Chapter J

Universal Gravitation

Blinn College - Physics 2325 - Terry Honan

J.1 - The Force of Gravity

Introduction

If Isaac Newton had merely written down his three laws of motion he would probably still be known as the most important physicist of all time. Instead, he went far beyond that; he solved all elementary problems involving mechanics and he found the force law describing gravity. His theory of gravity is the topic of this chapter.

The history of physics is largely a history of the unification of the fundamental forces. Before Newton there were separate notions of gravity. What was known as gravity before Newton was the force holding things to the surface of the Earth; we will call this terrestrial gravity. It was not clear that this force was the same as what was responsible for the motion of planets, moons and comets; this we will refer to as astronomical gravity. Newton, with his theory of universal gravitation showed that these two forces were actually the same; he unified terrestrial and astronomical gravity.

Origin of the Inverse Square Law

Newton was trying to understand the force law between point masses m_1 and m_2 separated by a distance r. The weight of a body is the gravitational force on it. Since the weight is proportional to the mass, the force on m_2 must be proportional to m_2.

$$F \propto m_2$$

Because of Newton's third law the magnitude of the force on m_1 is the same. Thus the force on m_2 must also be proportional to m_1.

$$F \propto m_1$$

How does the force vary with distance? It should also decrease with the distance r and should go to zero as r approaches infinity. Some possibilities are

$$F \propto \frac{1}{r}, \quad F \propto \frac{1}{r^2}, \quad F \propto \frac{1}{r^3} \text{ or } F \propto \frac{e^{-\alpha r}}{r^2}.$$

Newton settled on an inverse square law

$$F \propto \frac{1}{r^2}.$$

He deduced the inverse square law by comparing the ratio of the acceleration of the moon toward the earth to the acceleration due to gravity at the surface of the earth to the ratio of the radius of the moon's orbit to the radius of the earth. Suppose the force has the form $F \propto r^p$ for some power p, where an inverse square law gives $p = -2$. Since the acceleration of a body is proportional to the force $F \propto a$ we can conclude $a \propto r^p$. From this proportionality we can write the ratio

$$\frac{a_2}{a_1} = \left(\frac{r_2}{r_1}\right)^p.$$

We can find the power p using logs.

$$p = \frac{\ln(a_2/a_1)}{\ln(r_2/r_1)}$$

Let a_2 and r_2 refer to the moon's orbit. At Newton's time the acceleration of the moon toward the earth could be calculated. The period of the moon's orbit is $T_2 = 27.32$ days. The earth-moon distance, which was measured by parallax, is $r_2 = R_{\text{EM}} = 3.84 \times 10^8$ m. This gives

$$a_2 = \left(\frac{2\pi}{T_2}\right)^2 r_2 = \left(\frac{2\pi}{27.32 \times 24 \times 3600\ \text{s}}\right)^2 3.84 \times 10^8\ \text{m} = 0.002721\ \frac{\text{m}}{\text{s}^2}.$$

The acceleration at the surface of the earth is $a_1 = g = 9.80\ \mathrm{m/s^2}$ at a distance given by the earth's radius $r_1 = R_E = 6.37 \times 10^6\ \mathrm{m}$. Combining these expressions gives

$$p = \frac{\ln(a_2/a_1)}{\ln(r_2/r_1)} = \frac{\ln\left(\frac{0.002721}{9.80}\right)}{\ln\left(\frac{3.84\times10^8}{6.37\times10^6}\right)} = -2.00,$$

which verifies the inverse square law.

Newton realized that there was a crucial flaw in his logic here. He wanted a force law between particles; by a particle it is meant that the distance to an object is large compared to the size of the object. Clearly this is not correct for an object on the surface of the earth. Newton felt that the gravitational force for a spherical body, as long as one is outside of the body, is the same as if all of the mass is at the center. To prove this Newton had to first invent integral calculus and then do a difficult integral. We will refer to this result as the shell theorem and will discuss it shortly.

Newton's Law of Universal Gravitation

Newton's law of gravity is the attractive inverse square law between point masses. If m_1 and m_2 are point masses separated by distance r the magnitude of the force between them is

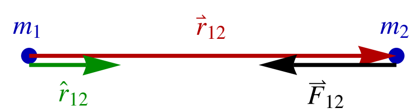
$$F = G\,\frac{m_1\, m_2}{r^2}.$$

This can be written as a vector expression. Let $\vec{r}_{12}$ be the vector from mass 1 to mass 2 and let $\vec{F}_{21}$ be the force on mass 2 due to mass 1.

$$\vec{F}_{12} = -G\,\frac{m_1\, m_2}{r_{12}^2}\,\hat{r}_{12}$$

$\hat{r}_{12}$ is the unit vector in the direction of the vector $\vec{r}_{12}$.

$$\hat{r}_{12} = \frac{\vec{r}_{12}}{\|\,\vec{r}_{12}\,\|} = \frac{\vec{r}_{12}}{r_{12}}$$



The constant G is known as Newton's universal gravitational constant. Gravitational forces between everyday objects are incredibly small. This is reflected in the small value of G in our SI system of units. Newton never knew the value of this constant. It was eventually measured to be

$$G = 6.673\times10^{-11}\,\frac{\mathrm{N\cdot m^2}}{\mathrm{kg^2}}.$$

The Force on a Mass Due to a Distribution of Masses

Suppose we want to find the force on a mass m due to a discrete distribution of masses, $m_1, m_2, \ldots$. Define the vector $\vec{r}_i$ to be from m_i to m. The force of m_i to m is

$$\vec{F}_i = -G\,\frac{m\, m_i}{r_i^2}\,\hat{r}_i$$

where $\hat{r}_i = \vec{r}_i / r_i$. The total force is the sum of all these parts $\vec{F} = \sum_i \vec{F}_i$. This gives

$$\vec{F} = -G\, m \sum_i \frac{m_i}{r_i^2}\,\hat{r}_i\,.$$

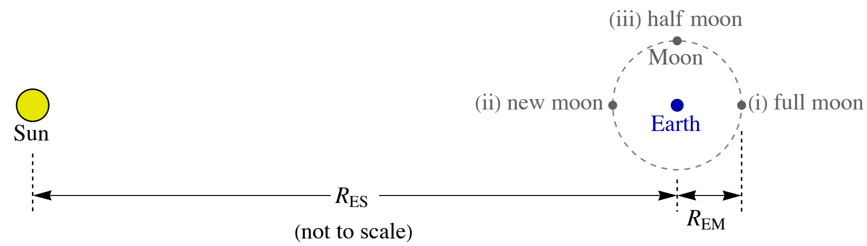
This is just using the idea that force is a vector and that forces add as vectors.

In our discussion of moments of inertia we dealt with continuous distributions. Break up a continuous distribution into an infinite number of infinitesimal pieces; take dm to be one of these infinitesimal pieces. The vector $\vec{r}$ is from dm to m.

$$\vec{F} = -G\, m \int \frac{\hat{r}}{r^2}\, dm\,.$$

Example J.1 - Net Force on Moon

What is the magnitude net force on the moon, due to the earth and sun, at a (i) full moon, (ii) new moon and (iii) half moon?



The masses of the earth, the moon and sun, and the earth-moon and the earth-sun distances are:

$$M_E = 5.97 \times 10^{24}\ \text{kg}\ ,\quad M_S = 1.99 \times 10^{30}\ \text{kg}\ ,\quad M_M = 7.35 \times 10^{22}\ \text{kg}\ ,$$
$$R_{\text{EM}} = 3.85 \times 10^{8}\ \text{m}\ \text{ and }\ R_{\text{ES}} = 1.50 \times 10^{11}\ \text{m}$$

Here assume a circular orbit. Looking at their numerical values we can also see that $R_{\text{EM}} << \text{R}_{\text{ES}}$, we will assume this, as well.

Solution

The magnitude of the force of the earth on the moon is

$$F_E = G\,\frac{M_E\,M_M}{R_{\text{EM}}^2} = 1.975 \times 10^{20}\ \text{N}\ .$$

The sun-moon distance varies, but with the $R_{\text{EM}} << \text{R}_{\text{ES}}$ assumption we can set the moon-sun distance to the earth-sun distance, $R_{\text{MS}} = R_{\text{ES}}$. The magnitude of the sun's force on the earth becomes

$$F_S = G\,\frac{M_S\,M_M}{R_{\text{ES}}^2} = 4.338 \times 10^{20}\ \text{N}\ .$$

(i) For the case of the full moon both forces $\vec{F}_S$ and $\vec{F}_E$ are in the same direction, to the left in the diagram. The magnitude of the net force is the sum of the two magnitudes.

$$F_{\text{net}} = F_S + F_E = 6.31 \times 10^{20}\ \text{N}$$

(ii) With the new moon, the forces $\vec{F}_S$ and $\vec{F}_E$ are in opposite directions, in the diagram: to the left for the sun and to the right for the earth. The magnitude of the net force is now the difference.

$$F_{\text{net}} = F_S - F_E = 2.36 \times 10^{20}\ \text{N}$$

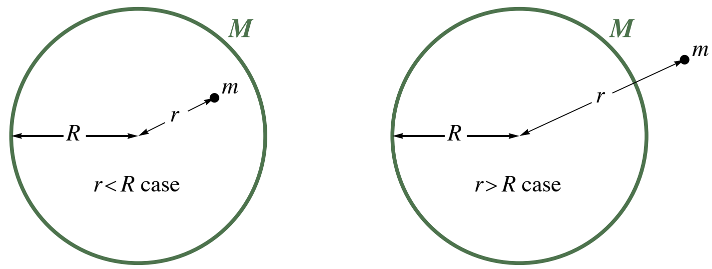
(iii) At the half moon phase, the forces $\vec{F}_S$ and $\vec{F}_E$ are perpendicular, since in the limit $R_{\text{EM}} << \text{R}_{\text{ES}}$ the hypotenuse of the earth-moon-sun triangle becomes parallel to the long side. The magnitude of the net force found by the Pythagorean theorem.

$$F_{\text{net}} = \sqrt{F_S^2 + F_E^2} = 4.77 \times 10^{20}\ \text{N}$$

Note that since $F_S > F_E$ the moon's trajectory must always curve toward the sun.

The Shell Theorem and Spherical Distributions

Consider a uniform spherical shell of mass M and radius R. If we consider a point mass m a distance r from the center then the shell theorem states that:

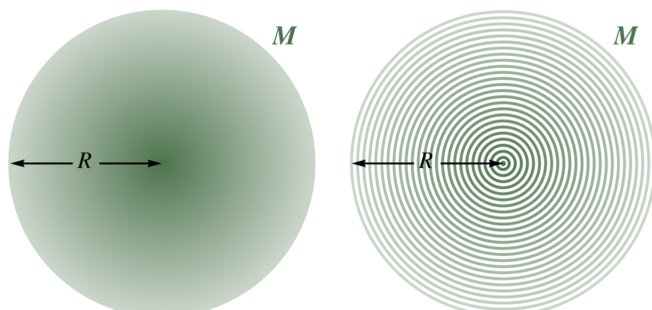


$$\text{for } r < R, \quad \vec{F} = \vec{0} \quad \text{and}$$

$$\text{for } r > R, \quad \vec{F} = -G\,\frac{M\,m}{r^2}\,\hat{r}$$

To state this in words: When m is outside the shell ($r > R$) the shell behaves as if all the mass is at the center. When m is inside the shell ($r < R$) the total gravitational force on the mass is zero. Clearly, by symmetry, at the exact center of the shell the force should be zero. It is a remarkable property of the inverse square law that it is zero everywhere inside. We will not prove the shell theorem this semester. It will be proven in the second semester course in the analogous case of electrostatics.

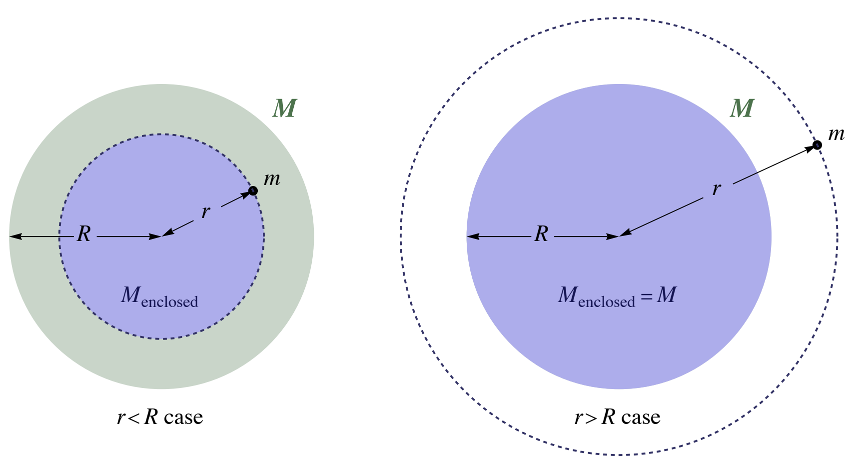
As a consequence of the shell theorem we can find the force on a mass m a distance r from the center of *any* spherical distribution. Break up the sphere into thin concentric shells.



If the shell is inside the radius r the shell behaves as if all its mass is at the origin. If it is outside it doesn't contribute to the force. This gives the result:

$$\vec{F} = -G\,\frac{m\,M_{\text{enclosed}}}{r^2}\,\hat{r}.$$

where M_{enclosed} is the total mass enclosed by a sphere of radius r.



The Gravitational Field

We will refer to the acceleration due to gravity at some position as the gravitational field $\vec{g}$; typically this no longer will be uniform. Since the force on a mass m is $\vec{F} = m\,\vec{g}$ we can define the field at some position by the following procedure. Add a test mass m_0 at some position. Define the field at that position to be the force divided by the test mass.

$$\vec{g} = \frac{\vec{F}}{m_0}$$

The result is then independent of the test mass.

Now consider a point mass M at the origin. The gravitational field at position $\vec{r}$ is found by placing a test mass m_0 there. The force on m_0 is

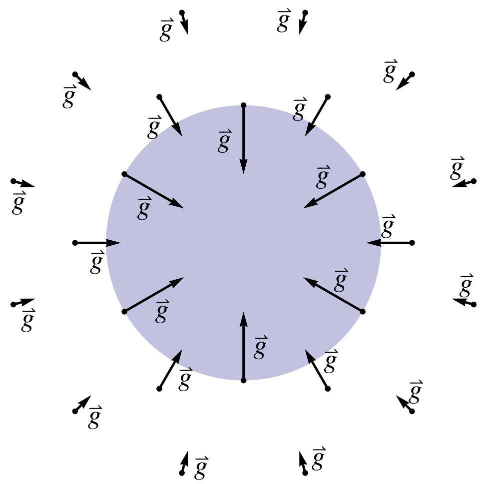
$$\vec{F} = -G\,\frac{m_0\,M}{r^2}\,\hat{r}.$$

Dividing m_0 into this gives

$$\vec{g} = -G\,\frac{M}{r^2}\,\hat{r} \quad \text{or} \quad g = \|\,\vec{g}\,\| = G\,\frac{M}{r^2}.$$

This also applies outside of a spherical body. At the surface of a spherical planet of mass M and radius R, the magnitude of the field is

$$g = G\,\frac{M}{R^2}.$$



Gravitational field at the surface and around a spherical planet

Example J.2 - Weight on a Different Planet

An astronaut has an earth weight of 120lb. What is her weight, in pounds, on a planet with twice the earth's radius and five times its mass.

Solution

The gravitational field (or acceleration due to gravity) g on the surface of a spherical planet of mass M and radius R is:

$$g = G\,\frac{M}{R^2}.$$

The weight W of an astronaut of mass m is related to g by $W = mg$. Both formulas apply to both cases, on the earth and the other planet. The astronaut's mass m is the same on both planets. The two formulas apply to each case. For the weights we see that the ratio of the weights equals the ratio of the g values.

$$W_1 = m\,g_1 \;\text{ and }\; W_2 = m\,g_2 \;\underset{\text{Dividing}}{\Longrightarrow}\; \frac{W_2}{W_1} = \frac{g_2}{g_1}$$

Similarly for g we see:

$$g_1 = G\,\frac{M_1}{R_1^2} \;\text{ and }\; g_2 = G\,\frac{M_2}{R_2^2} \;\underset{\text{Dividing}}{\Longrightarrow}\; \frac{g_2}{g_1} = \frac{M_2/M_1}{(R_2/R_1)^2}$$

Take case 2 to be the new planet and case 1 the earth.

$$\frac{W_2}{W_1} = \frac{M_2/M_1}{(R_2/R_1)^2} \implies \frac{W}{W_E} = \frac{M/M_E}{(R/R_E)^2} = \frac{5}{2^2}$$

We can solve for the result.

$$W = \frac{5}{2^2}\, W_E = \frac{5}{2^2} \times 120\,\text{lb} = 150\,\text{lb}$$

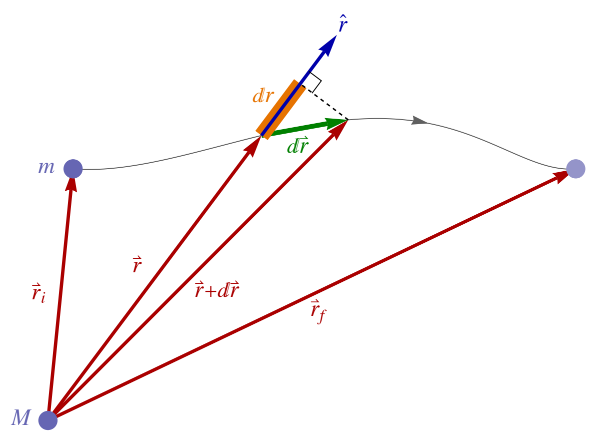
J.2 - Energy Considerations

Potential Energy of Two Masses

Recall that a conservative force is any force whose work is independent of path. We can define a potential energy for any conservative force by

$$\Delta U = -W = -\int \vec{F} \cdot d\vec{r}$$

We want to find the potential energy for two masses m and M separated by a distance r. To get this we will first consider M to be at a fixed position, which we will take as the origin, and move m in its presence along some path and find dU for that path.



Since the force on m due to M is $\vec{F} = -G\,\frac{Mm}{r^2}\,\hat{r}$, we get

$$\Delta U = -\int \vec{F} \cdot d\vec{r} = -\int_{\vec{r}_i}^{\vec{r}_f} \left(-G\,\frac{Mm}{r^2}\right) \hat{r} \cdot d\vec{r}$$

Multiplying a vector by a unit vector gives the component of the vector in the direction of the unit vector. $\hat{r} \cdot d\vec{r}$ becomes the radial component of $d\vec{r}$ which is just dr, the infinitesimal change in the radial variable and note this this is not the same as the magnitude of $d\vec{r}$.

$$\hat{r} \cdot d\vec{r} = dr \;\text{ where }\; dr = d\|\vec{r}\| = \left\|\vec{r} + d\vec{r}\right\| - \|\vec{r}\| \neq \left\|d\vec{r}\right\|$$

With the substitution of dr for $\hat{r} \cdot d\vec{r}$, the limits of the integral depend only on the radial distances of the endpoints and it is now a simple single-variable integral.

$$\Delta U = GMm \int_{r_i}^{r_f} \frac{dr}{r^2} = -GMm\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

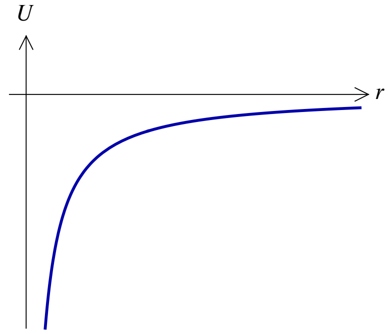
Clearly this result is independent of the path taken; this demonstrates the force is conservative and justifies the definition of the potential energy.

We are looking for some function $U(r)$, describing the potential energy as a function of position that satisfies $\Delta U = U(r_f) - U(r_i)$. This is unique up to an arbitrary constant. The simplest choice is

$$U(r) = -G\,\frac{Mm}{r}$$

where we chose to make the potential energy zero at infinity.

$$U(\infty) = 0 \quad \text{or more precisely} \quad \lim_{r\to\infty} U(r) = 0$$



Potential Energy of a Configuration

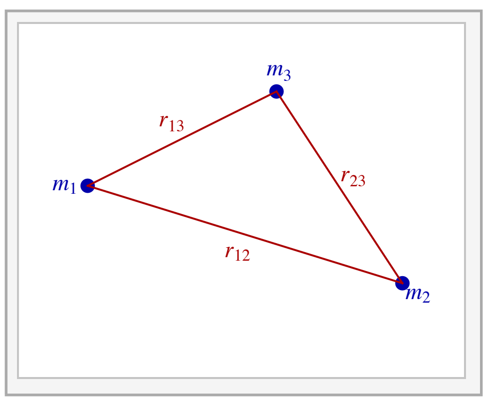
Now we consider the case of several point masses, m_1, m_2, ... Take the distance between m_1 and m_2 to be r_{12} and generally r_{ij} as the distance between the i^{th} and j^{th} masses. For two masses we have $U = -G\,m_1\,m_2/r_{12}$. For three masses we have this term plus energy terms between m_1 and m_3, and between m_2 and m_3. We get

$$U = -G\,\frac{m_1\,m_2}{r_{12}} - G\,\frac{m_1\,m_3}{r_{13}} - G\,\frac{m_2\,m_3}{r_{23}}.$$

The general expression is

$$U = -G\sum_{i<j}\frac{m_i\,m_j}{r_{ij}}.$$

The sum is over all indices i and j where we insist that $i < j$ to avoid double counting.



Interactive Figure - The potential energy for multiple point masses is found by summing over all pairs.

Energy and Escape Speed

The total energy is the sum of the kinetic energies of all masses and the total potential energy above. For a single mass m moving in the presence of a large mass M the total mechanical energy is

$$E = K + U = \frac{1}{2}\,m v^2 - G\,\frac{M m}{r}.$$

The potential energy is negative, going to zero at infinity. To escape the gravitational pull of M, m must have enough energy to reach infinity, or zero energy.

The escape speed is the critical speed needed to escape the gravity of a planet from the surface of the planet of mass M and radius R. This becomes

$$0 = E = \frac{1}{2}\,m v_{\text{esc}}^2 - G\,\frac{M m}{R}.$$

Solving for the escape speed we get

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}\,.$$

Example J.3 - Rocket Launched off a Spherical Planet

A rocket is launched off the surface of a spherical planet of mass M and radius R with a speed v_0 in a vertical (perpendicular to the surface) direction.

(a) If the speed v_0 is less than the escape speed, the rocket will reach some maximum distance from the center r_{max} and fall back. What is r_{max}?

Solution

$$E = \frac{1}{2}mv^2 - G\frac{Mm}{r}$$

m is the rocket's mass. It must scale out of the problem. Take the initial position to be at the surface and the final to be at r_{max}. The final kinetic energy is zero.

$$E_i = E_f \implies \frac{1}{2}mv_0^2 - G\frac{Mm}{R} = 0 - G\frac{Mm}{r_{\text{max}}} \implies r_{\text{max}} = \left(\frac{1}{R} - \frac{v_0^2}{2GM}\right)^{-1}$$

(b) For rocket speeds greater than the escape speed, the rocket will continue to infinity and still have kinetic energy. What is v_∞, the speed at infinity? (To be precise, this should be discussed in terms of limits: as $r \to \infty$, $v \to v_\infty$.)

Solution

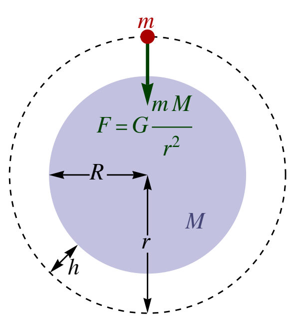
The expression for the energy is the same as is the initial values. The final potential energy is now zero.

$$E_i = E_f \implies \frac{1}{2}mv_0^2 - G\frac{Mm}{R} = \frac{1}{2}mv_\infty^2 + 0 \implies v_\infty = \sqrt{v_0^2 - 2G\frac{M}{R}}$$

Note that this can be written as $v_\infty = \sqrt{v_0^2 - v_{\text{esc}}^2}$.

J.3 - Orbits

Circular Orbits



We will now consider circular orbits of a satellite of mass m orbiting a much larger object of mass M, $m \ll M$. Take the larger object to be a sphere of radius R. If the satellite orbits at a height h above the surface, then the radius of the orbit is

$$r = R + h.$$

We now apply the second law to the satellite. The only force is gravity and the acceleration is centripetal. We get:

$$F_{\text{net},c} = ma_c \implies G\frac{Mm}{r^2} = ma_c.$$

In this expression the satellite mass cancels; this is the essence of weightlessness. Suppose one studies the orbit of the international space station, then m is the station's mass. If an astronaut is floating in the station and not touching anything, then m is the astronaut's mass. The cancellation of m implies that the astronaut and station have the same orbit; he will float relative to the station.

After canceling the masses we can then use the two expressions for the centripetal acceleration for uniform circular motion. This gives a pair of expressions, one that relates the speed and radius and the other relating the period and radius.

$$G\frac{M}{r^2} = a_c = \begin{cases} \frac{v^2}{r} \\ \left(\frac{2\pi}{T}\right)^2 r \end{cases}$$

Solving these expressions for the speed and period gives.

$$v = \sqrt{\frac{GM}{r}}$$

$$T^2 = \frac{4\pi^2}{GM} r^3$$

Example J.4 - Low-earth Orbit

On a planet without an atmosphere, it is possible to orbit just above the surface. For the earth the orbit must be above the atmosphere. Although the atmosphere gets less dense with the height above the surface, there is a point where the drag is small enough for an orbit. Consider a low earth orbit to be 200 km (about 120 mi) above the surface.

What are the period and speed of this low-earth orbit?

Solution

$$R_{\text{earth}} = 6.38\times10^6\text{ m}\,,\quad M_{\text{earth}} = 5.97\times10^{24}\text{ kg}$$

$$h = 200\text{ km} = 2.0\times10^5\text{ m} \implies r = R_{\text{earth}} + h = 6.58\times10^6\text{ m}$$

$$T^2 = \frac{4\pi^2}{G\,M_{\text{earth}}} r^3 \implies T = 5310\text{ s} = 89\text{ min}$$

Low-earth orbits can be higher but the height should not be much larger. These will have slightly longer periods, but low-earth orbits are close to 90 minutes.

$$v = \sqrt{\frac{G\,M_{\text{earth}}}{r}} = 7780\text{ m/s}$$

Example J.5 - Geostationary Orbit

It is useful for communication satellites, weather satellites and spy satellites to maintain a fixed position in the sky relative to the rotating earth. This is called a geostationary orbit. This requires an orbit directly over the equator with a period of 24 h.

(a) Take the period of a low-earth orbit to be 1.5 hours and the orbital radius to be the earth's radius, R_E. Estimate the radius of a geostationary orbit as a multiple of R_E.

Solution

$$T^2 = \frac{4\pi^2}{G\,M} r^3$$

We will use ratios to solve this, as we did in Example-J.2. We will use the period equation above for two cases: $r_1 = R_E$ with $T_1 = 1.5$ h and find the unknown r_2 with $T_2 = 24$ h. $M = M_{\text{earth}}$ for both cases.

$$T_1^2 = \frac{4\pi^2}{G\,M_{\text{earth}}} r_1^3 \text{ and } T_2^2 = \frac{4\pi^2}{G\,M_{\text{earth}}} r_2^3 \xRightarrow[\text{Dividing}]{} \left(\frac{T_2}{T_1}\right)^2 = \left(\frac{r_2}{r_1}\right)^3$$

$$r_2 = \left(\frac{T_2}{T_1}\right)^{2/3} r_1 \simeq \left(\frac{24\text{ h}}{1.5\text{ h}}\right)^{2/3} R_E = 16^{2/3}\,R_E \implies r_2 \simeq 6.4\,R_E$$

(b) Calculate this geostationary orbital radius precisely. While we are being precise, the orbital period is not one solar day but one sidereal day 23.93 h. (Note that a sidereal day is the time it takes for the earth to rotate once relative to the distant stars, as opposed to relative to the sun. There are 365.24 solar days in a year but, with one extra rotation relative to the distant stars, there are 366.24 sidereal days in a year.)

Solution

$$T^2 = \frac{4\pi^2}{G\,M_{\text{earth}}}\,r^3 \implies r = \left(G\,M_{\text{earth}}\,\frac{T^2}{4\pi^2}\right)^{1/3}$$

$$r = \left(\left(6.673\times10^{-11}\,\frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\right)(5.97\times10^{24}\,\text{kg})\,\frac{(23.93\times3600\,\text{s})^2}{4\pi^2}\right)^{1/3} = 4.22\times10^7\,\text{m}$$

Compare this with the radius of the earth.

$$R_{\text{earth}} = 6.38\times10^6\,\text{m} \implies r = 6.61\,R_{\text{earth}}$$

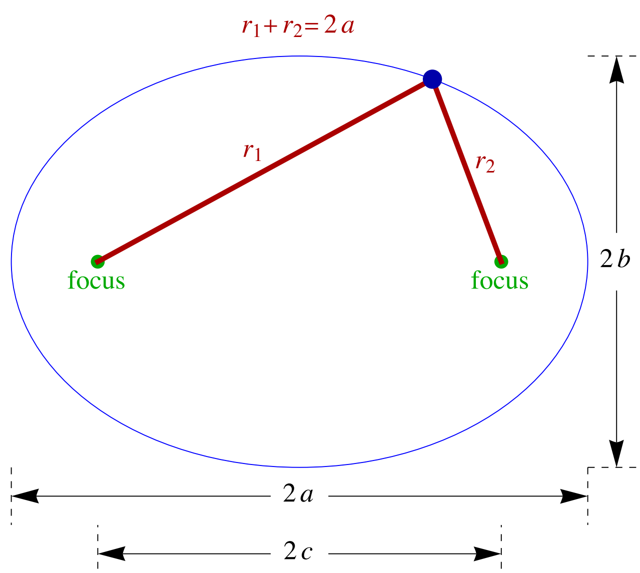
Kepler's Laws

Kepler came after both Copernicus and Galileo, and came before Newton. Kepler used data taken by someone else, Tycho Brahe, of the positions of the planets in the sky as functions of time. From this he realized that the circular orbits of Copernicus and Galileo could not fit the data. He settled on elliptical orbits and summarized his results with three laws.

After Kepler, Newton found his three laws of motion and his law of universal gravitation. He was able to derive Kepler's three law from his work. This was a remarkable feat. He was also able to prove that Kepler's laws implied that the gravitational force law had to be an inverse square law.

Kepler's First Law - Planets move in elliptical orbits about the sun, which is at a focal point of the ellipse.

Given two points, called focal points, an ellipse is the set of points in a plane such that the sum of the two distances from each focus is a constant.



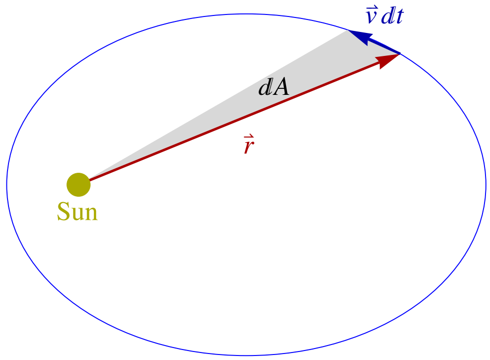
Interactive Figure

a is called the semimajor axis; it is half the largest distance between two points on the ellipse. The other dimension of the ellipse is the minor axis $2b$. Each focus is a distance of c from the center, where $c = e\,a$, with e known as the eccentricity. The eccentricity is the deviation from being a circle, $0 \le e < 1$. $e = 0$ is the circular case, where the two foci join and $a = b$. The variables a, b, c and e are related by

$$c = e\,a \text{ and } b = \sqrt{a^2 - c^2} = a\sqrt{1 - e^2}$$

Proving that orbits are elliptical is not difficult but is a bit beyond the scope of this course.

Kepler's Second Law - The radial vector from the sun to the planet sweeps out equal areas in equal time.



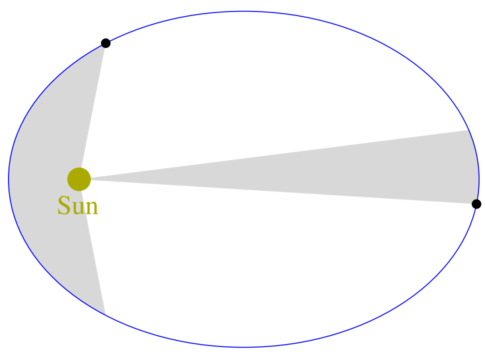
This is a simple consequence of the conservation of angular momentum. The area of a parallelogram formed by two vectors is $A\,B\sin\theta$ which is just the magnitude of the cross product of the vectors. In a time $d\,t$ a planet moves by $\vec{v}\,d\,t$. This traces out a triangle; this is half a parallelogram and thus has the infinitesimal area

$$d\,A = \frac{1}{2}\left\|\vec{r}\times\vec{v}\,d\,t\right\|$$

Since $\vec{L} = \vec{r}\times\vec{p}$ is constant, it follows that $d\,A/d\,t$ is constant.

$$\frac{d\,A}{d\,t} = \frac{1}{2}\left\|\vec{r}\times\vec{v}\right\| = \frac{1}{2\,m}\left\|\vec{r}\times\vec{p}\right\| = \frac{L}{2\,m} = \text{constant.}$$

This proves the result.

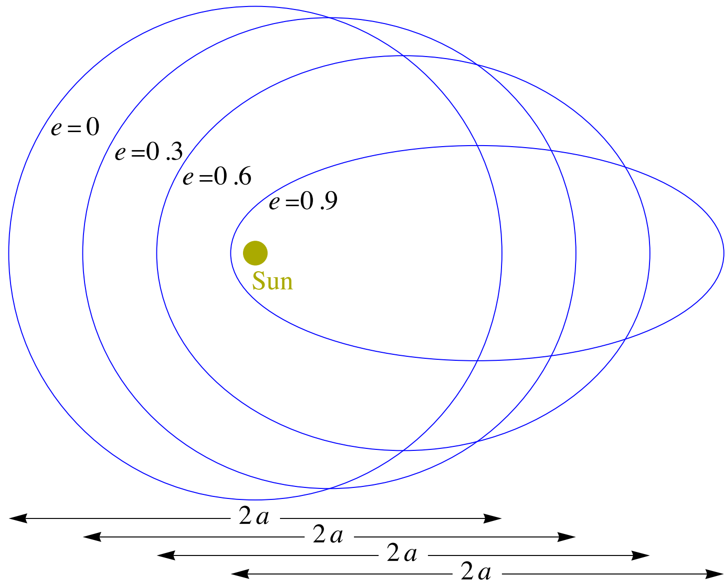


Interactive Figure

Kepler's Third Law - Period$^2 \propto$ (Semimajor axis)^3 or $T^2 \propto a^3$

Define the major axis as the largest distance between two points on an elliptical orbit. The semimajor axis, a, is half this. For a circular orbit, this is the same as the radius $a = r$. We saw that $T^2 \propto r^3$ for a circular orbit. The third law is then a generalization of this result, which we will not prove.

$$T^2 = \frac{4\,\pi^2}{G\,M}\,a^3.$$



Different orbits with the same period. The eccentricities vary but the semimajor axis and thus the period stays the same.