to be $W = \vec{F}\cdot\Delta\,\vec{r}$. We need to generalize this to a varying force acting on a general path that is allowed to curve. First let us consider an infinitesimal segment of the path $d\,\vec{r}$. Over a sufficiently small segment the path is straight and the force varies negligibly. It follows that the (in work over the infinitesimal segment is $\vec{\boldsymbol{F}}\cdot\boldsymbol{d}\vec{\boldsymbol{r}}$. A definite integral is an infinite sum over infinitesimal pieces. We denote this infinite sum with the integral symbol "$\int$" and write the work as

$$W = \int\vec{F}\cdot d\,\vec{r}.$$

The integral in this definition of work is of a vector field integrated along a general path. This is a mathematical object that a student will learn to evaluate in the third semester of the calculus sequence. We will not evaluate these general integrals this semester. We will discuss the integrals in general and only evaluate special cases. Some of these special cases will be discussed below.



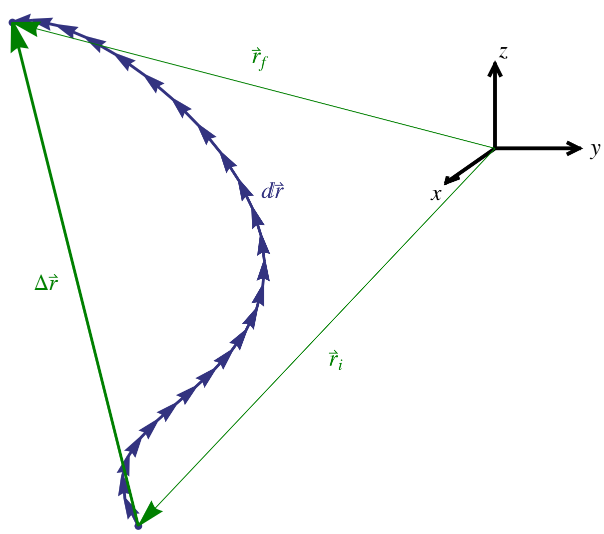
Interactive Figure

Constant Force

If a force is constant then we can take it out of the above general integral. This leaves the integral $\int d\vec{r}$ which is just the sum over all the infinitesimal pieces $d\vec{r}$; this becomes the vector $\Delta\vec{r}$ which is the vector from the starting position to the end of the path.

$$W = \int \vec{F} \cdot d\vec{r} = \vec{F} \cdot \int d\vec{r} = \vec{F} \cdot \Delta\vec{r}.$$

Note that this is the same result we had for a constant force over a straight-line path



Work Done by Gravity

An important special case of the previous result is the work done by gravity. The force is a uniform $\vec{F} = -m\,g\,\hat{y}$ and the work becomes

$$W_{\text{grav}} = \vec{F}\cdot\Delta\vec{r} = -m\,g\,\hat{y}\cdot\Delta\vec{r} = -m\,g\,\Delta y.$$

To derive this we used that $\hat{y}\cdot\Delta\vec{r}$ is the y component of $\Delta\vec{r}$, which is Δy.

Example F.3 - A Box on a Table

Consider a box of mass m and a table of height h.

(a) What is the work done by gravity when the box is moved from the table top to the floor?

Solution

We choose positive y to be upward so we have $\Delta y = -h$. It follows that

$$W_{\text{grav}} = -m\,g\,\Delta y = m\,g\,h.$$

(b) What is the work done by gravity when the box is moved from the floor back to the table top?

Solution

Now we have $\Delta y = +h$ and

$$W_{\text{grav}} = -m\,g\,\Delta y = -m\,g\,h.$$

(c) What is the total work done by gravity when the box is moved from the table top to the floor and then back to the table top?

Solution

The net vertical displacement is zero. $\Delta y = 0$. So

$$W_{\text{grav}} = -m\,g\,\Delta y = 0.$$

One Dimensional Work by a Varying Force

If we have $F(x)$ as a one dimension force as a function of position. The work when moving from position x_i to x_f the work is the one dimensional definite integral

$$W = \int_{x_i}^{x_f} F(x)\,dx.$$

This is the sort of one variable integral discussed in the first semester of calculus and the student will be responsible for such integrals. To evaluate a definite integral one uses the fundamental theorem of calculus.

The Fundamental Theorem of Calculus

The fundamental theorem of calculus relates integral and differential calculus. It is the rule one uses to evaluate a definite integral.

$$\int_a^b g(x)\,dx = G(x)\,\Big|_a^b = G(b) - G(a) \text{ where } G'(x) = g(x)$$

The definite integral of a function is the difference of some antiderivative of the function evaluated at the endpoints of the interval.

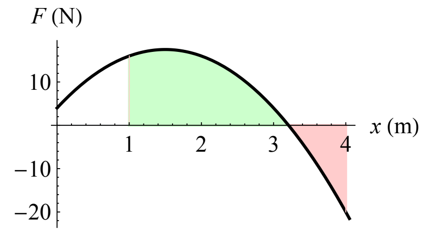
Example F.4 - Work Done by a Varying Force

A particle moves along the x-axis from 1m to 4m under the force

$$F(x) = 18\,x - 6\,x^2 + 4 \quad \text{(in SI units)}.$$

What is the work done by the force?

Solution



We evaluate the work integral using the fundamental theorem.

$$W = \int_{x_i}^{x_f} F(x)\,dx = \int_{1\,\mathrm{m}}^{4\,\mathrm{m}} \left(18\,x - 6\,x^2 + 4\right) dx$$
$$= \left(9\,x^2 - 2\,x^3 + 4\,x\right)\Big|_{1\,\mathrm{m}}^{4\,\mathrm{m}} = (144 - 128 + 16)\,\mathrm{J} - (9 - 2 + 4)\,\mathrm{J}$$
$$= 21\,\mathrm{J}$$

Hooke's Law and Elastic Forces

Hooke's law is the force law for a spring. Define $x = 0$ to be the relaxed position of a spring, the equilibrium position, and let x be the amount the spring is stretched from equilibrium. Hooke's law states that there is a proportionality between F and x, $F \propto x$. We can introduce a constant of proportionality k called the spring constant or force constant.

$$F = k\,x \ \text{(Force on spring)}$$

The spring constant is a property of a particular spring; a stiff spring has a large k and a loose spring has a small k. In this expression F is the force stretching the spring. Usually we consider the force of the spring itself when considering Hooke's law. By Newton's third law this is the negative of the force on the spring and we get

$$F = -k\,x \ \text{(Force of spring).} \tag{F.3}$$

Note that when the spring is stretched, x is positive and the force is negative. When the spring is compressed the force is positive and x is negative.

A force is said to be *elastic* when it satisfies Hooke's law. We will see that elasticity is quite common for sufficiently small deformations of almost anything. What is special about springs is they maintain their elasticity over large deformations. It should be clear that even a spring will violate Hooke's law when stretched a very large distance.

Work Done by a Spring

We may now derive an expression for the work done by a spring. Using the formula for the work done in one dimension by a varying force $F(x)$ and Hooke's law $F(x) = -k\,x$ we get a simple integral.

$$W_{\text{spring}} = \int_{x_i}^{x_f} F(x)\,dx = -k \int_{x_i}^{x_f} x\,dx.$$

Using the fact that $\frac{1}{2}x^2$ is the antiderivative of x we get

$$W_{\text{spring}} = -\frac{1}{2}\,k\left(x_f^2 - x_i^2\right). \tag{F.4}$$

Example F.5 - Hooke’s Law

It takes a force of magnitude 60 N to compress a spring by 4 cm.

(a) What is the force constant of the spring?

Solution

$$F = 60\,\mathrm{N} \ \text{ and } \ x = 0.04\,\mathrm{m}$$

We will use Hooke’s Law (F.3) to find the force constant. We will ignore the sign because only magnitudes are given.

$$F = k\,x \implies k = F/x = 1500\,\mathrm{N/m}.$$

(b) How much work is done compressing the spring? What is the work done *by* the spring?

Solution

The work done *by* the spring (F.4) is

$$W_{\text{spring}} = -\frac{1}{2}\,k\left(x_f^2 - x_i^2\right) = -\frac{1}{2}\,k\left(x^2 - 0^2\right) = -\frac{1}{2}\,F\,x = -1.2\text{ J}\,.$$

F.4 - Power

Power, in the most general sense, is the rate that something uses or provides energy.

$$\mathcal{P} = \frac{d\,\text{Energy}}{d\,t}.$$

The power delivered by a motor or engine is the rate that it can do work

$$\mathcal{P} = \frac{d\,W}{d\,t}.$$

If we write the work in terms of the force we get $d\,W = \vec{F}\cdot d\,\vec{r}$. Writing the infinitesimal displacement as $d\,\vec{r} = \vec{v}\,d\,t$ we get

$$\mathcal{P} = \vec{F}\cdot\vec{v}.$$

Units: The SI unit for power is: W = Watt = J / s

F.5 - Kinetic Energy and the Work-Energy Theorem

The Net Work

Newton's second law isn't a general statement about forces but is about the net force acting on a body, $\vec{F}_{\text{net}} = m\,\vec{a}$. $\vec{F}_{\text{net}}$ is the vector sum of all forces acting on a body; we use a free-body diagram to help us sum these forces. Let us symbolically write the net force in terms of all the forces acting on a body (all the forces in the free-body diagram) as

$$\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 + \ldots.$$

Each force acting on a body does work on the body. (Some of these works may be zero, however.) If the work done by $\vec{F}_i$ is labeled W_i then we can define the net work as the sum of all these works.

$$W_{\text{net}} = W_1 + W_2 + \ldots.$$

The Work-Energy Theorem

The work-energy theorem is a very important result. It is where the idea of energy comes into physics and it explains why work is a useful notion. We define the *kinetic energy* by

$$K = \frac{1}{2}\,m\,v^2.$$

The theorem is quite simple to state; it equates the net work to the change in the kinetic energy.

$$W_{\text{net}} = \Delta\,K$$

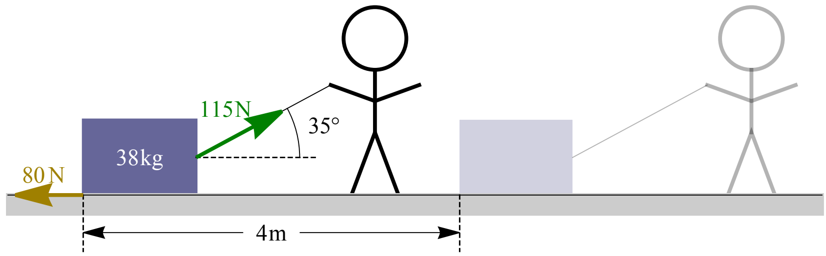
where the change is kinetic energy is $\Delta\,K = K_f - K_i = \frac{1}{2}\,m\,\left(v_f^2\ -\ v_i^2\right)$.

Example F.6 - Dragging a Crate (continued)

Before proving the theorem let us consider an example. We will continue the “Dragging a Crate” example (Example F.2).

A 38-kg crate initially at rest is dragged by a rope a distance of 4m along a horizontal floor. The rope has a tension of 115 N and makes an angle of 35° from horizontal. There is a backward friction force of 80 N acting on the crate. There are four forces acting on the crate:

tension, friction, the normal force and gravity.



In the earlier example (Example F.2) we found the work done by each force

$$W_T = 376.8\,\text{J}\ , \quad W_f = -320.0\,\text{J} \ \text{ and } W_N = 0 = W_{\text{grav}}$$

and we solved for the speed of the crate after moving.

(a) What is the net work?

Solution

$$W_{\text{net}} = W_T + W_f + W_N + W_{\text{grav}} = 376.8\,\text{J} - 320.0\,\text{J} + 0 + 0 = 56.8\,\text{J}$$

(b) Using the Work-Energy theorem find the speed of the crate after moving 4m.

Solution

$$W_{\text{net}} = \Delta K = \frac{1}{2} m \left(v_f^2 - v_i^2\right)$$

Using $m = 38$ kg and $v_i = 0$ we get the same value for the speed.

$$v_f = \sqrt{\frac{2}{m} W_{\text{net}}} = 1.73\,\text{m/s}$$

Proof of the Theorem

Consider a single mass moving along a general path. The net work is the sum of the works done by each force acting on a body. Since the integral of the sum is the sum of the integrals, the net work can be written as the work done by the net force.

$$W_{\text{net}} = \int \vec{F}_{\text{net}} \cdot d\vec{r}.$$

We then apply the second law $\vec{F}_{\text{net}} = m\,\vec{a}$ and use the definition of velocity to write $d\vec{r} = \vec{v}\,dt$. We then get

$$W_{\text{net}} = \int m\,\vec{a}\cdot\vec{v}\,dt.$$

We have now turned the integral over a contour into a simple integral over a single variable. The limits of this integration are the initial and final times.

$$W_{\text{net}} = \int_{t_i}^{t_f} \left(m\,\vec{a}\cdot\vec{v}\right) dt.$$

To evaluate the above integral we need to, using the fundamental theorem of calculus, find the antiderivative of the integrand $m\,\vec{a}\cdot\vec{v}$. This antiderivative is just the kinetic energy:

$$\frac{d}{dt}\left(\frac{1}{2} m\,v^2\right) = m\,\vec{a}\cdot\vec{v}.$$

To verify this, note that dot products satisfy the usual product rule:

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z \implies \frac{d}{dt}\left(\vec{A}\cdot\vec{B}\right) = \left(\frac{d}{dt}\vec{A}\right)\cdot\vec{B} + \vec{A}\cdot\left(\frac{d}{dt}\vec{B}\right)$$

and thus

$$\frac{d}{dt} v^2 = \frac{d}{dt}\vec{v}\cdot\vec{v} = 2\left(\frac{d}{dt}\vec{v}\right)\cdot\vec{v} = 2\,\vec{a}\cdot\vec{v}.$$

It is now clear that the kinetic energy is the antiderivative of $m\,\vec{a}\cdot\vec{v}$.

Using the fundamental theorem of calculus we get the work energy theorem

$$W_{\text{net}} = \int_{t_i}^{t_f} \left(m\,\vec{a}\cdot\vec{v}\right) d t = \frac{1}{2}\,m\,v(t_f)^2 - \frac{1}{2}\,m\,v(t_i)^2 = \Delta\,K.$$

F.6 - Conservative Forces and Potential Energy

When a body is moved in a uniform gravitational field $\vec{g}$ the work done by gravity is given by: $W = -m\,g\;\Delta\,y$ Note that if the body is moved along different paths with the same endpoints (starting and stopping points) the $\Delta\,y$ is the same and thus the work done by gravity is the same. In stretching a spring the work of the spring $W = -\frac{1}{2}\,k\left(x_f^2 - x_i^2\right)$ depends only on the endpoints x_i and x_f and not on the details of the path.

We will define such forces (where the work is independent of path) as conservative and we will then be able to define a new type of energy, called *potential energy* or U, for these forces. The entire effect of the work of these forces will be incorporated into considering the potential energy functions at the endpoints of the paths. Potential energy will be an easy and useful bookkeeping tool for keeping track of the work contributions for conservative forces in the work-energy theorem.

Conservative and Nonconservative Forces

A force is defined to be a conservative force when its work is independent of the path taken. Equivalently, we can say that for a conservative force the work around any closed path (a path that ends where it begins) is zero. Putting a circle in the integral sign implies that the path is closed and the definition of conservative is written:

$$\oint \vec{F}\cdot d\vec{r} = 0 \iff \vec{F}\text{ is conservative.}$$

Conservative Examples

Examples of conservative forces have already been mentioned. These are uniform gravitational forces and the elastic force of a spring. Other examples of conservative forces that will be encountered are nonuniform gravitational forces, which will be discussed later this semester, and the electrostatic force, which will be considered in the second semester course.

Dissipative friction is nonconservative.

Consider an object being dragged along a horizontal floor by a horizontal rope. If the dragging force is horizontal, the normal force is the weight and the magnitude of the force of kinetic friction is $f_k = \mu_k\,m\,g$ = constant. The direction of the force of friction opposes the direction of motion so the work done by friction becomes

$$W_f = \int \vec{f}_k\cdot d\vec{r} = -f_k \int d s.$$

Note that $d s$ is the magnitude of the infinitesimal displacement $d\vec{r}$, $d s = \left\|d\vec{r}\right\|$, and is then the infinitesimal arc length. The integral $\int d s$ is then the total arc length. It is clear that W_f depends on the path length and that kinetic friction isn't conservative.

Driving forces are nonconservative.

By driving forces we mean the force of a car's engine propelling a car or the force of a cyclist propelling a cycle. Consider a car that begins at some location, drives on some path and then returns to the same point. The car's engine does work in this process and thus cannot be conservative. A conservative force must give zero work for a closed path.

Other Nonconservative Forces

Any force that is not conservative is nonconservative, where our two conservative examples are gravity or the elastic force of a spring. If a tension force or normal force acts on a body then those also represent nonconservative forces; in some examples the tension and normal force will be present but do no work. If there is some external applied force acting on a body then that is nonconservative.

Potential Energy

For any conservative force we can define a potential energy, U. This idea is this: since the work depends only on the endpoints of a path and not the details of the path then we can write the work as the difference of some function evaluated only at the endpoints. We will define this function as the negative of the potential energy function. The reason for the sign will become clear later.

The definition of potential energy is

$$\Delta U = -W = -\int \vec{F} \cdot d\vec{r}.$$

The zero of potential energy is arbitrary. In some cases there will be standard choices of the zero position.

Potential Energy for Uniform Gravity

Since for gravity we have $W = -m\,g\,\Delta y$, we define gravitational potential energy by $\Delta U = m\,g\,\Delta y$. We can choose the zero of potential energy to be where $y = 0$ and then define the potential energy function as

$$U = m\,g\,y.$$

The zero of y is still arbitrary.

Elastic Potential Energy

The work done by a spring is given by $W = -\frac{1}{2}\,k\left(x_f^2 - x_i^2\right)$. When we take $\Delta U = -W$ we get

$$\Delta U = -W = \frac{1}{2}\,k\left(x_f^2 - x_i^2\right).$$

We want to find a function $U(x)$ that satisfies $\Delta U = U(x_f) - U(x_i)$. The easiest choice is

$$U = \frac{1}{2}\,k\,x^2.$$

In making this choice we take the zero position of potential energy to be the equilibrium position $x = 0$.

Work and Mechanical Energy

Let us now apply these ideas to the work-energy theorem. We begin by writing all forces acting on a body as

$$\vec{F}_{\text{net}} = \underbrace{\vec{F}_{\text{nc}}}_{\text{all nonconservative forces}} + \underbrace{\vec{F}_1 + \vec{F}_2 + \ldots}_{\text{conservative forces}}.$$

$\vec{F}_{\text{nc}}$ represents the sum of *all* nonconservative forces. The other forces are the conservative forces listed separately. We now make the same decomposition of the corresponding works.

$$W_{\text{net}} = W_{\text{nc}} + W_1 + W_2 + \ldots$$

Now we make the replacements $W_i = -\Delta U_i$. Plugging the above expression for W_{net} into the work-energy theorem $W_{\text{net}} = \Delta K$ and moving the ΔU_i terms to the right hand side gives:

$$W_{\text{nc}} = \Delta K + \Delta U_1 + \Delta U_2 + \ldots.$$

This result applies to a single mass. To apply it to a system containing multiple masses, like for instance Atwood's machine, we can sum this over every mass in the system. Now take W_{nc} to be the sum of the W_{nc} for all the masses. Call K_{tot} the sum of the kinetic energies of all masses and U_{tot} the sum over all the U_i for all the masses. We end up with the result

$$\begin{aligned} W_{\text{nc}} &= \Delta K_{\text{tot}} + \Delta U_{\text{tot}} \\ &= \Delta E_{\text{mech}} \end{aligned}$$

where we have defined the total mechanical energy of the system as $E_{\text{mech}} = K_{\text{tot}} + U_{\text{tot}}$. Usually the mechanical energy will just be written as E.

Nonconservative forces will usually consist of friction forces, which remove mechanical energy from a system, and driving forces like the work done by car's engine or a cyclist, which add mechanical energy.

F.7 - Conservation of Energy

There are two notions of conservation of energy we will consider. One will be of practical importance for problem solving. The other is a very fundamental notion.

Conservation of Mechanical Energy

The conservation of mechanical energy is a principle that will prove very useful in problem solving. Begin with the fundamental expression $W_{\text{nc}} = \Delta E_{\text{mech}}$. If in some problem there are no nonconservative forces (i.e. no dissipative friction or driving force) then we can conclude that $\Delta E_{\text{mech}} = 0$ or that

$$E_{\text{mech},i} = E_{\text{mech},f} \text{ or } E_i = E_f.$$

To solve such a problem we need to find an expression for E, the total mechanical energy. To do this we add a kinetic energy for each mass in the problem, add in gravitational potential energies for each mass and add an elastic potential energy for each spring.

Energy as a Fundamentally Conserved Quantity

Energy is a fundamentally conserved quantity. This means that it cannot be created or destroyed; we can just convert it from one form to another.

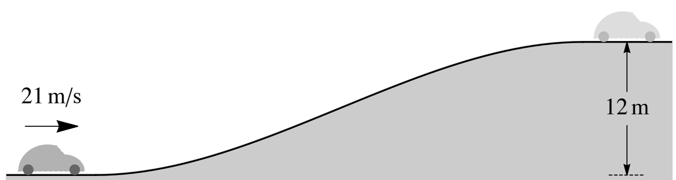
Consider the mechanical energy of a car $W_{\text{nc}} = \Delta E_{\text{mech}}$. The mechanical energy is not conserved due to the W_{nc} of friction and the engine. The energy lost to friction goes into heat. The energy from the engine comes from the energy stored in the chemical bonds of the fuel.

When solving problems that involve nonconservative forces we can rewrite $W_{\text{nc}} = \Delta E_{\text{mech}} = E_f - E_i$ as

$$E_i + W_{\text{nc}} = E_f$$

Written this way, problem solving is more similar for both types of problems, where W_{nc} is zero or not.

Example F.7 - A Rolling Car



A 1500kg car rolls in neutral up a 12m high hill. The car's speed at the bottom of the hill is 21 m/s.

(a) Suppose there is no friction. What is the speed of the car at the top of the hill?

Solution

$$m = 1500\text{kg}, \quad h = 12\text{m} \text{ and } v_i = 21\,\text{m/s}.$$

For a car we have $W_{\text{nc}} = W_{\text{friction}} + W_{\text{engine}}$ but since it is in neutral $W_{\text{engine}} = 0$. For part (a) we also have $W_{\text{friction}} = 0$.

Since there is just one mass and no springs, the mechanical energy is $E = \frac{1}{2}mv^2 + mgy$, and since $W_{\text{nc}} = 0$ mechanical energy is conserved. It is most convenient to choose the lowest point to be the zero of potential energy so we take $y_i = 0$ and $y_f = h$.

$$E = \frac{1}{2}mv^2 + mgy \text{ and } E_i = E_f \implies \frac{1}{2}mv_i^2 + 0 = \frac{1}{2}mv_f^2 + mgh$$

It follows that the velocity at the top is

$$v_f = \sqrt{v_i^2 - 2gh} = 14.3\,\text{m/s}$$

(b) Suppose now that there is friction and the speed of the car at the top is 9.5m/s . What is the work done by friction?

Solution

The car is still in neutral so we again have $W_{\text{engine}} = 0$.

$$W_{\text{nc}} = W_{\text{friction}} + W_{\text{engine}} = W_{\text{friction}} + 0$$

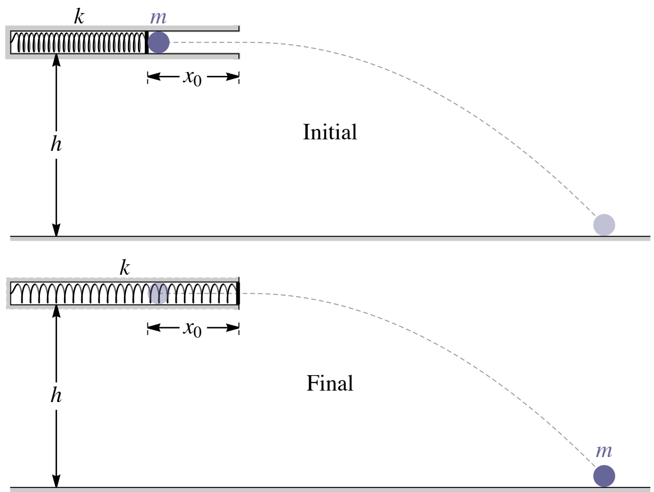
Since there is just one mass and no springs, the mechanical energy is $E = \frac{1}{2}mv^2 + mgy$, and since $W_{\text{nc}} = W_{\text{friction}}$ mechanical energy is conserved.

$$E = \frac{1}{2}mv^2 + mgy \text{ and } E_i + W_{\text{nc}} = E_f \implies \frac{1}{2}mv_i^2 + 0 + W_{\text{friction}} = \frac{1}{2}mv_f^2 + mgh$$

We can now find W_{friction}, which we expect to be negative. $v_f = 9.5\,\text{m/s}$.

$$W_{\text{friction}} = \frac{1}{2}mv_f^2 + mgh - \frac{1}{2}mv_i^2 = -86\,700\,\text{J}$$

Example F.8 - A Spring Gun



A ball of mass m is shot from a horizontal spring gun at a height h above the floor. The spring has a force constant k and is compressed by x_0 when cocked. What is the speed of the ball when it hits the floor? Assume no friction.

Solution

There is just one mass and thus one kinetic energy term $K = \frac{1}{2} m v^2$. There are two potential energy terms gravitational $U_{\text{grav}} = mgy$ and elastic $U_{\text{elastic}} = \frac{1}{2} k x^2$. The total energy is thus:

$$E = \frac{1}{2} m v^2 + mgy + \frac{1}{2} k x^2.$$

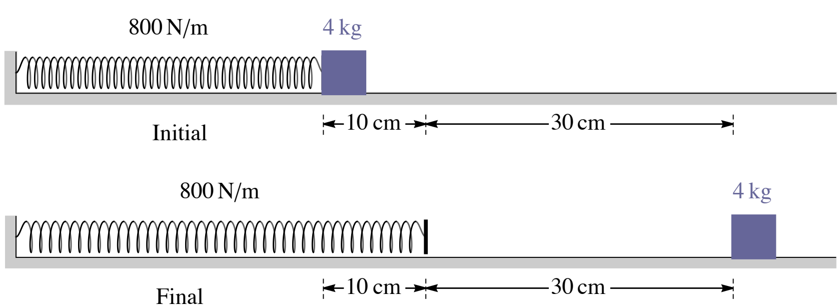
Since there is no friction and no non-conservative energy source then $W_{\text{nc}} = 0$ and mechanical energy is conserved. It is most convenient to choose the lowest point to be the zero of potential energy so we take $y_i = h$ and $y_f = 0$. The initial and final values of x, the compression of the spring from equilibrium are $x_i = x_0$ and $x_f = 0$.

$$E_i = E_f \implies 0 + mgh + \frac{1}{2} k x_0^2 = \frac{1}{2} m v_f^2 + 0 + 0$$

Solving for the final speed gives

$$v_f = \sqrt{\frac{k}{m} x_0^2 + 2gh}\,.$$

Example F.9 - A Spring and a Block



A 4-kg block is pushed along a (level) floor by a spring with a force constant of 800 N/m as shown. Initially the spring is compressed by 10 cm. After leaving the spring the block slides an additional distance of 30 cm before coming to a stop. What is the coefficient of kinetic friction between the block and the floor.

Solution

$$m = 4\,\text{kg}, \quad k = 800\,\text{N/m}, \quad x_0 = 0.10\,\text{m} \text{ and } d = 0.30\,\text{m}$$

Since the floor is level we can set $y = 0$ and omit the gravitational potential energy. This leaves just kinetic energy and elastic potential energy.

$$E = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$$

Because there is friction we have $W_{\text{nc}} = W_{\text{friction}}$. The normal force on the block is just its weight, $N = mg$. The total distance the block slides is $\Delta x = x_0 + d = 0.40\text{ m}$.

$$W_{\text{nc}} = W_{\text{friction}} = -f_k \Delta x = -\mu_k N \Delta x = -\mu_k mg (x_0 + d)$$

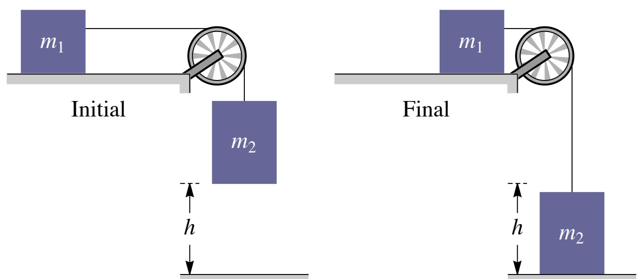
Note that the sign above follows from $\cos 180° = -1$. Both initial and final velocities are zero, $v_i = 0 = v_f$. We also have $x_i = x_0$ and $x_f = 0$.

$$E_i + W_{\text{nc}} = E_f \implies 0 + \frac{1}{2} k x_0^2 - \mu_k mg (x_0 + d) = 0 + 0$$

Solving for the coefficient of kinetic friction we get

$$\mu_k = \frac{k x_0^2}{2\, mg (x_0 + d)} = 0.255\ .$$

Example F.10 - Two Connected Masses



Two blocks of masses m_1 and m_2 are connected by a light string over an ideal pulley as shown. m_1 slides on a horizontal table and m_2 is initially a height h above the floor.

(a) Suppose there is no friction between m_1 and the table. What is the speed of m_2 when it hits the floor?

Solution

We have potential energies of $U = m_1 g y_1$ and $U = m_2 g y_2$ for the two masses. We can choose our zero value for y differently for each mass. Let us choose $y_1 = 0$ along the tabletop; this removes that potential energy term completely. For the hanging mass choose its lowest point to be the zero. So, $y_2 = h$ initially and $y_2 = 0$ at the floor. Both masses have kinetic energies. Because of our simple pulley arrangement both masses move the same distances, $\Delta x_1 = \Delta x_2 = \Delta x$, and thus will have the same speed: $v = v_1 = v_2$. The total kinetic energy becomes

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v^2 + \frac{1}{2} m_2 v^2 = \frac{1}{2} (m_1 + m_2) v^2$$

and the total mechanical energy is

$$E = \frac{1}{2} (m_1 + m_2) v^2 + m_2 g y_2$$

Since there is no friction for part (a) we have $W_{\text{nc}} = 0$ and then conservation of mechanical energy.

$$E_i = E_f \implies 0 + m_2 g h = \frac{1}{2} (m_1 + m_2) v_f^2 + 0$$

The final speed follows.

$$v_f = \sqrt{2 \left(\frac{m_2\, g}{m_1 + m_2} \right) h}$$

The expression was written so that the part inside the brackets is just the linear acceleration that we could have found with a more involved force analysis in Chapter D.

(b) Suppose now that there is a coefficient of kinetic friction μ_k between m_1 and the table. What is the speed of m_2 when it hits the floor?

Solution

Now that we have friction mechanical energy is no longer conserved. W_{nc} is just the work done by friction.

$$W_{\text{nc}} = W_{\text{friction}} = -f_k\,\Delta x = -(\mu_k\,N)\,\Delta x = -(\mu_k\,m_1\,g)\,h$$

We can then write

$$E_i + W_{\text{nc}} = E_f \implies (0 + m_2 g h) - \mu_k\,m_1\,g\,h = \frac{1}{2}(m_1 + m_2)\,v_f^2 + 0$$

Solving for v_f with the acceleration again in brackets gives

$$v_f = \sqrt{2\left(\frac{m_2\,g - \mu_k\,m_1\,g}{m_1 + m_2}\right)h}$$

F.8 - Force, Potential Energy and Stability

The basic relation between potential energy and force is

$$\Delta\,U = -\int \vec{F}\cdot d\vec{r}.$$

This is the rule for going from the force as a function of position to the potential energy. If we move along some path, the path of integration, then we get $\Delta\,U$ for the change in potential energy along that path. The infinitesimal form of this is

$$d\,U = -\vec{F}\cdot d\vec{r}.$$

This is the infinitesimal change in the potential energy when one moves along the infinitesimal displacement $d\vec{r}$.

Force from Potential Energy

The rule for going from force to potential energy leads to the question: can we do this in reverse and go from the potential energy to the force? The answer is yes. To get the x component of the force consider an infinitesimal displacement in the x direction $d\vec{r} = \hat{x}\,d x$. Using the fact that $\vec{F}\cdot\hat{x} = F_x$ we get

$$d\,U = -\vec{F}\cdot d\vec{r} = -\vec{F}\cdot\hat{x}\,d x = -F_x\,d x.$$

It is then a simple matter to solve for the x component of the force

$$F_x = -\frac{d\,U}{d x}.$$

This is simple but it is too simple, to the point of being incorrect when the potential energy is a function of more than one variable. For functions of more than one variable we take partial derivatives and not ordinary derivatives.

$$F_x = -\frac{\partial\,U}{\partial\,x},\quad F_y = -\frac{\partial\,U}{\partial\,y}\ \text{ and }\ F_z = -\frac{\partial\,U}{\partial\,z}.$$

To evaluate a partial derivative take the derivative with respect to the variable treating the other independent variables as constants. In a case when the potential energy is only a function of one variable the original expression with the ordinary derivative is correct, but we do not need the component subscript

$$F = -\frac{d\,U}{d x}.$$

As an example consider gravity. Take the potential to be a function of a horizontal and a vertical variable, x and y.

$$U(x,\,y) = m\,g\,y \quad\implies$$
$$F_x = -\frac{\partial\,U}{\partial\,x} = 0\ \text{ and }\ F_y = -\frac{\partial\,U}{\partial\,y} = -m\,g$$

This is the proper expression for the force. For another example consider the one dimensional case of the elastic potential energy of a spring.

$$U(x) = \frac{1}{2}k\,x^2 \implies F = -\frac{d\,U}{d x} = -k\,x$$

This is our familiar Hooke's law expression.

Example F.11 - Partial Derivatives and Potential Energy

(a) Given the function of two variables

$$f(x, y) = a\,x^3\,y - b\,y^2$$

Evaluate the partial derivatives: $\partial f / \partial x$ and $\partial f / \partial y$.

Solution

To evaluate the partial derivative with respect to one variable, take the ordinary derivative with respect to that variable treating the other variables as constant.

$$\frac{\partial f}{\partial x} = 3\,a\,x^2\,y \quad \text{and} \quad \frac{\partial f}{\partial y} = a\,x^3 - 2\,b\,y$$

(b) Now use the same function as a potential energy and find the force vector.

$$U(x, y) = a\,x^3\,y - b\,y^2$$

Evaluate the partial derivatives: $\partial U / \partial x$ and $\partial U / \partial y$.

Solution

To evaluate the partial derivative with respect to one variable, take the ordinary derivative with respect to that variable treating the other variables as constant.

$$F_x = -\frac{\partial U}{\partial x} = -3\,a\,x^2\,y \quad \text{and} \quad F_y = -\frac{\partial U}{\partial y} = -a\,x^3 + 2\,b\,y$$

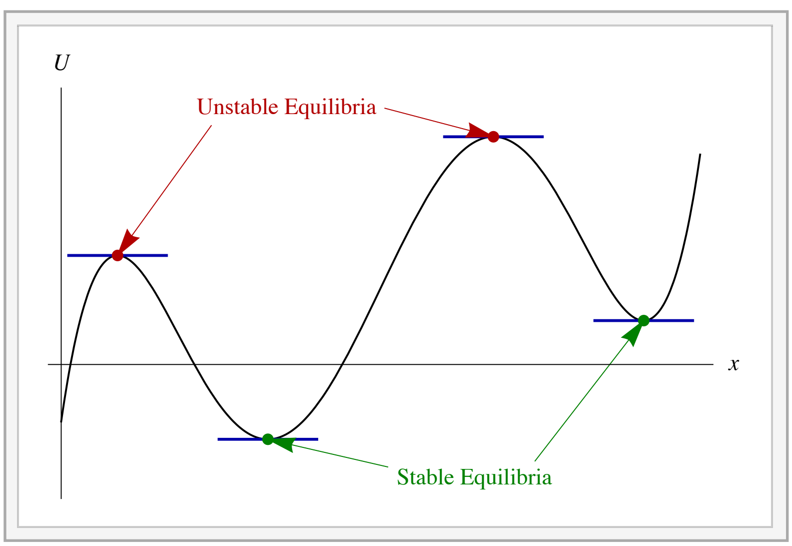
Write this as a vector.

$$\vec{F} = \langle F_x, F_y \rangle = \left\langle -3\,a\,x^2\,y\,,\, -a\,x^3 + 2\,b\,y \right\rangle$$

Stability and Potential Energy Diagrams

The expression $F = -dU/dx$ implies that the direction of the force is always toward lower potential energy. If we consider a graph of potential energy as a function of position then the force is toward lower U. For instance is the slope of U vs. x is positive then the force is negative and if the slope is negative the force is positive.

An equilibrium point is a point of zero force and thus zero slope. These are the local minima and maxima. If x is a small distance away from a local minima then the force is toward the minimum; we call this a *stable equilibrium*. Around a local maximum the opposite happens. When a small distance away from equilibrium the force is away; this is called an *unstable equilibrium*.



Interactive Figure