Chapter I

Rotational Dynamics and Equilibrium

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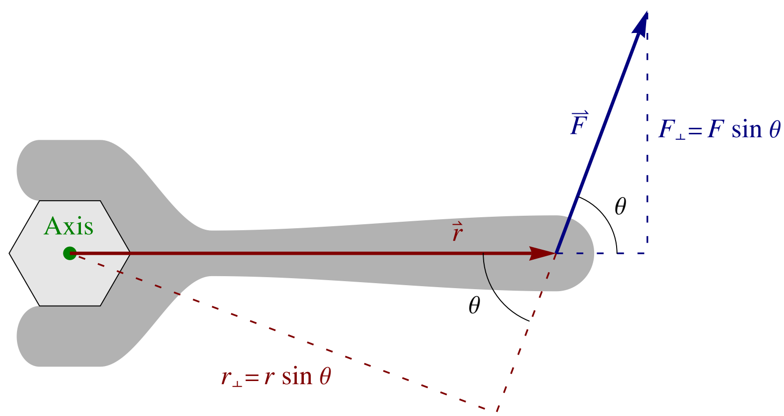
I.1 - The Vector Nature of Rotational Quantities

Torque about an Axis



We define torque as the rotational analog of force. Suppose you are trying to loosen a bolt. The axis of rotation is the center of the bolt. If you are unable to give sufficient torque with your hand you grab a wrench. Take $\vec{r}$ as the vector from the axis to where the force $\vec{F}$ is applied. Clearly the important part of the force is the component of the force perpendicular to the radial vector $\vec{r}$. Moreover the larger r is the larger the torque. This motivates the definition of torque

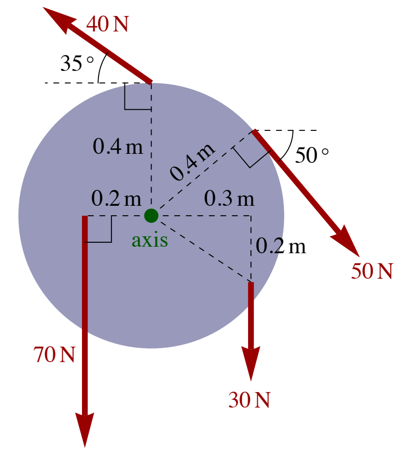
$$\tau = r\, F_\perp$$

If θ is the angle between $\vec{r}$ and $\vec{F}$ then we can write $F_\perp = F \sin\theta$. Similarly we can $r_\perp = r \sin\theta$ as the component of $\vec{r}$ perpendicular to $\vec{F}$. This gives us other ways of writing the torque.

$$\tau = r\, F_\perp = r\, F \sin\theta = r_\perp\, F$$

The sign of torque depends on the sign convention for kinematics. If a force tends to make something rotate in the positive direction then the torque is positive and similarly negative torques tend to make things rotate in the negative direction.

Example I.1 - Torques on a Disk



Four forces act on a disk with a 0.4-m radius.

(a) What is the torque of each force, taking counterclockwise as the positive sense of rotation?

Solution

The three equivalent formulas for torque are

$$\tau = r\,F_\perp = r\,F\,\sin\theta = \;r_\perp\,F.$$

For clarity, we will label the torques by τ_{30}, τ_{50}, τ_{40}, and τ_{70}. For τ_{30} the r is the hypotenuse of the two sides given, but we do not need its value. The part of r perpendicular to the force is $r_\perp = 0.30\text{ m}$. Since counterclockwise is positive this torque is negative since, acting by itself, it will make the disk rotate clockwise.

$$\tau_{30} = -r_\perp\,F = -(0.30\text{ m})\;30\text{ N} = -9.00\text{ Nm}$$

The 50-N force is perpendicular to the radial vector. The 50° angle is irrelevant here. It is also a clockwise and thus a negative torque.

$$\tau_{50} = -r\,F_\perp = -(0.40\text{ m})\;50\text{ N} = -20.00\text{ Nm}$$

The component of the 40-N force perpendicular to the radial vector is $F_\perp = (80\text{ N})\cos 35°$. Alternatively, we can identify the angle between the radial vector and the force is $90° - 35° = 55°$. This force make it rotate in the counterclockwise sense.

$$\tau_{40} = +r\,F_\perp = +(0.40\text{ m})\,(40\text{ N}\cos 35°) = 13.11\text{ Nm}$$

$$(\text{or }\;\tau_{40} = +r\,F\,\sin\theta = +(0.40\text{ m})\,(40\text{ N})\,\sin 55° = 13.11\text{ Nm})$$

The 70-N force is perpendicular to the radial vector and is counter-clockwise and thus a positive torque.

$$\tau_{70} = r\,F_\perp = (0.20\text{ m})\;70\text{ N} = 14.00\text{ Nm}$$

(b) What is the net torque on the disk, where net torque is the sum of the torques?

Solution

$$\tau_{\text{net}} = \tau_{30} + \tau_{50} + \tau_{70} + \tau_{80} = -1.89\text{ Nm}$$

The negative sign means that the net torque will cause a clockwise rotation.

(c) How would the answers to parts (a) and (b) be different if clockwise were chosen as the positive sense of rotation?

Solution

If clockwise were our positive sense of rotation then all torques would change signs.

Angular Velocity and Torque as Vectors

A rotation has a magnitude, the angle of rotation, and a direction, along the direction of the axis. Rotations are not vectors, though. Vector addition is commutative but rotations are not. It turns out that infinitesimal rotations do commute and *are* vectors. We can write an infinitesimal rotation as $d\vec{\theta}$. Since angular velocity about an axis requires only an infinitesimal rotation, $\omega = d\theta/dt$, we can define the angular velocity vector

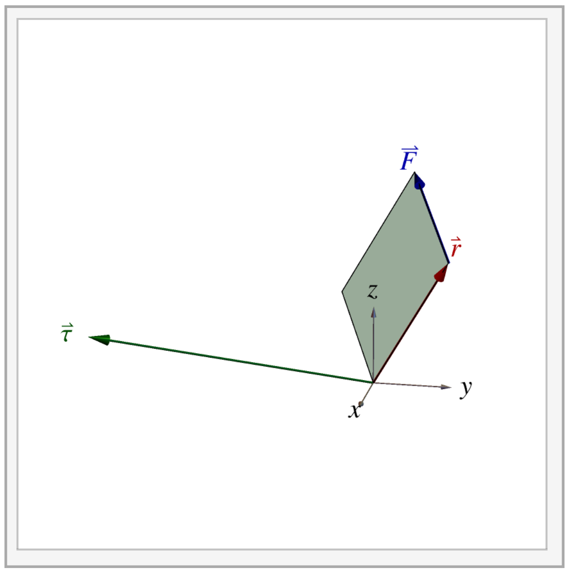
$$\vec{\omega} = \frac{d\vec{\theta}}{dt} \; .$$

There are two possible directions along an axis. We decide which direction by using the right hand rule. Wrap the fingers of your right hand in the direction of rotation. The thumb points in the direction of the vector. If angular velocity is a vector then we can also make angular acceleration a vector.

$$\vec{\alpha} = \frac{d}{dt} \vec{\omega}$$

Wit these considerations we can now make a vector out of the torque. We can assign its direction to the sense of rotation due to that torque. $\vec{r}$ and $\vec{F}$ are vectors; we will define the cross product so that the cross product of $\vec{r}$ and $\vec{F}$ is the torque $\vec{\tau}$.

$$\vec{\tau} = \vec{r} \times \vec{F}$$



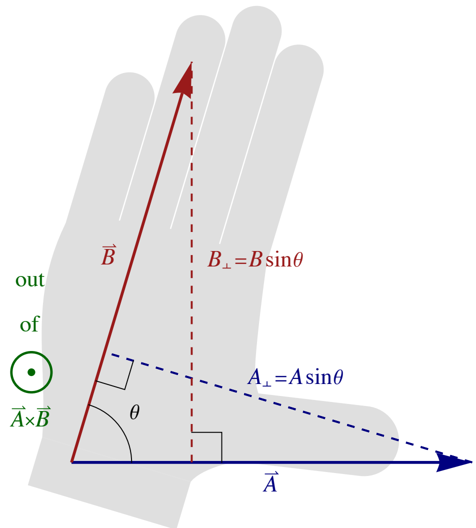
Interactive Figure

The Cross or Vector Product

The dot product, or scalar product, is a way of multiplying of two vectors that gives a scalar. The cross product, also known as the vector product, is a multiplication that gives a vector.

$$\vec{A} \times \vec{B} \text{ is a vector.}$$

The magnitude of this vector is $A\,B \sin\theta$. We will specify the direction with a unit vector $\hat{u}$. The two vectors $\vec{A}$ and $\vec{B}$ define a plane; their cross product is perpendicular to that plane. There are two unit vectors perpendicular to any plane; we use the right hand rule to find the correct one. Put your right thumb in the direction of the first entry $\vec{A}$ and your fingers in the direction of the second entry $\vec{B}$. The palm of your hand is in the direction $\hat{u}$, giving the direction of the cross product.



To find the direction of the cross product $\vec{A}\times\vec{B}$, put the thumb of your right hand in the direction of the first entry $\vec{A}$ and your fingers in the direction of the second entry $\vec{B}$. The palm of your right hand points in the direction of the cross product.

$$\vec{A}\times\vec{B} = A\,B\sin\theta\;\hat{u}\;\left(\hat{u}\text{ by right hand rule}\right)$$

Figure: The convention we use to represent the third dimension relative to some two-dimensional figure is to use a dot to represent "out of" and an × to represent "into". A useful way to remember this is with an arrow; if it points at you it is a dot and away an ×.

Properties of the Cross Product

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}\;\text{(antisymmetry } - \text{ not commutative)}$$

$$\left(\vec{A}\times\vec{B}\right)\times\vec{C} \neq \vec{A}\times\left(\vec{B}\times\vec{C}\right)\;\text{(not associative)}$$

$$\left(c\,\vec{A}\right)\times\vec{B} = c\left(\vec{A}\times\vec{B}\right) = \vec{A}\times\left(c\,\vec{B}\right)\;\text{(associative } w.r.t.\text{ scalar mult.)}$$

$$\left(\vec{A}+\vec{B}\right)\times\vec{C} = \vec{A}\times\vec{C} + \vec{B}\times\vec{C}\;\text{ and}$$

$$\vec{A}\times\left(\vec{B}+\vec{C}\right) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}\;\text{(distributive)}$$

The Cross Product and Components

With the dot product we were able to write it in terms of components. We can do the same for the cross product. As before we have to find the products of the basis unit vectors. Because of the antisymmetry property we get $\vec{A}\times\vec{A} = \vec{0}$. It follows then that

$$\vec{0} = \hat{x}\times\hat{x} = \hat{y}\times\hat{y} = \hat{z}\times\hat{z}.$$

Using our definition of the cross product we can see that the cross product of two perpendicular unit vectors is a third unit vector perpendicular to the two.

$$\vec{A}\times\vec{B} = A\,B\sin\theta\;\hat{u}\;\Longrightarrow\;\hat{v}\times\hat{w} = 1\cdot 1\cdot 1\;\hat{u}$$

The cross product of the unit vectors $\hat{x}$ and $\hat{y}$ is thus a unit vector perpendicular to the xy plane. This is either $\hat{z}$ or $-\hat{z}$. We insist that our coordinate system is right-handed; this means that

$$\hat{x}\times\hat{y} = \hat{z}.$$

For the other combinations of unit vectors there is a simple rule to keep track of their cross products. Arrange x, y and z around a circle.

$$\begin{matrix} & x & \\ z & & y \end{matrix}$$

If the order of the three coordinates has the same sense of rotation as x, y, z it gains a positive sign. If opposite it gets a minus sign.

$$\hat{y}\times\hat{z} = \hat{x}, \quad \hat{z}\times\hat{x} = \hat{y}\,,$$
$$\hat{y}\times\hat{x} = -\hat{z}, \quad \hat{x}\times\hat{z} = -\hat{y} \text{ and } \hat{z}\times\hat{y} = -\hat{x}$$

We can put all this together and get the cross product in terms of components.

$$\begin{aligned}\vec{A}\times\vec{B} &= \left(A_x\,\hat{x} + A_y\,\hat{y} + A_z\,\hat{z}\right)\times\left(B_x\,\hat{x} + B_y\,\hat{y} + B_z\,\hat{z}\right)\\ &= \hat{x}\left(A_y\,B_z - A_z\,B_y\right) + \hat{y}\,(A_z\,B_x - A_x\,B_z) + \hat{z}\left(A_x\,B_y - A_y\,B_x\right)\end{aligned}$$

The determinant method is a common way to write this. A determinant is a mathematical operation that completely antisymmetrizes a square matrix. On a 2×2 matrix we have:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\,d - b\,c\,.$$

On a higher order matrix we write it as an alternating sum of determinants of lower order. We will consider only the case of the cross product.

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{y}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{z}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$
$$= \hat{x}\left(A_y\,B_z - A_z\,B_y\right) - \hat{y}\,(A_x\,B_z - A_z\,B_x) + \hat{z}\left(A_x\,B_y - A_y\,B_x\right)$$

Note that there is not a discrepancy between the differing signs of the middle term for the y-component, since the factor multiplying $\hat{y}$ has terms in reverse order.

Example I.2 - Dynamics of a Particle - Net Torque

A particle of mass m moves along the path given by

$$\vec{r}(t) = \left\langle a\,t^2,\, b\,t,\, -c\,t^3\right\rangle\,,$$

where a, b and c are constants.

(a) What is the net torque about the origin that acts on the particle as a function of time? The net torque is the torque due to the net force.

Solution

The net force is found from the acceleration and the acceleration by differentiation.

$$\vec{v}(t) = \frac{d\,\vec{r}(t)}{d\,t} = \left\langle 2\,a\,t,\, b,\, -3\,c\,t^2\right\rangle$$

$$\vec{a}(t) = \frac{d\,\vec{v}(t)}{d\,t} = \langle 2\,a,\, 0,\, -6\,c\,t\rangle$$

$$\vec{F}_{\text{net}}(t) = m\,\vec{a}(t) = m\,\langle 2\,a,\, 0,\, -6\,c\,t\rangle$$

The torque about the origin is found by the cross product definition.

$$\vec{\tau}_{\text{net}}(t) = \vec{r}(t)\times\vec{F}_{\text{net}}(t) = m\left\langle a\,t^2,\, b\,t,\, -c\,t^3\right\rangle\times\langle 2\,a,\, 0,\, -6\,c\,t\rangle$$

$$\begin{aligned}\vec{\tau}_{\text{net}} &= m\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a\,t^2 & b\,t & -c\,t^3 \\ 2\,a & 0 & -6\,c\,t \end{vmatrix} = m\left\langle -6\,b\,c\,t^2 - 0,\, -\left(-6\,a\,c\,t^3 + 2\,a\,c\,t^3\right),\, 0 - 2\,a\,b\,t\right\rangle\\ &= m\left\langle -6\,b\,c\,t^2,\, 4\,a\,c\,t^3,\, -2\,a\,b\,t\right\rangle\end{aligned}$$

Angular Momentum of a Particle and Torque

We previously defined the torque as

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

We can similarly define the angular momentum of a particle as

$$\vec{L} = \vec{r} \times \vec{p}.$$

Both of these expressions are relative to an origin; $\vec{r}$ is the position vector. It is from the origin to the position to where the force is applied in the case of torque. It is from the origin to the position of the particle for the angular momentum.

The net torque on a particle (a point) is the torque due to the net force. If the particle is at position $\vec{r}$ then the net torque is $\vec{\tau}_{\text{net}} = \vec{r} \times \vec{F}_{\text{net}}$. It is straightforward to verify that the cross product satisfies the usually product rule for differentiation. Using this we can differentiate the angular momentum for a particle. This gives:

$$\frac{d}{dt}\vec{L} = \left(\frac{d}{dt}\vec{r}\right) \times \vec{p} + \vec{r} \times \frac{d}{dt}\vec{p} = \vec{v} \times m\,\vec{v} + \vec{r} \times \vec{F}_{\text{net}} = \vec{0} + \vec{\tau}_{\text{net}}.$$

This gives the analog of the momentum form of the second law $\vec{F}_{\text{net}} = d\vec{p}/dt$. This is

$$\vec{\tau}_{\text{net}} = \frac{d}{dt}\vec{L}.$$

Example I.3 - Dynamics of a Particle (continued) - Angular Momentum

We extend the previous example. A particle of mass m moves along the path given by

$$\vec{r}(t) = \left\langle a\,t^2,\, b\,t,\, -c\,t^3 \right\rangle,$$

where a, b and c are constants.

(b) What is the angular momentum of the particle about the origin as a function of time?

Solution

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \left\langle 2\,a\,t,\, b,\, -3\,c\,t^2 \right\rangle$$

The angular momentum about the origin is found by the definition.

$$\vec{L}(t) = \vec{r}(t) \times \vec{p}(t) = \vec{r}(t) \times m\vec{v}(t)$$

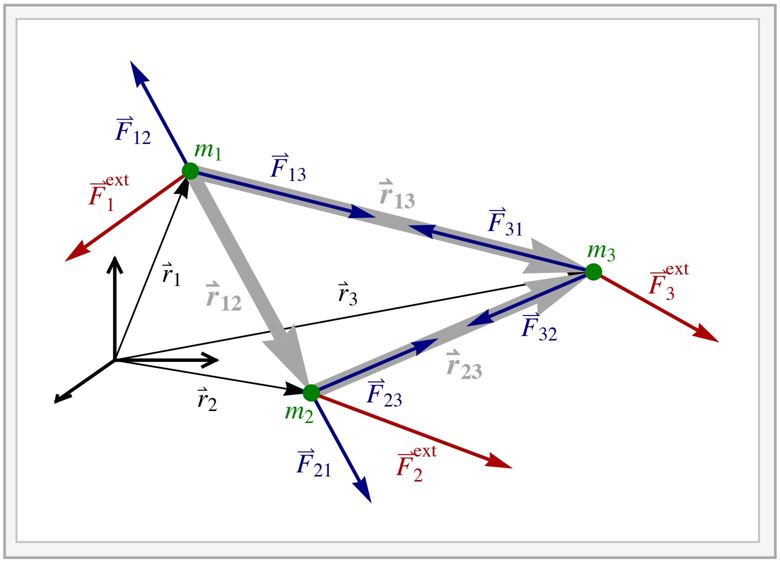
$$\vec{L}(t) = m \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a\,t^2 & b\,t & -c\,t^3 \\ 2\,a\,t & b & -3\,c\,t^2 \end{vmatrix} = m\left\langle -3\,b\,c\,t^3 + b\,c\,t^3,\, -\left(-3\,a\,c\,t^4 + 2\,a\,c\,t^4\right),\, a\,b\,t^2 - 2\,a\,b\,t^2 \right\rangle$$
$$= m\left\langle -2\,b\,c\,t^3,\, a\,c\,t^4,\, -a\,b\,t^2 \right\rangle$$

(c) For the results of parts (a) and (b) show that $\vec{\tau}_{\text{net}} = d\vec{L}/dt$.

Solution

$$\frac{d\vec{L}(t)}{dt} = m\left\langle -6\,b\,c\,t^2,\, 4\,a\,c\,t^3,\, -2\,a\,b\,t \right\rangle$$

A System of Particles



Interactive Figure

In the preceding chapter we considered a three particle system with masses m_1, $m_2$and m_3 at positions $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$. As before, we write the forces on m_1 as a sum of internal forces $\vec{F}_{21}$ and $\vec{F}_{31}$ and external forces $\vec{F}_1^{\text{ext}}$. The cross products of this with $\vec{r}_1$ gives the net torque. The torques for m_2 and m_3 break up similarly.

$$\vec{\tau}_{\text{net},1} = \vec{r}_1 \times \vec{F}_1^{\text{ext}} + \vec{r}_1 \times \vec{F}_{21} + \vec{r}_1 \times \vec{F}_{31} = \frac{d}{dt} \vec{L}_1$$

$$\vec{\tau}_{\text{net},2} = \vec{r}_2 \times \vec{F}_2^{\text{ext}} + \vec{r}_2 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{32} = \frac{d}{dt} \vec{L}_2$$

$$\vec{\tau}_{\text{net},3} = \vec{r}_3 \times \vec{F}_3^{\text{ext}} + \vec{r}_3 \times \vec{F}_{13} + \vec{r}_3 \times \vec{F}_{23} = \frac{d}{dt} \vec{L}_3$$

To concentrate on the bulk motion of our system we sum over these expressions. In the previous case with forces the internal forces canceled due to Newton's third law. Here we need to make an additional assumption that the forces are central forces; this is that $\vec{F}_{12}$, the force of mass 2 on mass 1, is directed parallel (or antiparallel) to the line between the masses.

$$\vec{F}_{21} \parallel \left(\vec{r}_1 - \vec{r}_2\right) \iff \left(\vec{r}_1 - \vec{r}_2\right) \times \vec{F}_{21} = \vec{0}$$

Now when we sum the net torques we get a cancellation of the internal torques. The internal torques cancel for all pairs of masses. The cancellation between m_1 and $m_2$follows from

$$\vec{r}_1 \times \vec{F}_{21} + \vec{r}_2 \times \vec{F}_{12} = \vec{r}_1 \times \vec{F}_{21} + \vec{r}_2 \times \left(-\vec{F}_{21}\right) = \left(\vec{r}_1 - \vec{r}_2\right) \times \vec{F}_{21} = \vec{0}.$$

The other internal forces cancel similarly. We end up with

$$\vec{\tau}_1^{\text{ext}} + \vec{\tau}_2^{\text{ext}} + \vec{\tau}_3^{\text{ext}} = \frac{d}{dt}\left(\vec{L}_1 + \vec{L}_2 + \vec{L}_3\right).$$

It should be clear how this could be generalized to four, or an arbitrary number, of particles. This gives the very fundamental result that for a system of particles

$$\vec{\tau}_{\text{net}}^{\text{ext}} = \frac{d}{dt} \vec{L}_{\text{tot}}.$$

Conservation of Angular Momentum

The conservation of angular momentum follows from the expression above. If there are no external torques on a system then the total angular momentum of the system is conserved.

$$\vec{\tau}_{\text{net}}^{\text{ext}} = \vec{0} \implies \frac{d}{dt} \vec{L}_{\text{tot}} = \vec{0} \implies \Delta \vec{L}_{\text{tot}} = \vec{0}$$

This derivation mirrors the conservation of linear momentum.

This is a very fundamental result. It has deep implications on the very large scale; in astrophysics it is crucial in the dynamics of planets, stars, solar systems and galaxies. It is also important on the very small scale; in particle accelerators where elementary particles are collided and created, angular momentum is always conserved.

I.2 - More on Rigid Bodies

Axes and Origins

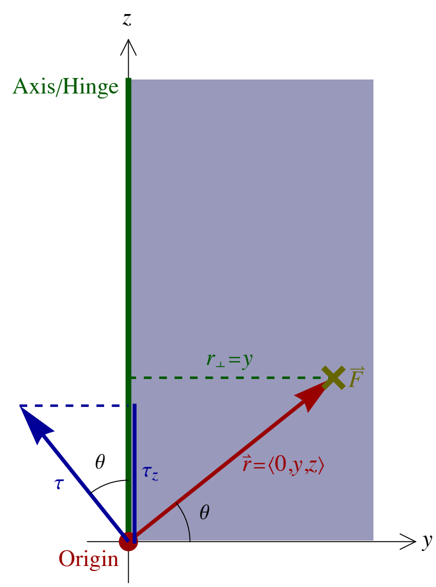
We began with a discussion of rigid bodies rotating about a fixed axis. Then we considered quantities like angular velocity, angular acceleration, torque and angular momentum as vectors. How are the two points of view related? Torque and angular momentum vectors are relative to an origin, where the position vector $\vec{r}$ is based at the origin. If the origin is chosen as some point on the axis then the vector relative to the axis is just the component in the direction of the axis. The torque about some origin is the vector

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

The torque about the z axis is just the z component of this $\tau_z = \tau$ where

$$\tau = r\,F_\perp = r\,F\sin\theta = r_\perp\,F.$$

Example I.4 - Axis and Origin



Consider a door in the yz-plane with the hinge as the axis in the z-direction. The origin is at the base of the door. A force of of magnitude F pushes into the door, in the negative-x direction.

$$\vec{F} = -F\,\hat{x}$$

The force acts at a point given by the position vector $\vec{r} = \langle 0, y, z\rangle$ from the origin.

(a) What is the torque relative to the axis?

Solution

The torque relative to an axis is $\tau = r_\perp\,F = y\,F$.

(b) What is the torque relative to the origin?

Solution

The torque relative to an origin is $\vec{\tau} = \vec{r} \times \vec{F}$.

$$\vec{\tau} = \vec{r} \times \vec{F} = \left(y\,\hat{y} + z\,\hat{z}\right) \times \left(-F\,\hat{x}\right) = -y\,F\,\hat{y} \times \hat{x} - z\,F\,\hat{z} \times \hat{x}$$

$$= y\,F\,\hat{z} - z\,F\,\hat{y} = \langle 0,\, -z,\; y\rangle\, F$$

(c) Show the the z-component of the torque about the origin is the torque about the axis.

Solution

It follows that $\tau_z = y\,F$ which is the torque about the z-axis.

Similarly, the angular momentum of a particle relative to an origin

$$\vec{L} = \vec{r} \times \vec{p}$$

can be written relative to an axis. If the axis is the z axis then L about the axis is just the z component of L about the origin. $L_z = L$ where

$$L = r\,p_\perp = r\,p\,\sin\theta = r_\perp p.$$

Angular Momentum of a Rigid Body

As before, we view our rigid body as a collection of point masses where the perpendicular distance form the axis to m_i is r_i. Since all the r_i are fixed we get the momentum related to the tangential velocity, which is then related to the angular velocity.

$$p_{i\perp} = m_i\, v_{it} = m_i\, r_i\, \omega$$

The angular momentum of the i^{th} mass becomes

$$L_i = r_i\, p_{i\perp} = r_i\, m_i\, v_{it} = m_i\, r_i^2\, \omega$$

The total angular momentum is the sum over all these terms $L = \sum_i L_i$. Using $I = \sum_i m_i\, r_i^2$ we get the angular momentum of a rotating rigid body

$$L = I\,\omega.$$

This is the result we had in our table relating rotations about a fixed axis to one dimensional linear motion.

Example I.5 - Angular Momentum of the Earth

The mass of the earth, the radius of the earth and the earth-sun distance are:

$$M_E = 5.97 \times 10^{24}\ \text{kg}\,,\quad R_E = 6.38 \times 10^6\ \text{m}\ \text{and}\ R_{\text{ES}} = 1.50 \times 10^{11}\ \text{m}.$$

Here assume a circular orbit.

(a) Estimate the rotational angular momentum of the earth, assuming it is a uniform sphere?

Solution

$$I = \frac{2}{5}\, M_E\, R_E^2 = 9.72 \times 10^{37}\ \text{kg m}^2$$

The angular velocity can be found from its rotational period of 1 day.

$$\omega_{\text{rot}} = \frac{2\,\pi}{T} = \frac{2\,\pi}{1\,\text{day}} = \frac{2\,\pi}{24 \times 3600\ \text{s}} = 7.27 \times 10^{-5}\ \text{s}^{-1}$$

The estimated angular momentum can then be found.

$$L = I\,\omega_{\text{rot}} = 7.07 \times 10^{33}\ \text{kg m}^2 / \text{s}$$

(b) Is the estimated result in part (a) too large or too small?

Solution

Because the earth is denser at its core the estimated moment is too large and thus the estimated angular momentum is too large.

(c) What is the orbital angular momentum of the earth as it orbits the sun?

Solution

Now we consider the angular momentum of a particle. First we find the speed from the orbital angular velocity

$$\omega_{\text{orbit}} = \frac{2\pi}{T} = \frac{2\pi}{1\,\text{yr}} = \frac{2\pi}{365.24\times 24\times 3600\ \text{s}} = 1.992\times 10^{-7}\ \text{s}^{-1}.$$

The tangential velocity gives the momentum, which is perpendicular to the radial vector.

$$v = v_t = r\,\omega \implies L = r\,p_\perp = r\,p = r\,m\,v = m\,r^2\,\omega$$

$$L = M_E\,R_{\text{ES}}^2\,\omega_{\text{orbit}} = 2.67\times 10^{40}\ \text{kg m}^2/\text{s}$$

Note that an alternative solution can be found using $L = I\omega$ and $I = \sum_i m_i\,r_i^2 = m\,r^2$.

Example I.6 - The Rotating Figure Skater

A figure skater spins about a vertical axis. With her arms out she has a moment of inertia of I_{out} and rotates at ω_{out}. When she brings his arms in, her moment is smaller, I_{in}. The moments are about the vertical axis of rotation.

(a) What is ω_{in}, her angular velocity with her arms in?

Solution

If the stool is frictionless then there is no net external torque acting, so angular momentum is conserved. The angular momentum is $L = I\,\omega$. It follows that

$$L_{\text{out}} = L_{\text{in}} \implies I_{\text{out}}\,\omega_{\text{out}} = I_{\text{in}}\,\omega_{\text{in}} \implies \omega_{\text{in}} = \frac{I_{\text{out}}}{I_{\text{in}}}\,\omega_{\text{out}}$$

Since $I_{\text{in}} < I_{\text{out}}$ it follows that he rotates faster $\omega_{\text{in}} > \omega_{\text{out}}$.

(b) Compare the kinetic energies K_{in} and K_{out}.

Solution

The kinetic energy is $K = (1/2)\,I\,\omega^2$. Using $L = I\,\omega$ we can write K in terms of L and I; since L is conserved this is useful.

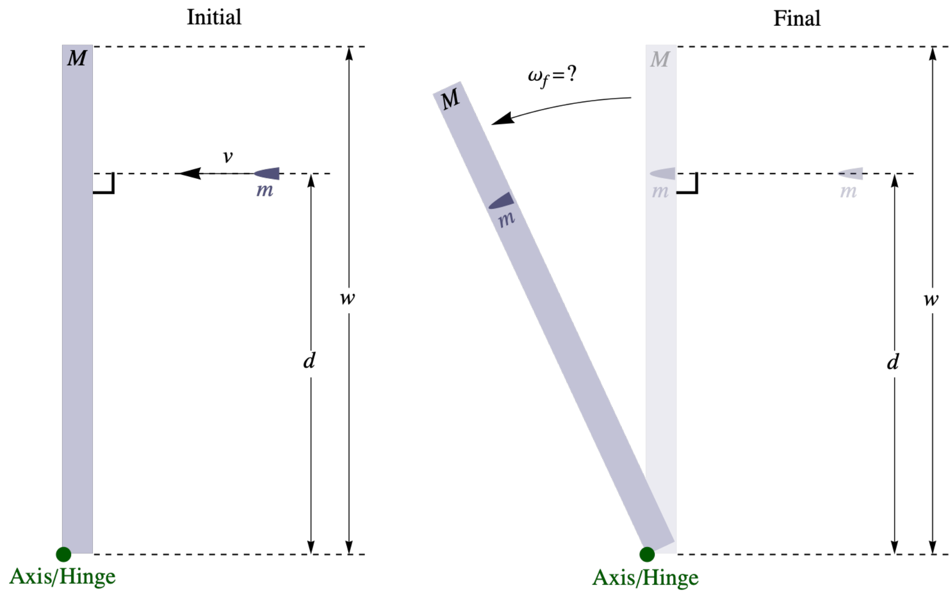
$$K = \frac{1}{2}\,I\,\omega^2 \ \text{and}\ L = I\,\omega \implies K = \frac{L^2}{2\,I}$$

Since $L_{\text{in}} = L_{\text{out}} = L$, it follows that $K_{\text{in}} > K_{\text{out}}$.

$$I_{\text{in}} < I_{\text{out}} \implies K_{\text{in}} = \frac{L^2}{2\,I_{\text{in}}} > \frac{L^2}{2\,I_{\text{out}}} = K_{\text{out}}$$

Where did the extra energy come from when she brings her arms in? View this from the perspective of the non-inertial rotating frame where there is the false centrifugal force acting outward. To bring her arms in she must do work against the centrifugal force; that is the source of the extra energy.

Example I.7 - Bullet Shot in Door



A bullet of mass m is shot at speed v toward a door. The bullet's velocity is perpendicular to the door and it hits the door at a distance d from the door's hinge. The door has mass M, height h and width w; assume that it swings without friction about the hinge. If the door is initially at rest then what is its angular velocity after the bullet embeds in it.

Solution

Given there is no friction in the hinge, there is no external torque about the hinge and angular momentum is conserved. Initially, there is no angular momentum in the door but the bullet does have angular momentum. We use the angular momentum of a particle:

$$L_i = r_\perp p = d\, m\, v.$$

After the bullet embeds in the door we have a rotating rigid body. The moment of inertia consists of the door's moment added to the bullet's contribution. The door is the same as a rod of length w; its height is irrelevant.

$$I_{\text{door}} = \frac{1}{3} M L^2 = \frac{1}{3} M w^2$$

The moment of inertia of the bullet after embedding comes from the moment for a discrete distribution.

$$I_{\text{bullet}} = \sum_i m_i r_i^2 = m\, d^2$$

The final angular momentum is $L_f = I_f\, \omega_f$ where $I_f = I_{\text{door}} + I_{\text{bullet}}$. Conservation of angular momentum gives ω_f.

$$L_i = L_f \implies d\, m\, v = \left(\frac{1}{3} M w^2 + m\, d^2\right)\omega_f \implies \omega_f = \frac{d\, m\, v}{\frac{1}{3} M w^2 + m\, d^2}$$

The Second Law

We can now, finally, derive the rotational equivalent of the second law $\tau_{\text{net}} = I\, \alpha$. Start with the momentum form of the second law.

$$\vec{\tau}_{\text{net}}^{\text{ext}} = \frac{d}{dt} \vec{L}_{\text{tot}}$$

Now take the component along the axis of rotation. When the system is the rigid body then the net external torque on the rigid body is just the torque on it. Similarly, the total angular momentum is just $I\, \omega$ the angular momentum of the body. We get

$$\tau_{\text{net}} = \frac{d}{dt} L.$$

Using $L = I\, \omega$ and $\alpha = d\omega / dt$ we get our result.

$$\tau_{\text{net}} = I\, \alpha$$

The Torque Due to Gravity

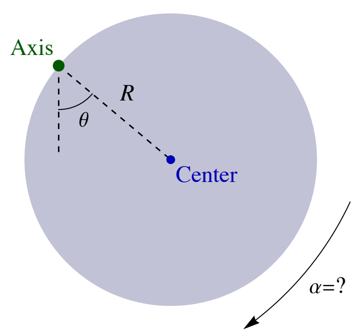
We saw earlier that to calculate the potential energy due to gravity we treat the object as if all the mass is at the center of mass. The same is true for finding the torque due to gravity.

$$\vec{\tau}_{\text{grav}} = \vec{r}_{\text{cm}} \times M\,\vec{g}$$

It is straightforward to verify this. Write the torque as the sum over the torques on all the masses in the body. Then use the definition of center of mass to get the result.

$$\vec{\tau}_{\text{grav}} = \sum_i \vec{r}_i \times m_i\,\vec{g} = \left(\sum_i m_i\,\vec{r}_i\right) \times \vec{g} = \left(M\,\vec{r}_{\text{cm}}\right) \times \vec{g} = \vec{r}_{\text{cm}} \times M\vec{g}$$

Example I.8 - Swinging Disk



A uniform disk of radius R swings without friction about a perpendicular axis through its rim. What is its angular acceleration as it swings through a position where the center is at an angle of θ from vertical, as shown?

Solution

We first need to draw a free-body diagram. When we draw free-body diagrams for torques we must draw the forces into the diagram carefully showing where they act. Here, the only contact force is at the axis; this force gives zero torque, since r is zero. The only torque comes from the weight mg, which acts at the center. The angle between the radial vector and the force is θ, so we can use the $\tau = rF\sin\theta$ formula. We choose our sense of rotation, clockwise, as positive, so the torque is positive.

$$\tau_{\text{net}} = \tau_{\text{grav}} = rF\sin\theta = R\,mg\sin\theta$$

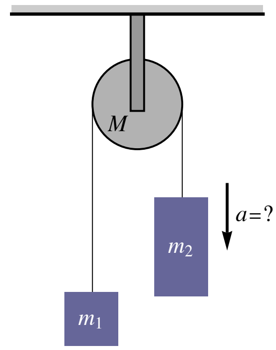
The moment of inertia can be found using the parallel-axis theorem.

$$I = I_{\text{cm}} + m\,d^2 = \frac{1}{2}\,m\,R^2 + m\,R^2 = \frac{3}{2}\,m\,R^2$$

The rotational second law gives us the angular acceleration.

$$\tau_{\text{net}} = I\,\alpha \implies R\,mg\sin\theta = \frac{3}{2}\,m\,R^2\,\alpha \implies \alpha = \frac{2\,g}{3\,R}\sin\theta$$

Example I.9 - Atwood's Machine with a Massive Pulley



In chapter D we solved Atwood's machine with an ideal pulley. Recall that an ideal pulley was frictionless and light, where light means that the pulley's mass is small compared to the other masses in the system. Now we will consider a pulley with mass; it will still be frictionless. With an ideal pulley the tension on both sides is the same. Here, with a massive pulley the tensions on either side are different. The different tensions are responsible for the angular acceleration of the pulley.

m_1 and m_2 are two masses connected by a light string over a frictionless pulley as shown. The pulley is a uniform disk of mass M. Take $m_1 < m_2$. What is the downward acceleration of m_2?

Solution

With our constrained system the motion of each mass is related. Let Δx_1 be the upward displacement of mass 1 and Δx_2 be the upward displacement of 2. The assumption of tension is that the rope or string does not stretch, so these must be equal. We also assume that the string does not slide on the pulley. This relates the rotational motion of the pulley to the linear motion of the hanging masses; the arc length $R\,\Delta\theta$, where R is the pulley's radius, must equal the hanging masses displacements.

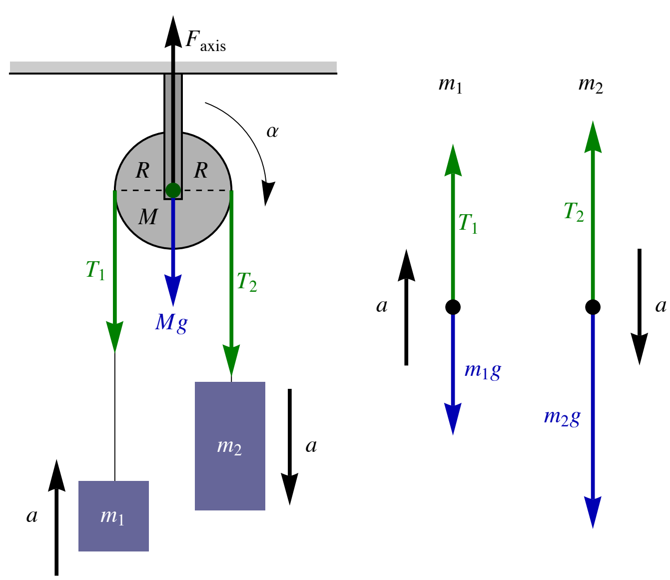
$$\Delta x_1 = \Delta x_2 = \Delta x = R\,\Delta\theta$$

Taking derivatives we can relate the velocities $v_1 = v_2 = v = R\,\omega$ and accelerations.

$$a_1 = a_2 = a = R\,\alpha$$

Note that R was not given. We introduce it in our solution, so it must cancel.

We need to draw a free-body diagram for each mass and for the pulley. For the pulley we draw the free-body diagram into the diagram, showing where the forces act. For the hanging masses this is the same as what we saw in Chapter D, except that the tensions are now different.



Applying the second law to the hanging masses gives a pair of equations. Here we choose the directinos of the accelerations as positive.

$$F_{\text{net},1} = m_1\,a_1 \implies T_1 - m_1\,g = m_1\,a$$

$$F_{\text{net},2} = m_2\,a_2 \implies m_2\,g - T_2 = m_2\,a$$

In Chapter D, where the tensions were equal this gave two linear equations with two unknows. Now we have three unknowns, a and the two tensions. There are four forces acting on the pulley, the two tensions, the pulley's weight $M g$ and an upward force F_{axis} acting at the axis; since these two forces act at the axis they produce no torque. We choose clockwise as positive, since that is the direction of our angular acceleration. The tensions are perpendicular to the radial vector so $\tau = r F_{\perp} = RF$. We now apply the rotational second law applied to the pulley.

$$\tau_{\text{net}} = I\,\alpha \implies R\,T_2 - R\,T_1 = I\,\alpha = \left(\frac{1}{2}\,M\,R^2\right)\alpha$$

Here we have added another equation but also added another unknown α. We can use $a = R\alpha$ to eliminate α in favor of a. We can also use the fact that the pulley is a uniform disk.

$$R\,T_2 - R\,T_1 = I\,\alpha = \left(\frac{1}{2}\,M\,R^2\right)\frac{a}{R} \implies T_2 - T_1 = \frac{M}{2}\,a$$

Adding this to the two second law expressions for the hanging masses we eliminate the tensions and we get our answer.

$$m_2\,g - m_1\,g = (m_1 + m_2 + M/2)\,a \implies a = \frac{m_2 - m_1}{m_1 + m_2 + M/2}\,g$$

I.3 - Equilibrium

The Conditions for Equilibrium

If a body is in equilibrium then there is no acceleration and there is no angular acceleration. This implies that the net force and the net torque *must* vanish.

$$\vec{F}_{\text{net}} = \vec{0} \text{ and } \vec{\tau}_{\text{net}} = \vec{0}$$

For the examples we will consider all possible rotation will be in a plane and thus we only need to consider torques relative to an axis.

The Origin (or Axis) is Arbitrary

When considering an equilibrium problem sometimes the choice of axis is clear. Often in isn't clear, though, and there isn't a natural choice. The key point is that the choice of origin or axis is arbitrary. When something is arbitrary then we have the luxury of making a choice that simplifies the problem.

The basic result is this: If the torques balance about one origin and the forces balance then the torques balance about any origin. If the vector from one origin to another is $\vec{r}_0$. If $\vec{r}_i'$ is the vector from the new origin at $\vec{r}_0$ to the mass m_i and $\vec{r}_i$ is from the first origin to the mass then

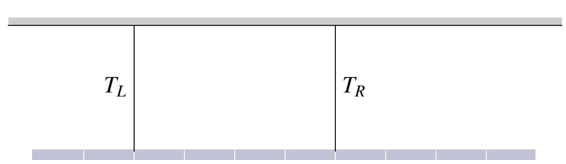
$$\vec{r}_i = \vec{r}_i' + \vec{r}_0.$$

Take the net torques about these axes to be $\vec{\tau}_{\text{net}}$ and $\vec{\tau}_{\text{net}}'$. If $\vec{\tau}_{\text{net}} = \vec{0}$ and $\vec{F}_{\text{net}} = \vec{0}$ then $\vec{\tau}_{\text{net}}' = \vec{0}$.

$$\vec{0} = \vec{\tau}_{\text{net}} = \sum_i \vec{r}_i \times \vec{F}_i = \sum_i \vec{r}_i' \times \vec{F}_i + \vec{r}_0 \times \sum_i \vec{F}_i = \vec{\tau}_{\text{net}}' + \vec{0}$$

This proves our result.

Example I.10 - Hanging Meter Stick

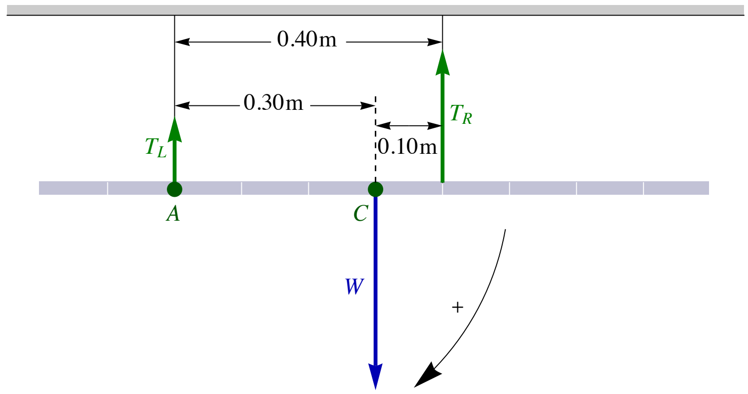


A horizontal uniform meter stick of weight W hangs from vertical strings at the 20-cm and 60-cm lines. What are both tensions, T_L and T_R?

Solution

The net torque and net force are both zero. The two tensions are the unknowns. Setting the net force to zero gives one equation, since all forces are vertical.

$$F_{\text{net}} = 0 \implies T_L + T_R = W$$



We may choose the axis anywhere. Any force that acts at the origin produces no torque. If we choose the axis to be when an unknown acts then the torque equation will not involve that unknown. We will choose the axis labeled A where T_L acts. Choosing clockwise as our positive sense of rotation we can write the torque equation and can solve for T_R

$$\tau_{\text{net},A} = 0 + (0.30\text{ m})\,W - (0.40\text{ m})\,T_R \implies T_R = \frac{3}{4}\,W$$

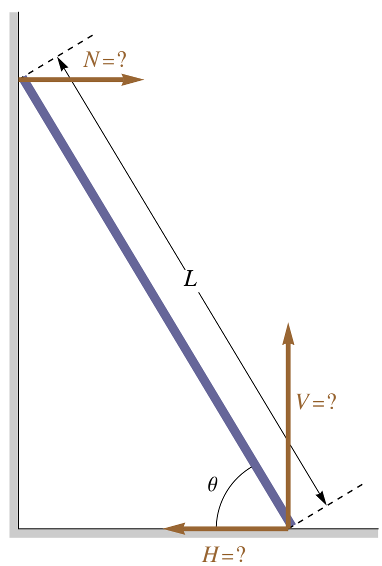
The force equation lets us find T_L.

$$T_L + T_R = W \implies T_L = W - T_R = \frac{1}{4}\,W$$

(Note that if we chose a different axis we would get the same answer. For instance, choosing the center C we get: $\tau_{\text{net},C} = (0.30\text{ m})\,T_L - (0.10\text{ m})\,T_R$. This leads to $3\,T_L = T_R$ and with the force equation we get the same solution.)

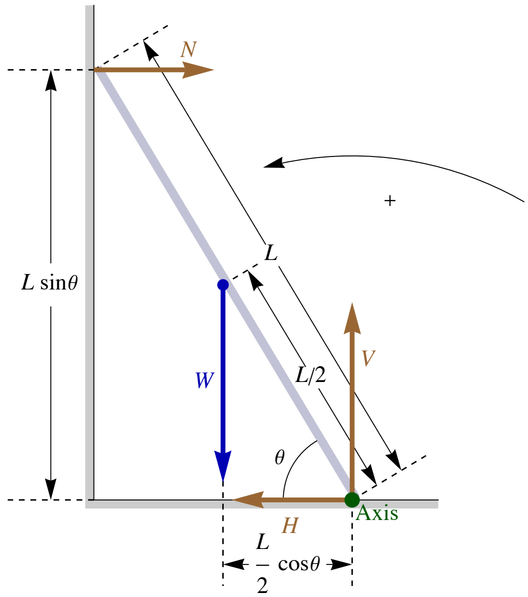
Example I.11 - Leaning Ladder

A uniform ladder of length L leans against a frictionless wall, making an angle of θ with the floor. What is the normal force N of the wall on the ladder and what are the horizontal and vertical components, H and V, of the force of the floor on the ladder?



Solution

The position of the axis is arbitrary but in this problem, given that two of the three unknowns act at the base of the ladder, that is the natural axis to choose; those two unknowns will not appear in our torque equation.



This is a two-dimensional problem so the force condition gives two equations.

$$F_{\text{net,hor}} = 0 \implies N = H$$
$$F_{\text{net,ver}} = 0 \implies V = W$$

For torques about our axis at the base of the ladder, we have two forces to consider.

Since the weight W is vertical $r_\perp$ is the horizontal part of r. Since we have $r = L/2$. Chosing counterclockwise as positive the torque due to the weight is positive.

$$\tau_W = r_\perp F = +\left(\frac{L}{2}\cos\theta\right) W$$

The normal force of the wall N is horizontal, so $r_\perp$ is the vertical part of $r = L$. It is clockwise and thus negative with our convention.

$$\tau_N = -r_\perp F = -(L\sin\theta)\, N$$

Our torque equation gives N and using $H = N$, gives H.

$$0 = \tau_{\text{net}} = \tau_W + \tau_N + \tau_H + \tau_V = +\left(\frac{L}{2}\cos\theta\right) W - (L\sin\theta)\, N + 0 + 0$$

Our full answer follows.

$$H = N = \frac{W}{2\tan\theta}$$
$$V = W$$