Chapter L

Waves

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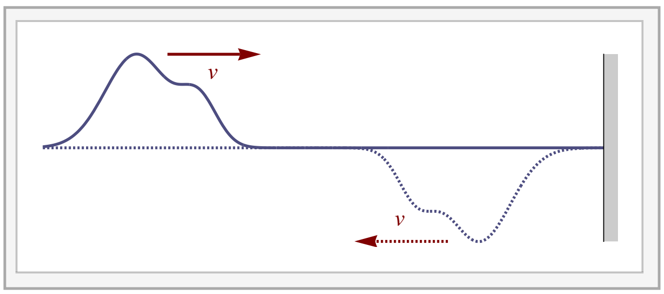
L.1 - General Considerations

A mechanical wave is a disturbance in a medium that propagates through the medium. We will generically label this disturbance by u. A wave carries energy. Consider a stretched string. Take x to be the position along the string and t is time. The equilibrium position of the string is its relaxed position. The disturbance y ($u = y$ for a string) is the perpendicular distance of a point on the string from equilibrium. When a pulse is put in the string it maintains its shape and travels the length of the string at a fixed speed. We will derive a formula for this speed; it will be $v = \sqrt{T/\mu}$, where T is the tension in the string and μ is the linear density of the string, its mass per length.



Interactive Figure

A wave is said to be *transverse* when the direction of the disturbance is perpendicular to the direction of propagation. Waves on a string are examples of transverse waves. Electromagnetic waves are also transverse; in the electromagnetic case the disturbance is the electric field, which is perpendicular to the direction of propagation. Note that electromagnetic waves are not mechanical waves; there is no medium and it can propagate in a vacuum. There is a plane of possible directions perpendicular to a direction of propagation. Choosing such a direction is choosing a *polarization*. Transverse waves can be polarized.

Consider a stretched spring. If a pulse is put into the spring then that pulse will propagate as a wave. Here a point on the spring moves back and forth a distance u from equilibrium but parallel to the direction of propagation x. When the disturbance is parallel to the direction of propagation we call the wave *longitudinal*. Longitudinal waves cannot be polarized.

Another example of a longitudinal wave is sound in a fluid. Here the molecules move back and forth parallel to the direction of wave propagation. We can view the disturbance of sound waves either in terms of displacement or in terms of pressure. Sound waves travel at the speed of sound. This varies with temperature; at 20 °C it is 343 m/s

Wave Type	**Disturbance – u**	**Transverse or Longitudinal**	**Wave Speed**
Waves on a string	$u = y(x, t)$	Transverse	$v = \sqrt{\frac{T}{\mu}}$
Electromagnetic Waves	$u = E$ = Electric field	Transverse	$v = c = 3.00\times 10^8\ \frac{\text{m}}{\text{s}}$
Compression Waves on a Spring	u = Parallel disp.	Longitudinal	no formula given
Sound Waves in a Fluid	u = Pressure or u = Displacement	Longitudinal	$v = v_{\text{sound}} = 343\ \frac{\text{m}}{\text{s}}$ (in air)

L.2 - The One Dimensional Wave Equation

We will take some generic wave variable to be u which is some disturbance that varies as a function of x, the position, and t, time. For example, consider waves on a stretched string. The position along the string is labelled by x, t is time and u is the distance of a point on the string from its equilibrium position. The one dimensional wave equation is

$$\frac{\partial^2}{\partial x^2} u = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} u,$$

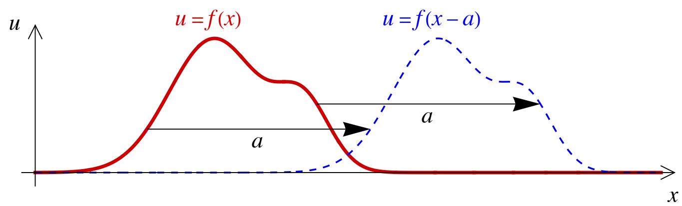
where v is the wave speed. This is a second order partial differential equation for $u(x, t)$.

The General Solution

The general solution of an ordinary differential equation (ODE), where we solve for functions of one variable, involves arbitrary constants. The general solution of a partial differential equation (PDE), where we are solving for functions of several variables, will involve arbitrary functions. The general solution is

$$u(x, t) = f(x - v\,t) + g(x + v\,t),$$

where f and g are arbitrary functions. To understand this general solution, consider the function $u(x) = f(x)$. If we shift this by a in the positive direction we get $u(x) = f(x - a)$. We can now see that $u(x, t) = f(x - v\,t)$ describes a pulse of arbitrary shape $u(x) = f(x)$ moving in the positive direction with speed v. $u(x, t) = g(x + v\,t)$ corresponds to a pulse of a different arbitrary shape moving in the opposite direction at the same speed.



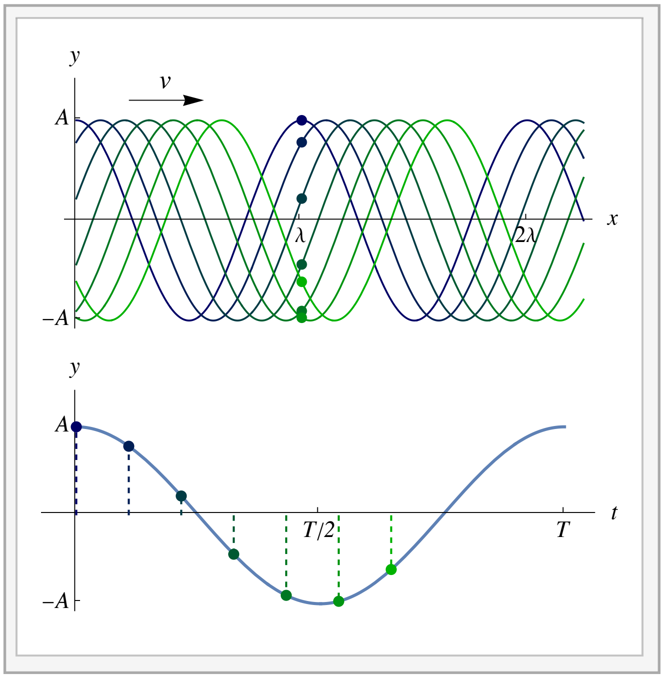
To verify this is a solution we will plug $u(x, t) = f(x - v\,t)$ into the wave equation. We need to evaluate the partial derivatives using the chain rule:

$$\frac{\partial^2}{\partial x^2} f(x - v\,t) = \frac{\partial}{\partial x} f'(x - v\,t) = f''(x - v\,t) \text{ and}$$

$$\frac{\partial^2}{\partial t^2} f(x - v\,t) = (-v) \frac{\partial}{\partial t} f'(x - v\,t) = (-v)^2 f''(x - v\,t).$$

Inserting $u = f(x - v\,t)$ into the wave equation we can see now that it is a solution for any function f. If we replace v with $-v$ it is still a solution and since f is arbitrary we can replace it with g; it follows that $u = g(x + v\,t)$ is also a solution. Since the derivative of the sum of two functions is the sum of the derivatives the sum of our f and g solutions must also be a solution. This verifies that our expression for the general solution is indeed a solution. For it to be the general solution then *any* solution can be written in this form; to verify this is beyond the scope of the class.

Sinusoidal Waves



Interactive Figure

We often consider waves where the shape of the pulse f (or g) are sinusoidal.

$$f(x) = A \cos k\, x$$

A is called the amplitude. k is called the wave number; this is related to the wavelength λ, which is the spatial period of the function. Since the period of sine is 2π and the period of f is λ we get $k\lambda = 2\pi$ or

$$k = \frac{2\pi}{\lambda}.$$

If we take this function f and move it in the positive or negative direction we get $f(x \mp v\, t) = A \cos[k\,(x \mp v\, t)]$ or

$$u(x, t) = A \cos(k\, x \mp \omega\, t),$$

where the angular frequency ω and wave number are related to the wave speed by $k\, v = \omega$.

If we choose some point on the string x_0 then at that position moves as

$$A \cos(\omega\, t + \phi).$$

This is our expression for simple harmonic motion. Since the angular frequency is related to the frequency by $\omega = 2\pi f$ the wave speed can also be written in terms of the frequency and wavelength.

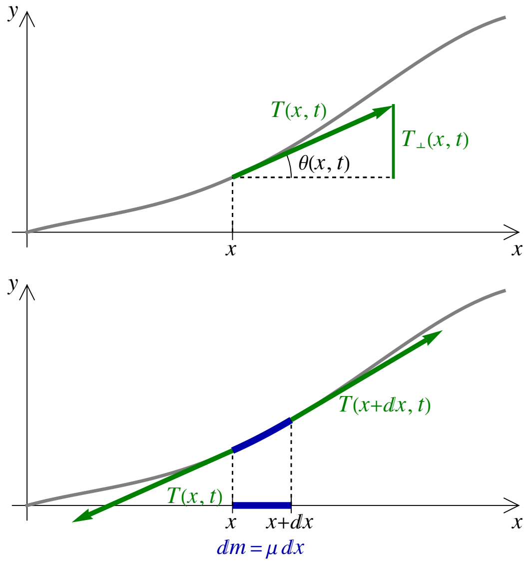
$$v = \frac{\omega}{k} = f\,\lambda$$

L.3 - The Speed of Waves on a String

Consider a stretched string. Take x to be the position along the string and t to be time. The perpendicular distance of a point on the string from equilibrium is y. y is a function of x and t; it is the disturbance, so we will use $y(x, t)$ instead of $u(x, t)$. We will study the physics of this stretched string assuming that our disturbance y is small. Applying Newton's second law to an infinitesimal segment of a string will give an equation of the form of a one dimensional wave equation. From this we will be able to read off the speed of the waves.

Consider an infinitesimal segment of width dx between x and $x + dx$. Since μ is the mass per length, the infinitesimal mass of this segment is $\mu\, dx$. Take the angle of the string from the x-axis to be θ. The tangent of the angle is the slope of the curve, dy/dx, but since y is a function of more than one variable we must use partial derivatives: $\tan\theta = \partial y/\partial x$. The assumption that our disturbance is small is equivalent to the assumption that the angle θ is small. For small angles we have:

$$\theta \simeq \sin\theta \simeq \tan\theta = \frac{\partial y}{\partial x}.$$



The tension in the string is T. We are interested in the perpendicular motion of the string, where perpendicular means the y direction. The perpendicular component of the tension is

$$T_\perp = T\sin\theta = T\,\frac{\partial y}{\partial x}$$

The infinitesimal net force dF_{net} on an infinitesimal segment of string of width dx is

$$dF_{\text{net}} = T_\perp(x+dx,\,t) - T_\perp(x,\,t) = \frac{\partial T_\perp}{\partial x}\,dx = T\,\frac{\partial^2 y}{\partial x^2}\,dx.$$

Note that with our small angle approximation the component of the net force parallel to the string (the x component) vanishes.

The acceleration of the infinitesimal segment is $\partial^2 y/\partial t^2$ and its mass is $\mu\,dx$. Applying the second law gives

$$F_{\text{net}} = m\,a \implies T\,\frac{\partial^2 y}{\partial x^2}\,dx = (\mu\,dx)\,\frac{\partial^2 y}{\partial t^2}.$$

This gives

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{T/\mu}\,\frac{\partial^2 y}{\partial t^2}.$$

If we compare this with the one dimensional wave equation

$$\frac{\partial^2}{\partial x^2}\,u = \frac{1}{v^2}\,\frac{\partial^2}{\partial t^2}\,u,$$

we can see that with $u = y$ we get the wave equation with the speed given by

$$v = \sqrt{\frac{T}{\mu}}.$$

L.4 - Power Carried by Waves on a String

Waves carry energy. When a sinusoidal wave travels down a string the flow of energy is constant. If some quantity of energy flows in a time then the energy per time or power is constant. Now consider some point on the string. If the wave is moving in some direction at the speed v, then the energy moves past the point at the same rate. In a time dt all the energy in an infinitesimal segment of width $dx = v\,dt$ will pass the point.

The energy of a particle in simple harmonic motion satisfies

$$E = \frac{1}{2}\, m\, v_{\max}^2$$

Since a point on the string moves in simple harmonic motion we can apply this formula to dx. The infinitesimal mass of the segment is

$$dm = \mu\, dx = \mu\, v\, dt$$

and the infinitesimal energy in the segment is

$$dE = \frac{1}{2}\, dm\, v_{\max}^2 = \frac{1}{2}\, \mu\, v\, dt\;\, v_{\max}^2 .$$

The power is given by $\mathcal{P} = dE/dt$. The maximum speed of a point in simple harmonic motion is $v_{\max} = \omega A$. It follow that the power travelling down a string due to a sinusoidal wave is

$$\mathcal{P} = \frac{1}{2}\, \mu\, A^2\, \omega^2\, v.$$

Example L.1 - Waves on a Steel Wire

A wave of the form

$$y(x,\, t) = (0.020\text{ m})\, \sin\!\left[\left(105\text{ s}^{-1}\right)t + \left(3.0\text{ m}^{-1}\right)x\right]$$

travels down a steel wire with a linear density of 0.014226 kg/m.

(a) What are the frequency and wavelength of the wave? Also, what is the wave speed and what is the direction of the wave?

Solution

From the form of the function we can read off the amplitude A, the angular frequency ω and wave number k. Also, the linear density μ is given.

$$A = 0.020\text{ m}\,,\quad \omega = 105\text{ s}^{-1},\quad k = 3.0\text{ m}^{-1}\quad\text{and}\quad \mu = 0.014226\text{ kg/m}$$

The frequency, wavelength and speed follow from formulas for sinusoidal waves.

$$f = \frac{\omega}{2\pi} = 16.7\text{ Hz},\quad \lambda = \frac{2\pi}{k} = 2.09\text{ m}\quad\text{and}\quad v = \frac{\omega}{k} = 35\,\frac{\text{m}}{\text{s}} = f\,\lambda$$

The solution for a pulse is $f(x \mp v\,t)$, where the negative sign means the pulse is moving in the positive-x direction and positive implies the negative-x direction. Since the relative sign between ωt and $k x$ terms is positive, the wave is moving in the negative-x direction.

(b) What is the maximum speed of a point on the wire as the wave passes.

Solution

A point on a string (or wire) moves in simple harmonic motion as a sinusoidal wave passes. The maximum speed for simple harmonic motion is

$$v_{\max} = \omega\, A = 2.1\text{ m/s}\,.$$

Note that the wave speed is quite distinct from the speed of a point on the wire.

(c) What is the tension in the wire?

Solution

The tension in the wire can be found from the wave speed v and the linear density μ.

$$v = \sqrt{\frac{T}{\mu}} \quad\Longrightarrow\quad T = \mu\, v^2 = 17.4\text{ N}$$

(d) The volume density (mass/volume) of steel is $8050\text{ kg}/\text{m}^3$. What is the diameter of the wire?

Solution

The cross-sectional area of the wire can be found from the linear density μ and the volume density $\rho = 8050\ \mathrm{kg}/\mathrm{m}^3$.

$$\mu = \frac{m}{\ell} \implies \rho = \frac{m}{V} = \frac{m}{\mathrm{Area}\times\ell} = \frac{\mu}{\mathrm{Area}} \implies \mathrm{Area} = \frac{\mu}{\rho} = 1.767\times 10^{-6}\ \mathrm{m}^2$$

$$\mathrm{Area} = \pi\left(\frac{d}{2}\right)^2 \implies d = 2\sqrt{\frac{\mathrm{Area}}{\pi}} = 1.50\ \mathrm{mm}$$

(e) At what rate does energy flow down the wire?

Solution

The rate of energy flow is the power. Note that v is the wave speed.

$$\mathcal{P} = \frac{1}{2}\mu A^2 \omega^2 v = 1.10\ \mathrm{W}$$