Chapter H

Rotational Kinematics and Energy

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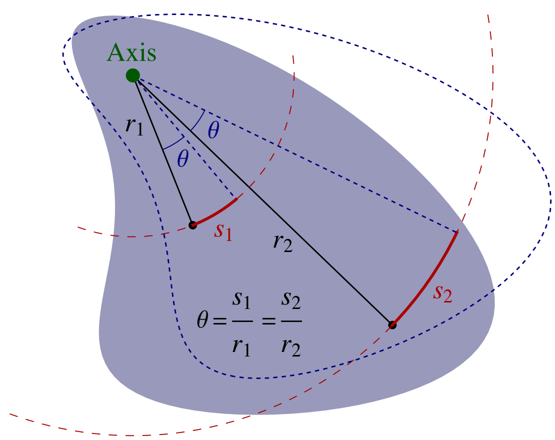
H.1 - Kinematics of Rotations about a Fixed Axis

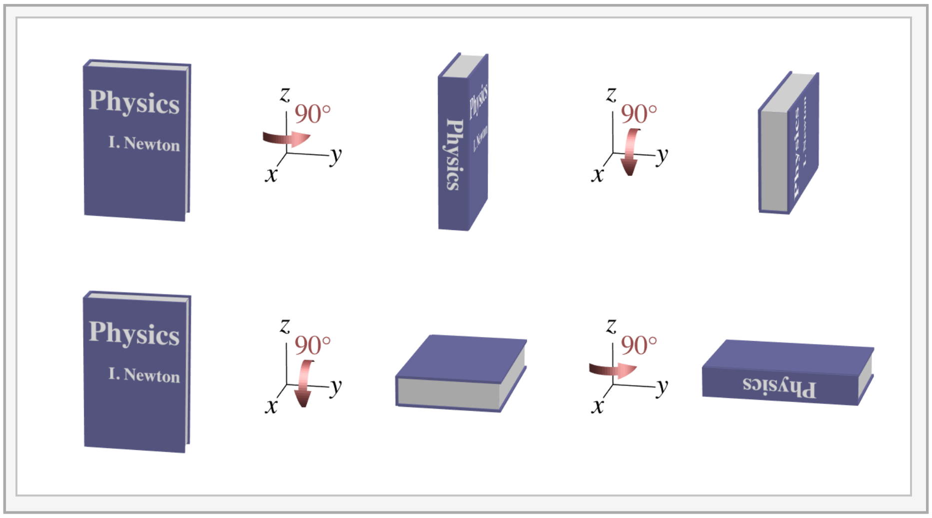
Rigid Bodies and Rotations in General

The distance between any two positions in a rigid body is fixed. A book can be viewed as a rigid body as long as it is kept closed; when it is opened then the distance between a point on the back cover and a point on the front cover varies and it is not rigid.

A rotation is described by an axis and an angle. An axis is a line. The axis of rotation of a door is its hinge. The axis of a tire is its axle. Often in a planar diagram we will draw an axis as a point. The axis is then the line perpendicular to that plane through the point. A rotation is about some axis and by some angle. Note that when a rigid body rotates different points move different distances. The distance a point moves s is proportional to the (perpendicular) distance from the axis r, but the ratio s/r is the same for any two points. This ratio is just the angle of rotation in radians.



In general, three dimensional rotations about different axes do no commute; this means that changing the order of two rotations gives a different answer. The interactive diagram below demonstrates this. Rotate the book by a 90° angle about the vertical axis and then by 90° about a horizontal axis. Then, starting from the same orientation as before, the two rotations are repeated in the opposite order. The book ends up in a different orientation.



Interactive Figure - Rotations about different axes do not commute.

Kinematical Variables

To understand rotational kinematics it is essential to appreciate the analogy to one dimensional kinematics. Recall that a one dimensional vector is a real number and that its direction is given by its sign. The rotational analog of the position is the angle of rotation. The other kinematical variables follow:

	One Dimensional Linear Motion	**Rotations about a Fixed Axes**
Position	x (m)	θ (angle in rad)
Velocity	v (m/s)	ω (angular velocity in rad/s)
Average	$v_{\text{ave}} = \frac{\Delta x}{\Delta t}$	$\omega_{\text{ave}} = \frac{\Delta \theta}{\Delta t}$
Instantaneous	$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$
Acceleration	a $\left(\text{m}/\text{s}^2\right)$	α $\left(\text{angular acceleration in rad}/\text{s}^2\right)$
Average	$a_{\text{ave}} = \frac{\Delta v}{\Delta t}$	$\alpha_{\text{ave}} = \frac{\Delta \omega}{\Delta t}$
Instantaneous	$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$

Constant Angular Acceleration

Since the rotational variables θ, ω and α are interrelated the same as x, v and a, we can find expressions for constant angular acceleration.

One Dimensional Linear Motion	**Rotations about a Fixed Axes**
$v = v_0 + a\,t$	$\omega = \omega_0 + \alpha\,t$
$\Delta x = \frac{1}{2}\,(v_0 + v)\ t$	$\Delta \theta = \frac{1}{2}\,(\omega_0 + \omega)\ t$
$\Delta x = v_0\,t + \frac{1}{2}\,a\,t^2$	$\Delta \theta = \omega_0\,t + \frac{1}{2}\,\alpha\,t^2$
$v^2 = v_0^2 + 2\,a\ \Delta x$	$\omega^2 = \omega_0^2 + 2\,\alpha\ \Delta \theta$

Example H.1 - Decelerating Ceiling Fan

In 8 s, a ceiling fan slows uniformly from 20 rev/min to 8 rev/min.

(a) What is the angular acceleration of the fan?

Solution

We need to convert our angular velocities, the rates of rotation, from rev/min to rad/s.

$$\omega_0 = 20\,\frac{\text{rev}}{\text{min}} \times \frac{2\,\pi\,\text{rad}}{\text{rev}} \times \frac{1\text{ min}}{60\text{ s}} = 2.0944\,\frac{\text{rad}}{\text{s}}$$

$$\omega = 8\,\frac{\text{rev}}{\text{min}} \times \frac{2\,\pi\,\text{rad}}{\text{rev}} \times \frac{1\text{ min}}{60\text{ s}} = 0.83776\,\frac{\text{rad}}{\text{s}}$$

We also know $t = 8$ s and we are looking for α. Use the first equation for constant angular acceleration: $\omega = \omega_0 + \alpha\,t$.

$$\omega = \omega_0 + \alpha\,t \implies \alpha = \frac{\omega - \omega_0}{t} = -0.157\,\frac{\text{rad}}{\text{s}^2}$$

(b) How many times did the fan rotate while slowing?

Solution

The number of rotations is related to the rotation angle $\Delta\theta$.

$$\#\text{ of rotations } = \frac{\Delta\theta}{2\,\pi}$$

We can use any constant angular acceleration equation that involves $\Delta\theta$ to find this, since we have already found α. To find the answer without reference to the α found in part (a) we will use the second equation.

$$\Delta\,\theta = \frac{1}{2}\,(\omega_0 + \omega)\;t = 11.729\text{ rad} \implies \frac{\Delta\theta}{2\,\pi} = 1.87$$

Relation Between Linear and Rotational Variables

In chapter E we described circular motion in terms of centripetal and tangential coordinates, where the centripetal direction is toward the center of the circle and the tangential direction is in the direction of motion. A point on a rigid body a distance r from the center moves in a circle of radius r, so that discussion is relevant here.

The velocity is purely tangential

$$\vec{v} = v_t\,\hat{u}_t.$$

v_t is just the speed, which is just $v_t = \frac{ds}{dt}$ where $d\,s$ is the infinitesimal arc length. Since by the definition of angles in radians with is related to the infinitesimal angle $d\,s = r\,d\,\theta$. Since $d\,\theta/d\,t = \omega$ we get

$$\vec{v} = v_t\,\hat{u}_t = r\,\omega\,\hat{u}_t \text{ or in other words } v_t = r\,\omega \text{ and } v_c = 0.$$

In chapter E we saw the acceleration had the form

$$\begin{aligned}\vec{a} &= a_c\,\hat{u}_c + a_t\,\hat{u}_t \\ &= \frac{v^2}{r}\,\hat{u}_c + \frac{d\,v}{d\,t}\,\hat{u}_t\,.\end{aligned}$$

We can now rewrite the centripetal acceleration in terms of the rotational variables

$$a_c = \frac{v^2}{r} = \frac{(r\,\omega)^2}{r} = \omega^2\,r.$$

The tangential component of acceleration can be written similarly.

$$a_t = \frac{d\,v}{d\,t} = \frac{d\,r\,\omega}{d\,t} = r\,\frac{d\,\omega}{d\,t} = r\,\alpha.$$

Summarizing for the acceleration

$$\vec{a} = a_c\,\hat{u}_c + a_t\,\hat{u}_t$$

$$= \omega^2 \, r \; \hat{u}_c + r \, \alpha \; \hat{u}_t \, .$$

Example H.2 - Decelerating Ceiling Fan (continued)

(c) While the fan is slowing there is an instant when $\omega = 0.85$ rad/s. At that instant, what are the tangential velocity and the magnitude of the acceleration of the tip of the fan, 0.90 m from the axis.

Solution

$$\omega = 0.85 \text{ rad/s}\,, \quad r = 0.90 \text{ m} \;\text{ and }\; \alpha = -0.157 \, \frac{\text{rad}}{\text{s}^2} \quad \text{(from part (a))}$$

We can solve for the tangential velocity.

$$v_t = r \, \omega = 0.765 \text{ m/s}$$

To find the magnitude of the acceleration we use the Pythagorean theorem with the perpendicular components a_c and a_t.

$$a_c = \omega^2 \, r = 0.65025 \, \frac{\text{m}}{\text{s}^2} \;\text{ and }\; a_t = r \, \alpha = -0.14137 \, \frac{\text{m}}{\text{s}^2} \quad \Longrightarrow \quad a = \sqrt{a_c^2 + a_t^2} = 0.665 \, \frac{\text{m}}{\text{s}^2}$$

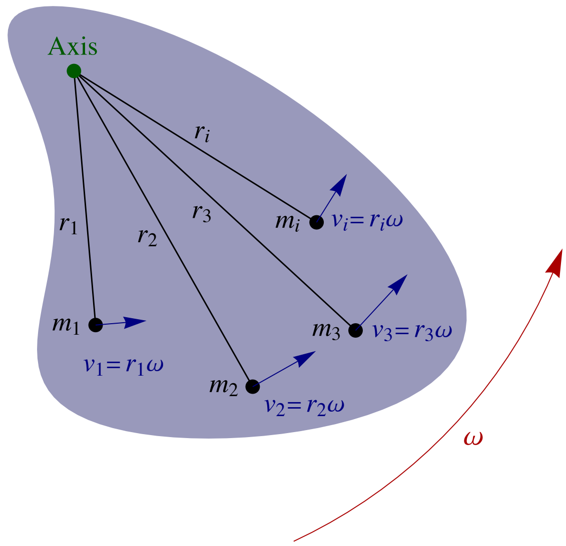
H.2 - Dynamics of Rigid Bodies Rotating about an Axis

Summary and Analogy with One Dimensional Motion

	One Dimensional Linear Motion	**Rotations about a Fixed Axes**
Kinematics	$x,\ v,\ a$	$\theta,\ \omega,\ \alpha$
Force	F	τ (torque)
Inertia	m	I (moment of inertia)
Momentum	$p = m\,v$	$L = I\,\omega$ (angular momentum)
Second Law	$F_{\text{net}} = m\,a$ $F_{\text{net}} = \frac{d}{dt}\,p$	$\tau_{\text{net}} = I\,\alpha$ $\tau_{\text{net}} = \frac{d}{dt}\,L$
Conservation of Momentum	$F_{\text{net}}^{\text{ext}} = 0$ $\Longrightarrow \Delta\,p_{\text{tot}} = 0$	$\tau_{\text{net}}^{\text{ext}} = 0$ $\Longrightarrow \Delta\,L_{\text{tot}} = 0$
Kinetic Energy	$K = \frac{1}{2}\,m\,v^2$	$K = \frac{1}{2}\,I\,\omega^2$
Work	$W = \int F\,dx$	$W = \int \tau\,d\theta$
Work–Energy Theorem	$W_{\text{net}} = \Delta\,K$	$W_{\text{net}} = \Delta\,K$
Power	$\mathcal{P} = \frac{dW}{dt} = F\,v$	$\mathcal{P} = \frac{dW}{dt} = \tau\,\omega$

This table is an extension of the preceding tables for kinematics. Now we consider dynamics. Dynamical quantities are things like force and mass. The rotational analog of force is called torque and the rotational analog of mass is the moment of inertia. These two quantities are undefined in the table; their definitions follow. For all the other quantities, the above table serves as the definitions of the variables.

Kinetic Energy and the Definition of the Moment of Inertia



Consider a rigid body consisting of point masses m_i. The perpendicular distance from the axis to the i^{th} mass is r_i. If the rigid body rotates with angular velocity ω then the speed of the i^{th} mass is

$$v_i = r_i\,\omega.$$

The total kinetic energy is the sum of the kinetic energies of all the masses. Using the above expression for the speed we get

$$K = \frac{1}{2}\sum_i m_i\,v_i^2 = \frac{1}{2}\sum_i m_i\,r_i^2\,\omega^2 = \frac{1}{2}\left(\sum_i m_i\,r_i^2\right)\omega^2.$$

Using our desired expression for the kinetic energy $K = \frac{1}{2} I\,\omega^2$ we get the expression for moment of inertia for a rigid body about some axis.

$$I = \sum_i m_i\,r_i^2.$$

This expression is for a discrete distribution; this means that the distribution is a collection of point masses.

For a continuous distribution we replace the sum with an integral. Break up the distribution into in infinite number of infinitesimal pieces. Take dm to be the mass of one of the infinitesimal pieces. The perpendicular distance from the axis to dm is r. The total mass is M which is just the sum (integral) of all the infinitesimal pieces.

$$M = \int dm$$

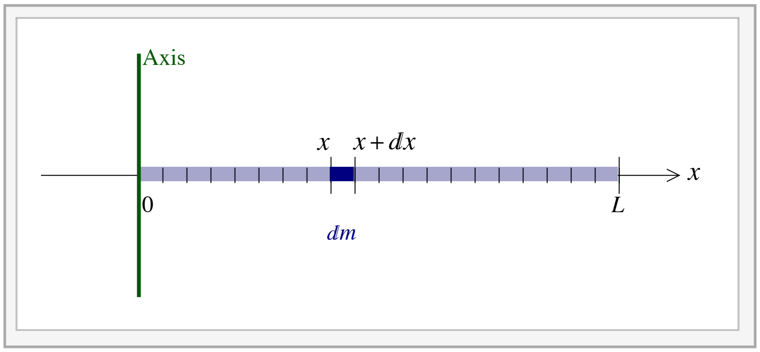
The moment of inertia is the sum of all the $r^2\,dm$. This gives

$$I = \int r^2\,dm.$$

Moments of Inertia for Uniform Bodies

We can use the formula above to calculate the moments of inertia for certain simple geometric shapes with uniform mass distributions. By uniform distribution of mass we mean the density is constant throughout the body.

Thin Rod about Perpendicular Axis Through End



Interactive Figure

Consider a rod of negligible thickness, mass M and length L. Take the x axis to be along the rod with $x = 0$ at the axis. To integrate over the whole rod we will choose x as our integration variable. The limits of integration are

$$0 \leq x \leq L.$$

dm is the mass between x and $x + dx$. Since the distribution is uniform we can conclude that the fraction of the mass is the same as the fraction of the length. The fraction of the length is dx/L. This gives:

$$dm = \frac{M}{L}\,dx.$$

The perpendicular distance from the axis to x is $r = |x|$.

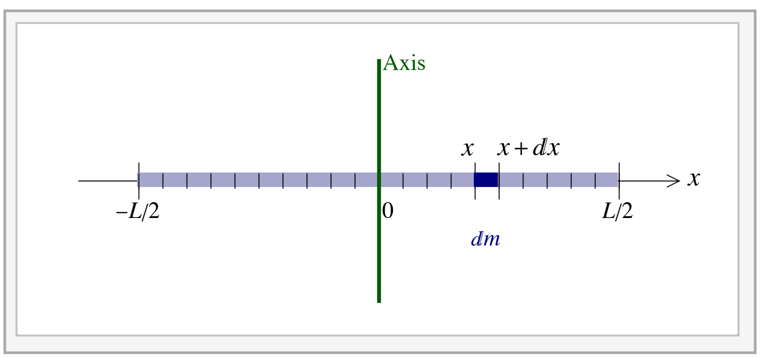
Putting all this together we can write the moment as an integral.

$$I = \int r^2\,dm = \int_0^L x^2\,\frac{M}{L}\,dx = \frac{M}{L}\left(\frac{1}{3}\,L^3 - 0\right)$$

This gives our result

$$I = \frac{1}{3}\,M\,L^2.$$

Thin Rod about Perpendicular Axis Through Center



Interactive Figure

This is the same as before except that our limits of integration are different. The limits of integration are

$$-\frac{L}{2} \leq x \leq \frac{L}{2}.$$

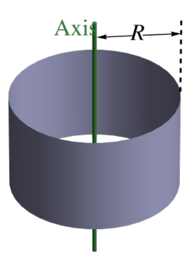
The expression for dm is the same and the integral becomes:

$$I = \int r^2\,dm = \int_{-L/2}^{L/2} x^2\,\frac{M}{L}\,dx = \frac{M}{L}\,\frac{1}{3}\left(\frac{L^3}{8} - -\frac{L^3}{8}\right).$$

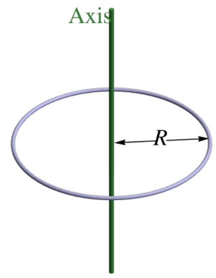
We then get

$$I = \frac{1}{12} M L^2.$$

Hoop or Thin-shelled Hollow Cylinder about Perpendicular Axis through Center



Hollow Cylindrical Shell



Hoop

First consider a hoop of mass M and radius R rotating about a perpendicular axis through the center. r is the distance from the axis to the infinitesimal mass dm. All the mass is at the same distance

$$r = R = \text{constant}.$$

It is possible to find I without actually performing an integral.

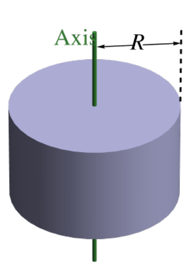
$$I = \int r^2 \, dm = \int R^2 \, dm = R^2 \int dm.$$

Since $M = \int dm$ we get

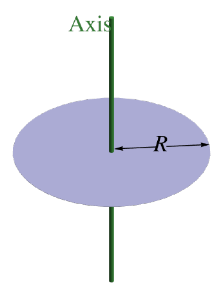
$$I = M R^2.$$

Now consider a thin-shelled hollow cylinder about the central axis. It is still true that all the mass is the same perpendicular distance of $r = R$ from the axis and the above formula still applies.

Disk or Solid Cylinder about Perpendicular Axis through the Center



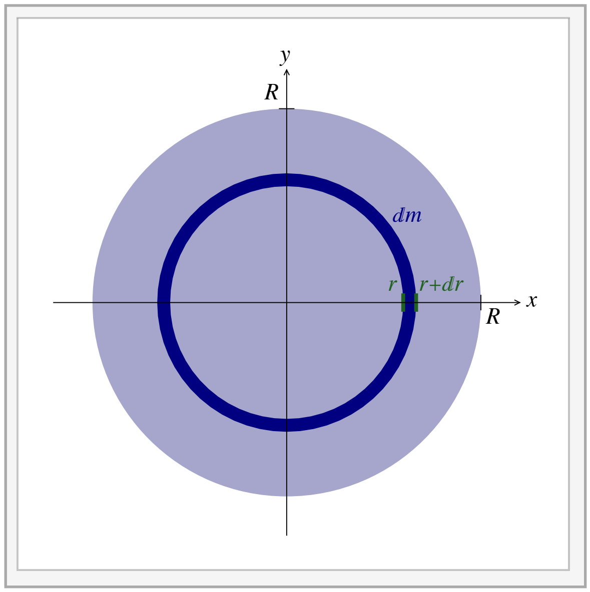
Solid Cylinder



Disk

It should now be clear that the moment for a disk should be the same as a solid cylinder. We can break up a disk into concentric thin rings of radius r with thickness dr. The limits of integration become

$$0 \le r \le R.$$



Interactive Figure

The infinitesimal area of a thin ring can be written as the length of the ring, which is the circumference $2\pi r$ multiplied by its thickness dr.

$$dA = 2\pi r\, dr$$

The uniform distribution implies that

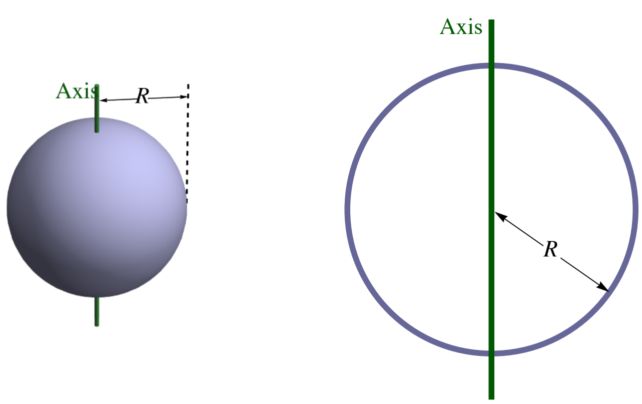
$$dm = \frac{M}{A_{\text{tot}}}\, dA = \frac{M}{\pi R^2}\, 2\pi r\, dr \implies dm = \frac{2M}{R^2}\, r\, dr.$$

$$I = \int r^2\, dm = \int_0^R r^2\, \frac{2M}{R^2}\, r\, dr = \frac{2M}{R^2} \int_0^R r^3\, dr = \frac{2M}{R^2}\left(\frac{1}{4} R^4 - 0\right).$$

The final result is

$$I = \frac{1}{2} M R^2.$$

Thin-shelled Hollow Sphere about Axis through Center



The integral for the moment is $I = \int r^2\, dm$, where r is the distance from an axis. The r used in three dimensions is not the distance from an axis but the distance from an origin $\sqrt{x^2 + y^2 + z^2}$. To avoid confusion between the different r we will refer to this last value as ρ.

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

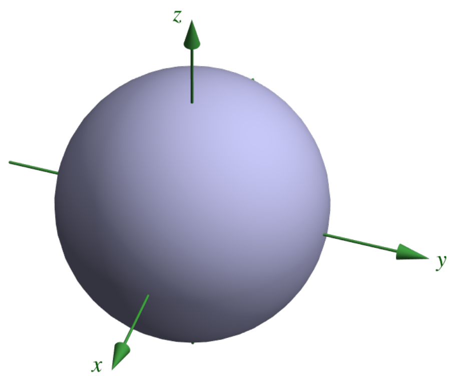
For *any* spherical distribution our calculation is made awkward because the integral involves r, but the distribution is symmetric with respect to

ρ. It is possible to perform the necessary integrals but here we will introduce a trick that will greatly simplify the integration. The perpendicular distance from the z axis is $\sqrt{x^2+y^2}$, so the moment of inertia about the z axis is

$$I_z = \int r^2\, dm = \int \left(x^2+y^2\right) dm.$$

We can similarly find the moments about the x and y axes.

$$I_x = \int r^2\, dm = \int \left(y^2+z^2\right) dm$$
$$I_y = \int r^2\, dm = \int \left(x^2+z^2\right) dm$$



Since we have spherical symmetry these three moments must be the same $I_x = I_y = I_z$. It follows that we can write the moment as 1/3 the sum of the three. Doing this we get an integral in terms of ρ instead of r.

$$I = \frac{1}{3}\left(I_x+I_y+I_z\right) = \frac{2}{3}\int \left(x^2+y^2+z^2\right) dm = \frac{2}{3}\int \rho^2\, dm$$

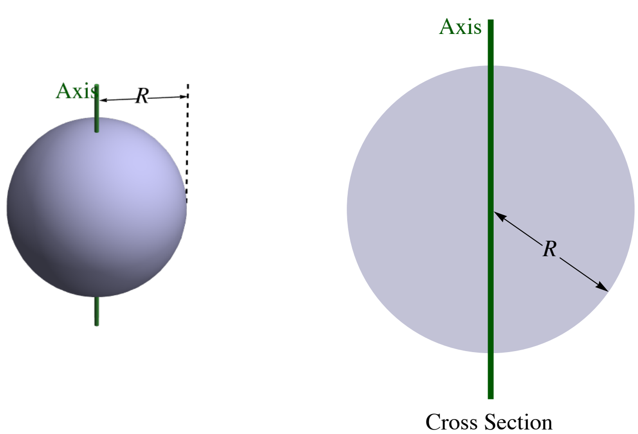
Applying this general spherical trick to the case of a thin-shelled hollow sphere the resulting integral is trivial. All the mass in this case is at the same distance, the radius of the sphere $\rho = R =$ constant.

$$I = \frac{2}{3}\int \rho^2\, dm = \frac{2}{3}\int R^2\, dm = \frac{2}{3} R^2 \int dm$$

The final result becomes

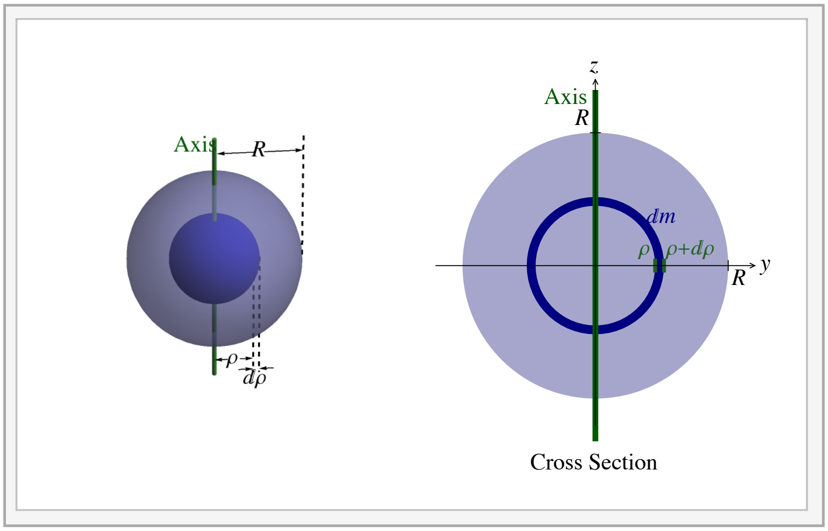
$$I = \frac{2}{3} M R^2$$

Solid Sphere about Axis through Center



Now we consider a uniform solid sphere. We can use the general spherical trick discussed above in this case as well. Here we break up the sphere into thin concentric spheres. Take the integration variable to be ρ and its limits to be

$$0 \le \rho \le R.$$



Interactive Figure

The infinitesimal volume of a thin sphere can be written as the area of the sphere, which is the area $4\pi\rho^2$ multiplied by its thickness $d\rho$.

$$dV = 4\pi\rho^2\, d\rho$$

The uniform distribution implies that

$$dm = \frac{M}{V_{\text{tot}}}\, dV = \frac{M}{\frac{4}{3}\pi R^3}\, 4\pi\rho^2\, d\rho \implies dm = \frac{3M}{R^3}\,\rho^2\, d\rho.$$

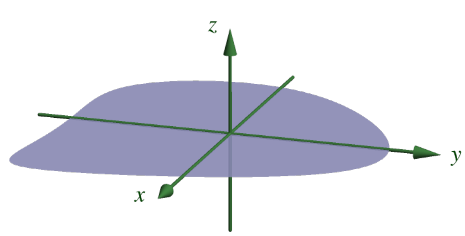
Inserting this into our integral we get

$$I = \frac{2}{3}\int \rho^2\, dm = \frac{2}{3}\int_0^R \rho^2\, \frac{3M}{R^3}\,\rho^2\, d\rho = \frac{2M}{R^3}\int_0^R \rho^4\, d\rho.$$

The final result becomes

$$I = \frac{2}{5} M R^2.$$

Planar Objects and the Rectangle



I_x, I_y and I_z are the moments of inertia about the x, y and z axes. For any planar (flat) object in the xy-plane

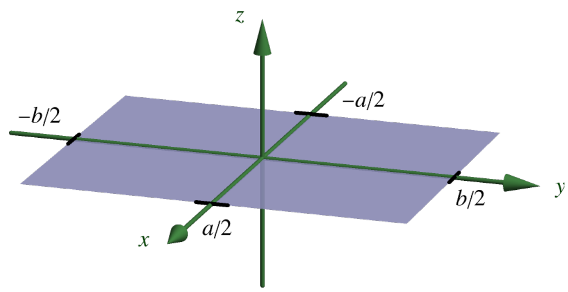
$$I_z = I_x + I_y.$$

The proof is simple. If the object is entirely in the xy-plane then $z = 0$ for the entire mass distribution. Since r is the perpendicular distance form the axis in the definition $I = \int r^2\, dm$, we have

$$I_x = \int \left(y^2 + z^2\right) dm = \int y^2\, dm \text{ and } I_y = \int \left(x^2 + z^2\right) dm = \int x^2\, dm.$$

The proof follows easily.

$$I_z = \int \left(x^2 + y^2\right) dm = \int x^2\, dm + \int y^2\, dm = I_y + I_x$$



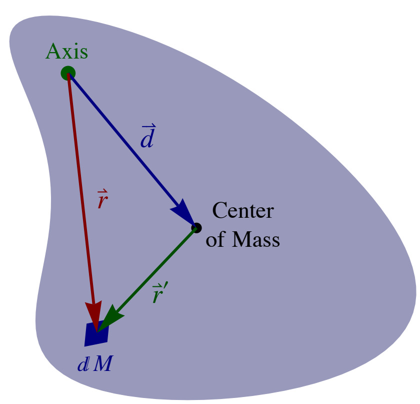
As an example, consider rectangular plate about perpendicular axis through the center, Take the plate to have dimensions $a \times b$ and take the a to be the length in the x direction and b to be the y length. I_x and I_y are equivalent to the moments of uniform rods about the center or length b and a.

$$I = I_z = I_x + I_y = \frac{1}{12} M\, b^2 + \frac{1}{12} M\, a^2 \implies I = \frac{1}{12} M \left(a^2 + b^2\right).$$

Parallel-Axis Theorem

The parallel-axis theorem relates the moment of a rigid body about some axis to the moment about the axis parallel to the first and passing through the center of mass. If I is the moment about an axis, I_{cm} is the moment about the parallel axis that passes through the center of mass (the center of mass axis) and d is the distance between the two axes, then the parallel axis theorem is

$$I = I_{\text{cm}} + M\, d^2.$$



To prove this take $\vec{r}$ to be the perpendicular vector from the original axis to the infinitesimal mass dm. Take $\vec{r}'$ to be the perpendicular vector from the center of mass axis to dm. Take $\vec{d}$ as the perpendicular vector from the original axis to the center of mass axis. These vectors are related by

$$\vec{r} = \vec{r}' + \vec{d}.$$

Squaring this gives

$$r^2 = \left(\vec{r}\,' + \vec{d}\right) \cdot \left(\vec{r}\,' + \vec{d}\right) = r'^2 + d^2 + 2\,\vec{d} \cdot \vec{r}\,'.$$

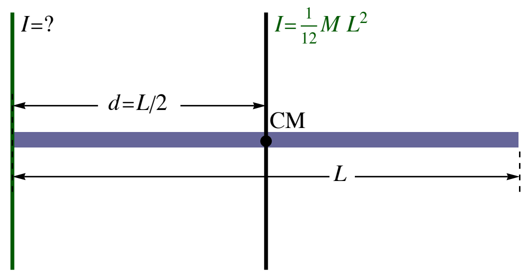
With this we can rewrite the moment of inertia

$$I = \int r^2\, dm = \int r'^2\, dm + d^2 \int dm + 2\,\vec{d} \cdot \int \vec{r}\,'\, dm.$$

The first term is just I_{cm} the second is $M\,d^2$ and the third must be zero for our result to be true. $\int \vec{r}\,'\, dm$ is related to the center of mass in the $\vec{r}\,'$ coordinates and this is just zero $\int \vec{r}\,'\, dm = M\,\vec{r}\,'_{\text{cm}} = \vec{0}$

Example H.3 - The Uniform Thin Rod

We have two formulas for the moment of inertia for a thin rod about perpendicular axis, one through the center and the other through the end. With the parallel-axis theorem we can derive the expression about one end from the other. Prove $I = \frac{1}{3}\,M\,L^2$ about an end using $I = \frac{1}{12}\,M\,L^2$ about a perpendicular axis through the center.



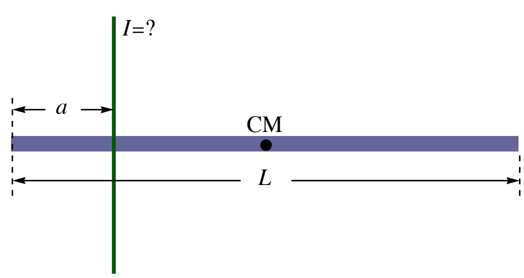
Solution

The distance from the center to the end is $d = L/2$. Here we have $I_{\text{cm}} = \frac{1}{12}\,M\,L^2$ so using $\frac{1}{3} = \frac{1}{12} + \frac{1}{4}$ we get our result.

$$I = I_{\text{cm}} + M\,d^2 = \frac{1}{12}\,M\,L^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}\,M\,L^2$$

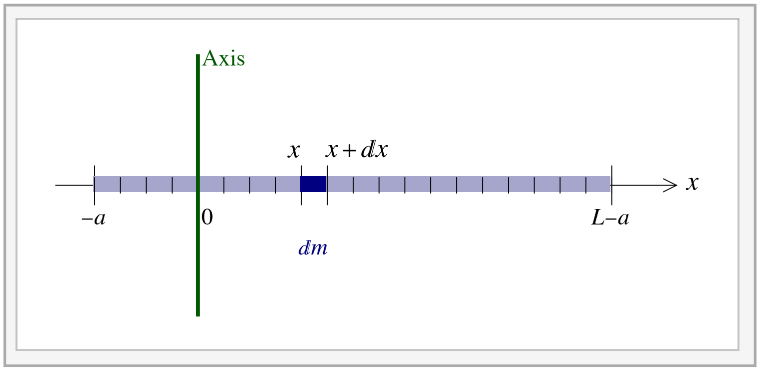
Example H.4 - Uniform Rod by Two Methods

What is the moment of inertia of a uniform rod of length L and mass M about a perpendicular axis through a point a distance a from one end?



(a) Solve this by explicitly doing the integral.

Solution



Interactive Figure

We have solved the integration problem for the rod in two previous cases: for the axis at one end where the limits of integration were 0 and L, and for the axis through the center where the limits were $\pm L/2$. Now we will again use x as our integration variable but now with limits $-a$ and $L-a$.

$$-a \le x \le L-a$$

The dm is the same as the previous cases

$$dm = \frac{M}{L}\,dx$$

and, as before, the perpendicular distance from the axis to x is $r = |x|$. Putting all this together and evaluate the integral.

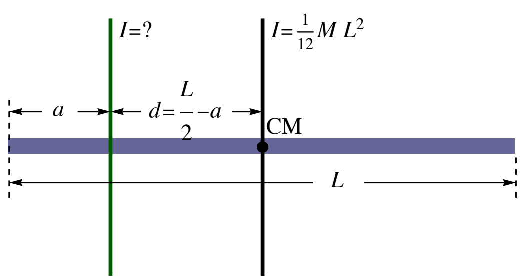
$$I = \int r^2\,dm = \int_{-a}^{L-a} x^2\,\frac{M}{L}\,dx = \frac{M}{L}\left(\frac{1}{3}(L-a)^3 - \frac{1}{3}(-a)^3\right)$$

Simplifying this, we get

$$I = M\left(\frac{L^2}{3} - a\,L + a^2\right).$$

(b) Solve this using the parallel-axis theorem.

Solution



The distance from the center to the end is $d = L/2 - a$. Here we have $I_{\mathrm{cm}} = \frac{1}{12}\,M\,L^2$.

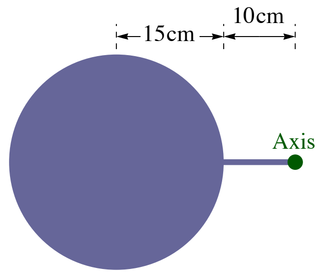
$$I = I_{\mathrm{cm}} + M\,d^2 = \frac{1}{12}\,M\,L^2 + M\left(\frac{L}{2} - a\right)^2$$

Simplifying this we get the same result.

$$I = \frac{1}{12}\,M\,L^2 + \frac{1}{4}\,M\,L^2 - 2\,M\,\frac{L}{2}\,a + M\,a^2 = M\left(\frac{L^2}{3} - a\,L + a^2\right)$$

Example H.5 - Ball on a Light Rod

A uniform solid ball has a mass of 3.5 kg and a radius of 15 cm. It is a the end of a 10-cm light (massless) rod. What is the moment of inertia of the ball about the far end of the rod?



Solution

We will use the parallel axis theorem with $I_{\text{cm}} = \frac{2}{5} M R^2$ and with $d = 25$ cm.

$$M = 3.5\,\text{kg}, \quad R = 0.15\,\text{m}, \quad d = 0.25\,\text{m}$$

$$I = I_{\text{cm}} + M\,d^2 = \frac{2}{5} M\,R^2 + M\,d^2 = 0.250\,\text{kg m}^2$$

H.3 - Energy and Rigid Bodies

Gravitational Potential Energy

It is a straightforward matter to find the potential energy of a rigid body.

$$U = \sum_i m_i\,g\,y_i = g \sum_i m_i\,y_i = g\,M\,y_{\text{cm}}$$

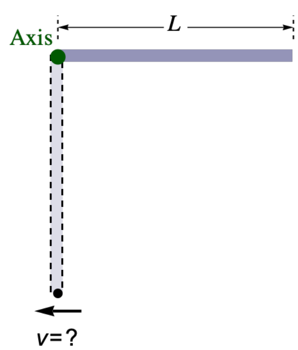
Here M is the total mass and y_{cm} is the height of the center of mass. It follows that the total potential energy of a rigid body is

$$U = M\,g\,y_{\text{cm}}.$$

This is easy to interpret. When calculating the potential energy of a rigid body we treat the body as if all its mass is at its center of mass.

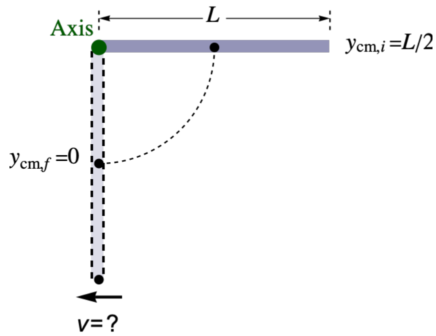
Example H.6 - Swinging Rod

A uniform rod of length L swings without friction about an axis at one end. It is released from rest from a horizontal position. What is the speed of the tip as it swings through the position directly below the axis"



Solution

We will use conservation of energy to find the angular velocity of the rod below and from that find the linear velocity of the tip.



We have kinetic energy $K = \frac{1}{2} I \omega^2$ in the rotating rod and potential energy $U = m g y_{cm}$, where y_{cm} is the height of the center of the rod. The mechanical energy is conserved and given by.

$$E = \frac{1}{2} I \omega^2 + m g y_{cm}$$

The initial kinetic energy is zero and the initial and final heights of the center of mass are $y_{cm,i} = L/2$ and $y_{cm,f} = 0$, choosing the lower point as the zero.

$$E_i = E_f \implies 0 + m g \frac{L}{2} = \frac{1}{2} I \omega_f^2 + 0$$

For our uniform thin rod we have $I = \frac{1}{3} m L^2$

$$m g \frac{L}{2} = \frac{1}{2} I \omega_f^2 = \frac{1}{2} \left(\frac{1}{3} m L^2 \right) \omega_f^2 \implies \omega_f = \sqrt{\frac{3 g}{L}}$$

The velocity can be found from the angular velocity using the tangent velocity formula, $v_t = r \omega$. Since r is the distance from the axis we have $r = L$.

$$v = r \omega = L \omega_f = L \sqrt{\frac{3g}{L}} = \sqrt{3 g L}$$

Rotation with Translation - Rolling Motion

Now consider the case of a body with both rotational and translational motion. As examples think of a spinning ball and of a rolling object. For a system of particles we derived the result

$$K_{tot} = K'_{tot,cm} + \frac{1}{2} M v_{cm}^2.$$

K_{tot} is the total kinetic energy of the system and $K'_{tot,cm}$ is the total kinetic energy in the center of mass frame. This center of mass energy is, for a rotating rigid body, just

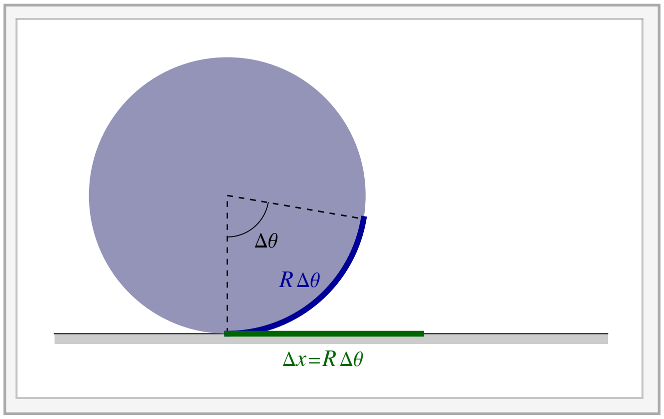
$$K'_{tot,cm} = \frac{1}{2} I_{cm} \omega^2.$$

It follows that we can write

$$K_{tot} = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2.$$

In the case of a rolling body we have a rolling constraint that a body rolls without slipping. This is that the arc length along the rolling radius of the body is the same as the distance Δx it moves along the surface it rolls on. If it rotates by an angle $\Delta\theta$ then the arc length is $R \Delta \theta$. The constraint becomes

$$R \Delta \theta = \Delta x.$$



Interactive Figure

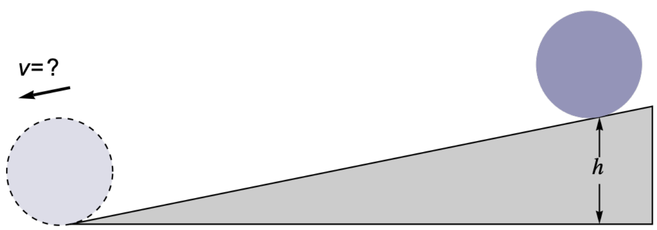
Since the velocity is $v = dx/dt$ and the angular velocity is $\omega = d\theta/dt$ the rolling constraint becomes

$$R\,\omega = v.$$

Taking another derivative gives the acceleration and angular acceleration and we get

$$R\,\alpha = a.$$

Example H.7 - Rigid Body Races



Different objects, a uniform solid sphere, a uniform hollow spherical shell, a uniform solid cylinder and a uniform hollow cylindrical shell, are rolled down an incline. The objects have varying masses and radii. Which will move fastest at the bottom of the incline, which we take to be of height h? Find the speed of each. Assume that they roll without slipping.

Solution

We will solve all four cases by writing $I = \kappa\,M\,R^2$ where the table below gives the different κ values.

Object $I = \kappa\,M\,R^2$	κ
Solid Sphere	2/5
Solid Cylinder	1/2
Hollow Spherical Shell	2/3
Hollow Cylndrical Shell	1

We will see that the mass M and radius R scale out of the problem and the speed at the bottom will only depends on κ and h. The only potential energy here is gravitational potential energy and that is determined by the position of the center of mass.

$$U = M\,g\,y_{\text{cm}}$$

For the total kinetic energy we have translational and rotational terms.

$$K_{\text{tot}} = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2}\,M\,v^2 + \frac{1}{2}\,I\,\omega^2$$

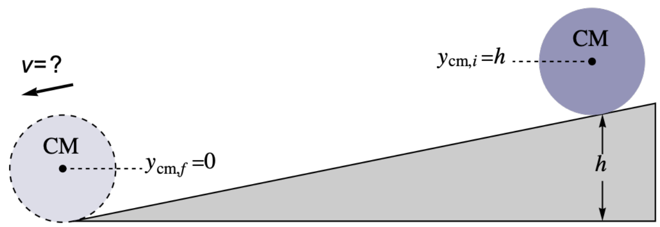
Our rolling without slipping constraint $R\,\Delta\,\theta = \Delta\,x$ implies that $R\,\omega = v$. Using $I = \kappa\,M\,R^2$ and $\omega = v/R$ we get:

$$K_{\text{tot}} = \frac{1}{2}\,M\,v^2 + \frac{1}{2}\,I\,\omega^2 = \frac{1}{2}\,M\,v^2 + \frac{1}{2}\left(\kappa\,M\,R^2\right)\left(\frac{v}{R}\right)^2 = \frac{1}{2}\,(1+\kappa)\,M\,v^2.$$

It follows that our total mechanical energy is

$$E = K_{\text{tot}} + U = \frac{1}{2}\,(1+\kappa)\,M\,v^2 + M\,g\,y_{\text{cm}}.$$

Choosing the lowest position of the center of mass to be $y_{\text{cm}} = 0$ we get $y_{\text{cm},i} = h$ and $y_{\text{cm},f} = 0$. Out initial kinetic energy is zero.



Conservation of mechanical energy gives:

$$E_i = E_f \implies 0 + M\,g\,h = \frac{1}{2}\,(1+\kappa)\,M\,v^2 + 0 \implies v = \sqrt{\frac{2\,g\,h}{1+\kappa}}\,.$$

It should now be clear that the smaller the κ, the larger the speed at the bottom. The order from fastest to slowest is: solid sphere, solid cylinder, hollow sphere and hollow cylinder.