Chapter A

Electric Charges and Electric Fields

Blinn College - Physics 2326 - Terry Honan

A.1 - Electric Charge

Properties of Charge

There are two types of electric charge. Like charges repel and unlike attract.

We can label the types with signs. The choice of positive and negative is arbitrary.

Electric charge is conserved.

The other fundamentally conserved quantities are: energy, linear momentum and angular momentum; these were discussed in the first semester mechanics course. A conserved quantity cannot be created or destroyed. To charge a body one can add charges to it or remove charges from it.

Electric charge is quantized.

Any charge is an integer multiple of the fundamental charge e. The unit of charge is C = coulombs, which is considered a fundamental unit in the SI system.

$$Q = n\,e, \quad \text{where } e = 1.602\times 10^{-19}\text{ C} \quad \text{and } n \text{ is an integer}$$

The historical definition of the Coulomb will be given later, when we discuss the magnetic force between two conductors. Since the May 2019 redefinition of the SI system, the coulomb is defined by setting the value for the fundamental charge e to be exactly $1.60217634\times 10^{-19}$C.

Normal matter consists of protons, neutrons and electrons with charges shown here.

Particle	Charge
proton	$+e$
neutron	0
electron	$-e$

It is clear that any combination of these will satisfy the charge quantization rule $Q = n\,e$, where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

> **Units:** The SI unit for charge is: C

Aside on Quarks and Particles

Fractionally Charged Quarks

Quarks are particles with fractional charge that combine to form a class of particles called hadrons, which includes protons and neutrons. Normal matter consists of combinations of two quarks: the *up* quark u and the *down* quark d, and electrons. The electron is not made up of quarks and is believed to be fundamental.

Quark	Charge
u	$+\frac{2}{3}\,e$
d	$-\frac{1}{3}\,e$

Although, it appears that the existence of fractionally charge quarks violates the charge quantization condition, fractionally charged quarks only occur in integer-charged combinations in nature. Combinations with three quarks form a class of particles called baryons, with neutrons and protons as examples. In addition to quarks there are antiquarks, which are the antiparticle counterpart of quarks; antiparticles are particles of the same mass but have the opposite charge. Three antiquarks make antibaryons. Also, a quark q and an antiquark $\overline{q}$ combine to form a class of particles called mesons.

Particle	Quark Content
proton	$u\,u\,d$
neutron	$u\,d\,d$
antiproton	$\overline{u}\,\overline{u}\,\overline{d}$
antineutron	$\overline{u}\,\overline{d}\,\overline{d}$
positive pion	$u\,\overline{d}$
negative pion	$\overline{u}\,d$

In addition to the up and down quarks there are four more quarks: the strange quark s, the charmed quark c, the bottom quark b and the top quark t. The charmed and top have the same charge as the up quark and the strange and bottom have the same charge as the down. These exotic quarks do not occur in usual matter but may be created by sufficiently powerful accelerators.

Fundamental Forces and Elementary Particles

The fundamental forces are: electromagnetism, the weak nuclear force, the strong nuclear force and gravity. Elementary particles break up into two categories: fermions which satisfy the Pauli Exclusion principle and bosons which do not. The fermions are the particles of matter and the bosons are the particles of interaction. The current working theory of particle physics, which has remained intact since the mid-1970s, is known as the *Standard model*. This encapsulates the fundamental elementary particles, which are described below, and their forces of interaction. The standard model unifies the weak and electromagnetic forces into what is called the electroweak force. It also describes the strong nuclear force as a separate interaction, but it does not include gravity, which is described by general relativity. The standard model is just provisional, until we can find a better theory; this would need to incorporate gravity, explain dark energy, dark matter and many other mysteries.

Fundamental fermions consist of quarks and leptons. They break up into three families, where the masses get increasingly large with the later families. Each fundamental fermion has an corresponding antiparticle.

	First Family	Second Family	Third Family
Quarks	u (up quark)	c (charm quark)	t (top quark)
	d (down quark)	s (strange quark)	b (bottom quark)
Leptons	ν_e (electron neutrino)	ν_μ (muon neutrino)	ν_τ (tau neutrino)
	e^- (electron)	μ^- (muon)	τ^- (tau)

Fundamental Fermions - the Particles of Matter

It is still a mystery why there are three families.

In our quantum description of interaction, all forces are associated with particles. These forces and their particles are listed below.

Fundamental Interaction			Particle
Standard Model	Electroweak Force	Electromagnetism	γ (photon)
		Weak Nuclear Force	$W^+,\ W^-,\ Z^0$ (W and Z bosons)
		Strong Nuclear Force	g (eight gluons)
		Gravity	gravitons

Fundamental Forces and their Bosons

All of these particles have been experimentally observed, except the graviton. Only recently, in 2016, have we observed gravity waves; this is a very weak and subtle thing to detect. The graviton is the quantum of this gravitational wave and we are nowhere near having the capability to see individual gravitons. In the standard model all particles with mass, the W and Z bosons and the quarks and leptons, get their mass through what is known as the Higgs mechanism. In the simplest implementation of the Higgs mechanism there is a single residual observable particle, called the Higgs particle or Higgs boson; this particle was observed in 2012.

Generation of Particle Masses
H^0 (Higgs boson)

Particles get their mass through the Higgs mechanism,
which produces the Higgs boson, as a by-product.

A.2 - Coulomb's Law

The first of the fundamental forces to be understood was gravity; this was described by Newton with his law of universal gravitation. Coulomb successfully described the electrostatic force between charges by analogy to gravity. We will first review Newtonian gravity before discussing Coulomb's law.

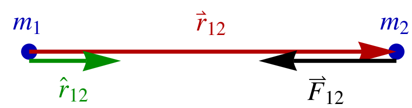
Review of Gravity

Newton's law of gravity is an inverse square law between point masses. If m_1 and m_2 are point masses separated by distance r the magnitude of the force between them is

$$F = G\,\frac{m_1\,m_2}{r^2}.$$

This can be written as a vector expression. Let $\vec{r}_{12}$ be the vector from mass 1 to mass 2 and let $\vec{F}_{12}$ be the force on mass 2 due to mass 1.

$$\begin{aligned}\vec{F}_{12} &= -G\,\frac{m_1\,m_2}{r_{12}^2}\,\hat{r}_{12}\\ &= -G\,\frac{m_1\,m_2}{r_{12}^3}\,\vec{r}_{12}\end{aligned}$$



Note that a unit vector is denoted by a "^" placed over the symbol and any other vector has a "⇀" over it. A vector $\vec{A}$ divided by its magnitude $A = \|\vec{A}\|$ is a unit vector $\hat{A} = \frac{\vec{A}}{A}$. So it follows that the unit radial vector $\hat{r}$ is the position vector divided by r, its magnitude. The two vector expressions above are equivalent because

$$\hat{r} = \frac{\vec{r}}{r} \implies \frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3}$$

Coulomb's Law

Coulomb found the force law for electrostatics by analogy to gravity. Mass is the gravitational analog of charge. There are two key differences between the electric and gravitational cases. Electric charge can be positive or negative but mass is always positive. The force between two masses is attractive but the force between like charges is repulsive. The electric force is an inverse square law between point charges. The magnitude of the force is

$$F = k_e\,\frac{|Q_1|\,|Q_2|}{r^2},$$

where the absolute values guarantee a positive result. The constant k_e is a universal constant, like Newton's gravitational constant G. It is related to another constant ε_0, which is usually taken as more fundamental.

$$\varepsilon_0 = 8.85\times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}$$

$$k_e \equiv \frac{1}{4\,\pi\,\varepsilon_0} = 8.99\times 10^{9}\,\frac{\mathrm{N}\cdot\mathrm{m}^2}{\mathrm{C}^2}$$

Example A.1 - Gravitational versus Electrical Attraction for a Hydrogen Atom

In addition to the attractive electric force between a proton and an electron there is also gravitational attraction as well. Is it important to consider gravity when studying the physics of the hydrogen atom?

What is the ratio of the gravitational to electric attraction between a proton and an electron?

Solution

The values of the relevant constants are :

$$
\begin{aligned}
k_e &= 8.99\times 10^9\ \mathrm{N\cdot m^2/C^2}\\
G &= 6.673\times 10^{-11}\ \mathrm{N\cdot m^2/kg^2}\\
e &= 1.602\times 10^{-19}\ \mathrm{C}\\
m_{\text{electron}} &= 9.11\times 10^{-31}\ \mathrm{kg}\\
m_{\text{proton}} &= 1.673\times 10^{-27}\ \mathrm{kg}
\end{aligned}
$$

The electric and gravitational forces have magnitudes :

$$
\begin{aligned}
F_{\text{elec}} &= k_e \frac{|Q_1|\cdot|Q_2|}{r^2} = k_e\, e^2 / r^2\\
F_{\text{grav}} &= G \frac{m_1\, m_2}{r^2} = G\, m_{\text{electron}}\, m_{\text{proton}} / r^2
\end{aligned}
$$

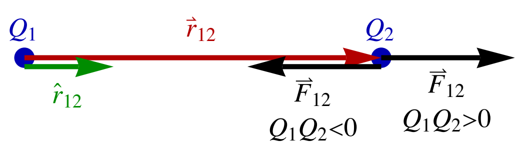
We now take the ratio. Note that because both forces vary as inverse square laws, the ratio does not depend on the distance between particles, r.

$$\frac{F_{\text{grav}}}{F_{\text{elec}}} = \frac{G\, m_{\text{electron}}\, m_{\text{proton}} / r^2}{k_e\, e^2 / r^2} = \frac{G\, m_{\text{electron}}\, m_{\text{proton}}}{k_e\, e^2} = 4.41\times 10^{-40}$$

The above number is dimensionless and is thus independent of units. This small numeric value shows that unless 40 digit accuracy is needed, we can ignore gravity when studying the hydrogen atom. The smallness of this value poses a fundamental question : How does some underlying theory that unifies gravity with the other forces, the strong nuclear force and the electroweak force, give rise to such a small dimensionless number. There is no such unified theory now but it is considered an ultimate goal of physics.

To get a vector expression for the force we must include both possibilities, attractive and repulsive. In the case of like charges $Q_1\, Q_2 > 0$ and unlike charges give $Q_1\, Q_2 < 0$.

$$
\begin{aligned}
\vec{F}_{12} &= k_e \frac{Q_1\, Q_2}{r_{12}^2}\, \hat{r}_{12}\\
&= k_e \frac{Q_1\, Q_2}{r_{12}^3}\, \vec{r}_{12}
\end{aligned}
$$

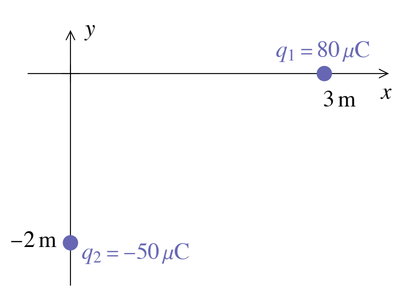


Note that the above expressions differ from the gravitational result by a sign. This is because in the gravitational case there are only like "charges" (mass is positive) and it is attractive but in the electric case, like charges repel.

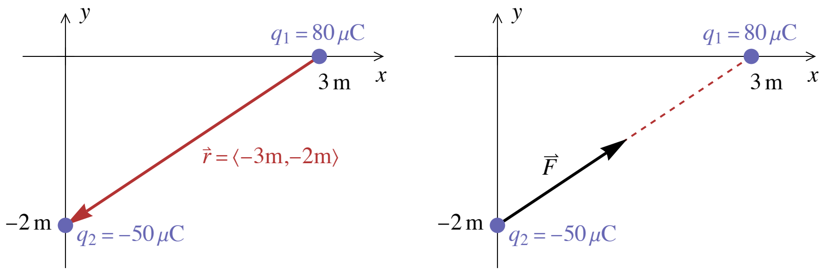
To find the force on a charge due to a distribution of charge one adds the forces due to each charge in the distribution. Force is a vector and this addition is then vector addition.

Example A.2 - Two Point Charges

An $80\,\mu$C is at $(3\,\mathrm{m},\, 0)$ and a $-50\,\mu$C is at $(0,\, -2\,\mathrm{m})$. What is the force on the -50 μC charge?



Solution



The vector from $Q_1 = 80\,\mu\text{C}$ to $Q_2 = -50\,\mu\text{C}$ is $\vec{r} = \langle -3\,\text{m},\, -2\text{m} \rangle$, where we choose the vector form the other charge to the charge we are calculating the force on. Two equivalent forms of Coulomb's force law are

$$\vec{F} = k_e\, Q_1\, Q_2\, \frac{\hat{r}}{r^2} \quad \text{and} \quad \vec{F} = k_e\, Q_1\, Q_2\, \frac{\vec{r}}{r^3}.$$

Since these are equivalent one should choose the form that is easiest. If the vector $\vec{r}$ is along an axis then the unit vector form is easier. As an example, if $\vec{r}$ is in the negative y direction then, $\hat{r} = -\hat{y} = \langle 0,\, -1 \rangle$. If $\vec{r}$ is in some other direction, then $\hat{r} = \vec{r}/r = \vec{r}\big/\big\|\vec{r}\big\|$ is awkward.

This is the case here, since $\vec{r} = \langle -3\,\text{m},\, -2\text{m} \rangle$.

$$\begin{aligned}
\vec{F} &= k_e\, Q_1\, Q_2\, \tfrac{\vec{r}}{r^3} \\
&= 9.0 \times 10^9\, \tfrac{\text{N}\cdot\text{m}^2}{\text{C}^2} \left(80 \times 10^{-6}\,\text{C}\right)\left(-50 \times 10^{-6}\,\text{C}\right) \tfrac{\langle -3\,\text{m}, -2\,\text{m} \rangle}{\left[(3\,\text{m})^2 + (2\,\text{m})^2\right]^{3/2}} \\
&= \langle 2.30,\, 1.54 \rangle\,\text{N}
\end{aligned}$$

Electrostatics and the Need for Fields

Coulomb's law describes the interaction of charged particles. It turns out that this is only correct in the context of electrostatics. Electrostatic is the study of electric charges, electric forces and fields where nothing is allowed to move. Since currents are moving charges, we do not allow for currents in electrostatics. Coulomb's law, and Newton's law of universal gravitation, appear to involve forces that acts instantaneously over arbitrarily large distances. This is known as action-at-a-distance. Since the introduction of Newtonian gravity, many philosophers of science were troubled by this notion.

The problem of action-at-a-distance became more serious with Einstein's introduction of the theory of special relativity in 1905. With special relativity, if one can send information faster than light, then one can send information backward in time; this would generate many causal paradoxes, not to mention science fiction plots. By information we mean merely one bit of information, a zero or a one. To demonstrate this in the context of Coulomb's law consider the following. Alice and Bob are some distance apart and both hold electric charges in their hands. If Alice moves her charge toward Bob then, according to Coulomb's law, Bob would instantly feel in increase in the force on his charge. To see how this can convey information, when Bob feels a larger force, call this a one and when a smaller force call that a zero; thus by moving her charge at regular time intervals, Alice may send information to Bob *instantaneously*.

It turns out however that in the nineteenth century, well before Einstein's introduction of special relativity, the problem of action-at-a-distance was removed for the case of the electric force through the introduction of electric and magnetic fields and with Maxwell's equations, which described the behavior of the fields both together and with electric charges. With Maxwell's equations, information may not be conveyed faster than light speed.

This is one of the big ideas of physics. Particles do not interact directly; all interaction is mediated by fields. Particles create fields, fields propagate by dynamical rules at a finite speed no faster than the speed of light and then fields exert forces on particles. In the case of electromagnetism, charged particles create electric and magnetic fields, as described by Maxwell's equations. Also, Maxwell's equations show how electric and magnetic field propagate. Electric and magnetic fields then exert forces on charged particles. So for the case of Alice and Bob, when Alice moves her charge, it creates a disturbance in the electromagnetic field (both fields combined). That disturbance propagates at the speed of light and when that disturbance hits Bob's position he feels a change in the force. It is not instantaneous.

The issue of action-at-a-distance was still present in Newtonian gravity at Einstein's time; with special relativity, Newton's law of universal gravitation then needed to be replaced. Einstein resolved this with a revolutionary new theory, General Relativity, which is a field theory of gravity that describes gravitational attraction in terms of the curvature of space and time.

A.3 - The Electric Field

The gravitational analog of electric field $\vec{E}$ is the gravitational field $\vec{g}$. To define the gravitational field, we find the force $\vec{F}$ on a test mass m_0 and divide the test mass into it.

$$\vec{g} = \frac{\vec{F}}{m_0}.$$

With a larger test mass the force is larger but when we divide out by the mass the result for the gravitational field is the same.

We define the electric field similarly. Find the force $\vec{F}$ on a test charge q_0 and divide the test charge into it.

$$\vec{E} = \frac{\vec{F}}{q_0}.$$

This is then independent of the value of the the charge. The direction of the electric field is the direction of the force on a positive test charge; thus, it points toward negative charges and away from positive charges.

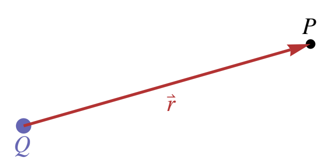
Field of a Point Charge

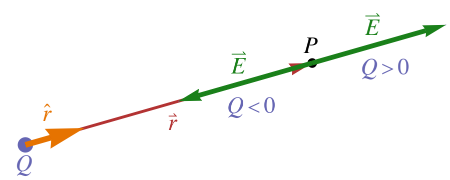
The electric field for a point charge Q can be found using the prescription above. Take the vector $\vec{r}$ to be the vector from Q to some point P. To get $\vec{E}$ due to Q at P, first find the force on a test charge q_0 placed at P using Coulomb's law.

$$\vec{F} = k_e \frac{Q\, q_0}{r^2}\, \hat{r} = k_e \frac{Q\, q_0}{r^3}\, \vec{r}$$

Dividing by q_0 gives

$$\vec{E} = k_e\, Q\, \frac{\hat{r}}{r^2} = k_e\, Q\, \frac{\vec{r}}{r^3}$$

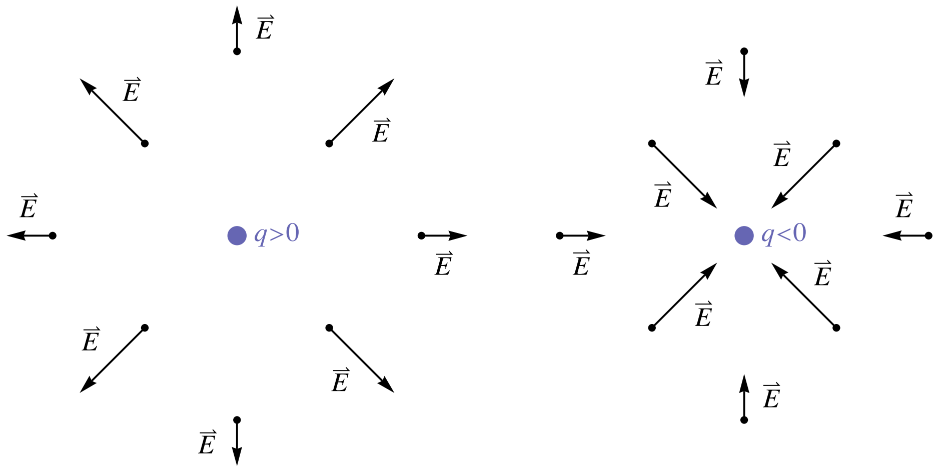




The magnitude of the field is given by

$$E = \|\vec{E}\| = k_e \frac{|Q|}{r^2}$$

where the absolute value of the charge forces the result to be positive, as the magnitude of a vector must. Since $\hat{r}$ is the radial unit vector (pointing away from the charge) then the direction of $\vec{E}$ is given by $\pm\hat{r}$, where is sign is the same as for Q.



The electric field is directed away from positive charges and toward negative.

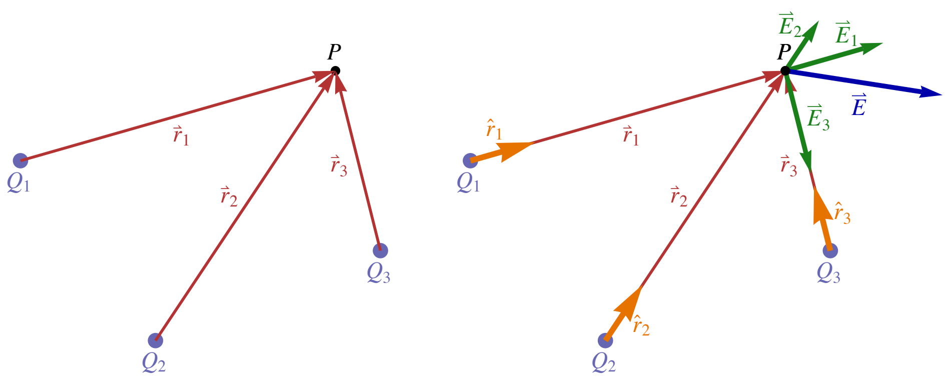
Principle of Superposition

To find the field for more complex charge configurations we use the principle of superposition. A charge distribution may be viewed as a collection of point charges. The electric field due to that distribution is the sum (or integral) over the fields due to point charges. We will consider first the case of a discrete distribution, where there is a collection of point charges, and then the case of a continuous distribution, where the charge is spread continuously over a line, surface or a volume.

Discrete Distribution

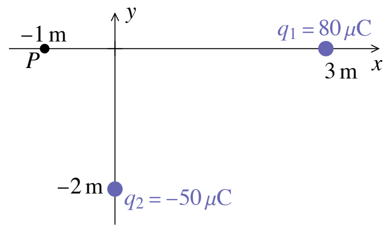
We want the field at a point P due to charges q_i. Take the vector $\vec{r}_i$ to be the vector from q_i to P. $\hat{r}_i$ is the unit vector in the direction of $\vec{r}_i$. The field $\vec{E}$ is the sum of the fields due to each charge.

$$\vec{E} = k_e \sum_i Q_i \frac{\hat{r}_i}{r_i^2} = k_e \sum_i Q_i \frac{\vec{r}_i}{r_i^3}$$

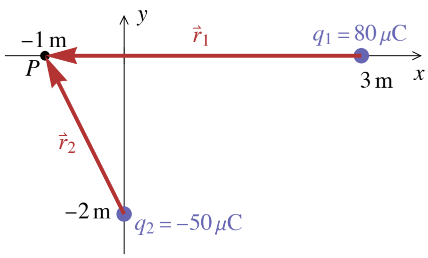


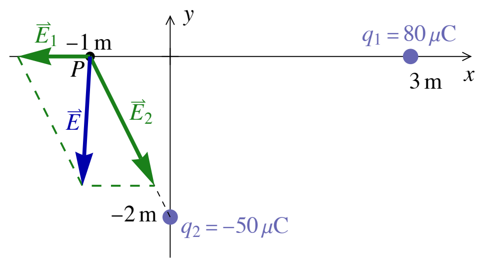
Example A.3 - Two Point Charges (Continued)

An 80μC is at (3 m, 0) and a -50μC is at (0, −2 m). What is the electric field at (−1 m, 0)?



Solution





P is the point where we evaluate the field, so here: $P = (-1\text{ m},\ 0)$. The vector from $Q_1 = 80\,\mu\text{C}$ to P is $\vec{r}_1 = \langle -4\text{ m}, 0\rangle$ and from $Q_2 = -50\,\mu\text{C}$ to P we have $\vec{r}_2 = \langle -1\text{ m}, 2\text{m}\rangle$. where we choose the vector form the other charge to the charge we are calculating the force on. The equivalent forms of the inverse square law for $\vec{E}_i$ are

$$\vec{E}_i = k_e\, Q_i\, \frac{\hat{r}_i}{r_i^2} = k_e\, Q_i\, \frac{\vec{r}_i}{r_i^3}$$

Of the equivalent forms we choose the easiest. Since the vector $\vec{r}_1$ is along an axis then the unit vector form is easier for it.

$$\vec{r}_1 = \langle -4\text{ m}, 0\rangle \implies \hat{r}_1 = \langle -1, 0\rangle = -\hat{x} \text{ and } r_1 = 4\text{ m} \implies \vec{E}_1 = k_e\, Q_1\, \frac{\hat{r}_1}{r_1^2}$$

$\vec{r}_2$ is not along an axis so the $\vec{r}_i / r_i^3$ form is easiest.

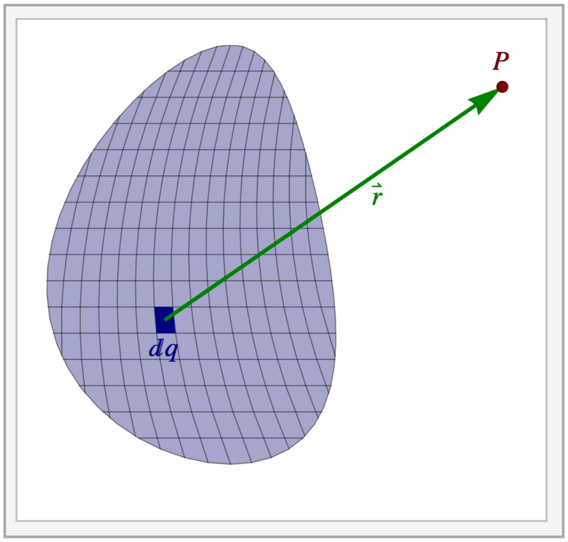
$$\vec{r}_2 = \langle -1\text{ m}, 2\text{m}\rangle \implies \vec{E}_2 = k_e\, Q_2\, \frac{\vec{r}_2}{r_2^3}$$

Summing $\vec{E}_1$ and $\vec{E}_2$ gives the total electric field.

$$\begin{aligned}
\vec{E} = \vec{E}_1 + \vec{E}_2 &= k_e\, Q_1\, \tfrac{\hat{r}_1}{r_1^2} + k_e\, Q_2\, \tfrac{\vec{r}_2}{r_2^3} \\
&= 9.0\times 10^9\ \tfrac{\text{N}\cdot\text{m}^2}{\text{C}^2}\left(80\times 10^{-6}\text{ C}\right)\tfrac{\langle -1, 0\rangle}{(4\text{ m})^2} \\
&\qquad +9.0\times 10^9\ \tfrac{\text{N}\cdot\text{m}^2}{\text{C}^2}\left(-50\times 10^{-6}\text{ C}\right)\tfrac{\langle -1\text{ m}, 2\text{ m}\rangle}{\left[(1\text{ m})^2+(2\text{ m})^2\right]^{3/2}} \\
&= \langle -45\,000,\ 0\rangle\text{ N/C} + \langle 40\,250,\ -80\,500\rangle\text{ N/C} \\
&= \langle -4750,\ -80\,500\rangle\text{ N/C}
\end{aligned}$$

Continuous Distribution

We want the field at a point P due to a continuous distribution of charge. In this case we break up the distribution into an infinite number of infinitesimal pieces, where the charge of an infinitesimal piece is dq.



Interactive Figure

Summing over the dq consists of evaluating an integral. Take Q to be the total charge; writing this as the sum over all dq gives

$$Q = \int dq.$$

Take the vector $\vec{r}$ to be the vector from dq to P. The field is

$$\vec{E} = k_e \int \frac{\hat{r}}{r^2}\, dq = k_e \int \frac{\vec{r}}{r^3}\, dq.$$

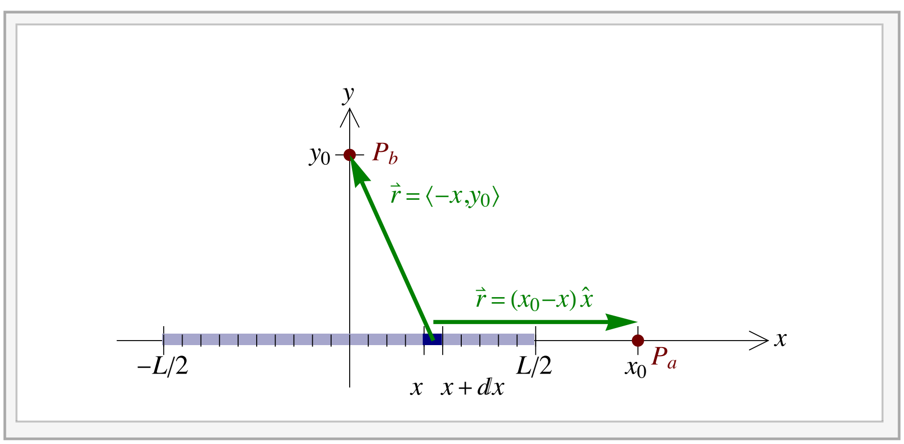
The above integrals, although they look like indefinite integrals, are vaguely defined definite integrals, where the integration variables and limits must be determined on a case by case basis.

Using Integrals to Calculate Electric Fields

1. Mathematically break up the distribution into an infinite number of infinitesimal pieces. This requires choosing an integration variable and its limits.
2. Find dq, the infinitesimal charge of each infinitesimal piece. You must write the dq in terms of the differential d of the integration variable.
3. Write an expression for $\vec{r}$, the vector from dq to the point P where the value of the field is wanted. Note that r is the magnitude of $\vec{r}$.
4. Put the pieces together to form a properly-defined definite integral and evaluate the integral.

Example A.4 - Uniform Line of Charge

Consider a line of uniform charge Q along the x-axis from $-L/2$ to $L/2$.



Interactive Figure

(a) What is the electric field along the x-axis at x_0, where $x_0 > L/2$?

Solution

Choose x as the integration variable with limits: $-\frac{L}{2} \le x \le \frac{L}{2}$.

The charge between x and $x + dx$ is: $dq = \frac{Q}{L}\, dx$.

The vector from dq (at x) to x_0 is: $\vec{r} = \langle x_0 - x,\, 0\rangle = \hat{x}\,(x_0 - x)$. Thus: $\hat{r} = \hat{x}$ and $r = \|\vec{r}\,\| = |x_0 - x| = x_0 - x$

$$\vec{E} = k_e \int \frac{\hat{r}}{r^2}\, dq = k_e \int_{-L/2}^{L/2} \frac{\hat{x}}{(x_0 - x)^2}\, \frac{Q}{L}\, dx = \hat{x}\, k_e\, \frac{Q}{L} \int_{-L/2}^{L/2} \frac{dx}{(x_0 - x)^2}$$

This is an integral of the form: $\int u^{-2}\, du$ where $u = x_0 - x$. Since $du = -dx$ and $\int u^{-2}\, du = -1/u + C$ we can evaluate the integral. (Note that the signs cancel.)

$$\vec{E} = \hat{x}\, k_e\, \frac{Q}{L} \int_{-L/2}^{L/2} \frac{dx}{(x_0 - x)^2} = \hat{x}\, k_e\, \frac{Q}{L}\, \frac{1}{x_0 - x}\Bigg|_{-L/2}^{L/2} = \hat{x}\, k_e\, \frac{Q}{L} \left(\frac{1}{x_0 - L/2} - \frac{1}{x_0 + L/2} \right)$$

(b) What is the electric field along the positive y-axis at y_0?

Solution

It is the same charge distribution so the first two steps are the same: $-\frac{L}{2} \le x \le \frac{L}{2}$ and $dq = \frac{Q}{L}\, dx$.

The vector from dq, at $(x,\, 0)$ to $(0,\, y_0)$ is: $\vec{r} = \langle -x,\, y_0\rangle = \hat{y}\, y_0 - \hat{x}\, x$.

$$\vec{E} = k_e \int \frac{\vec{r}}{r^3}\, dq = k_e \int_{-L/2}^{L/2} \frac{\langle -x,\, y_0\rangle}{\left(x^2 + y_0^2\right)^{3/2}}\, \frac{Q}{L}\, dx = k_e\, \frac{Q}{L} \left\langle -\int_{-L/2}^{L/2} \frac{x}{\left(x^2 + y_0^2\right)^{3/2}}\, dx,\; y_0 \int_{-L/2}^{L/2} \frac{1}{\left(x^2 + y_0^2\right)^{3/2}}\, dx \right\rangle$$

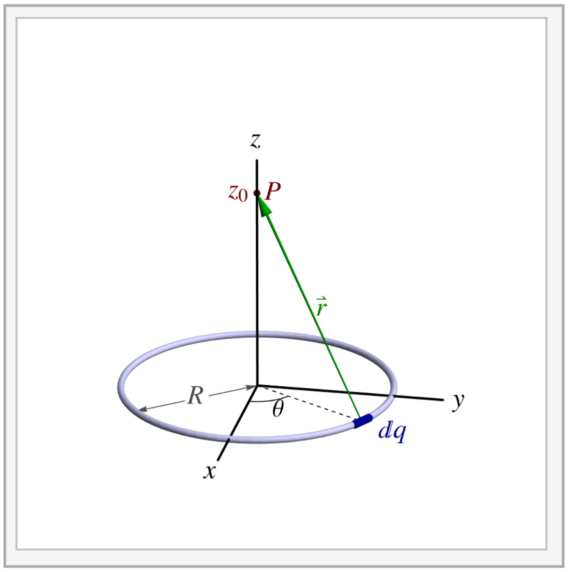
By symmetry the integral for the x-component is zero, leaving only the y-component.

$$\vec{E} = \hat{y}\, k_e\, \frac{Q}{L}\, y_0 \int_{-L/2}^{L/2} \frac{dx}{\left(x^2 + y_0^2\right)^{3/2}}$$

This is of the tabulated form: $\int \frac{dx}{\left(x^2 + a^2\right)^{3/2}} = \frac{1}{a^2}\, \frac{x}{\sqrt{x^2 + a^2}} + C$. We can then evaluate the integral.

$$\vec{E} = \hat{y}\, k_e\, \frac{Q}{L}\, \frac{1}{y_0}\, \frac{x}{\sqrt{x^2 + y_0^2}}\Bigg|_{-L/2}^{L/2} = \hat{y}\, k_e\, \frac{Q}{y_0}\, \frac{1}{\sqrt{(L/2)^2 + y_0^2}}$$

Example A.5 - Ring of Uniform Charge



Interactive Figure

Consider a thin ring of radius R with uniform charge Q in the xy-plane with the center at the origin

(a) What is the electric field at the origin?

Solution

Because of symmetry $\vec{E} = \vec{0}$.

(b) What is the electric field along the positive z-axis at z_0?

Solution

Because of symmetry we can conclude that the field must be in the z-direction.

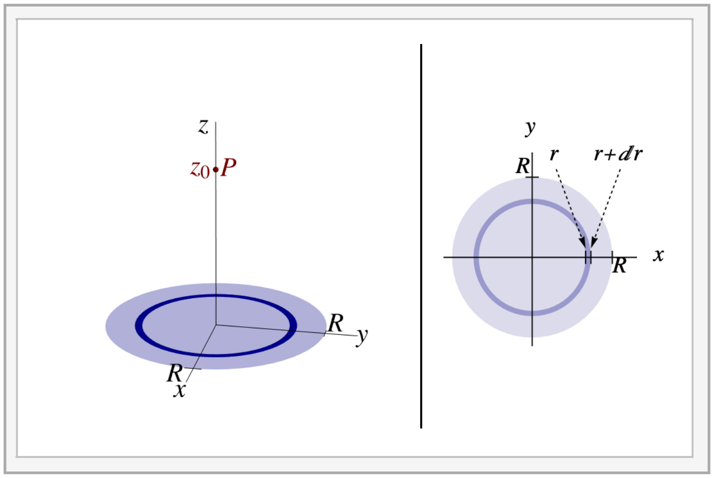
$$\vec{E} = k_e \int \frac{\vec{r}}{r^3}\, dq = \hat{z}\, E_z$$

$$= \hat{z}\, k_e \int \frac{z_0}{\left(R^2 + z_0^2\right)^{3/2}}\, dq = \hat{z}\, k_e \frac{z_0}{\left(R^2 + z_0^2\right)^{3/2}} \int dq = \hat{z}\, k_e \frac{z_0\, Q}{\left(R^2 + z_0^2\right)^{3/2}}$$

Example A.6 - Disk of Uniform Charge

Consider a thin disk of radius R with uniform charge Q in the xy-plane with the center at the origin. What is the electric field along the positive z-axis at z_0?

Solution



Interactive Figure

We begin with the field due to a uniform ring.

$$\vec{E} = \hat{z}\, k_e \frac{z_0\, Q}{\left(R^2 + z_0^2\right)^{3/2}}$$

We then break up the disk into concentric rings and sum over them by integrating. We will choose our integration variable to be r, varying from 0 to R: $0 \le r \le R$. The distance from the center of each ring is the same z_0 and the field is still in the $\hat{z}$-direction. Each ring now has a radius r and a charge $d q$. We now need to make the following substitutions:

$$Q \to d q\,,\quad R \to r\,,\quad z_0 \to z_0 \text{ and } \hat{z} \to \hat{z}$$

and we get

$$\vec{E} = \hat{z}\, k_e \int \frac{z_0\, d q}{\left(r^2 + z_0^2\right)^{3/2}}$$

The charge between r and $r + d r$ is $d q$. With a uniform charge, the fraction of the charge is the same as the fraction of the area $d q / Q = d A / A_{\text{tot}}$. The (infinitesimal) area is the length of the curve, the circumference $2\pi r$ times $d r$, the thickness: $d A = 2\pi r\, d r$. The total area is $A_{\text{tot}} = \pi R^2$. This gives is $d q$.

$$d q = \frac{Q}{A_{\text{tot}}} d A = \frac{Q}{\pi R^2} 2\pi r\, d r = \frac{2\,Q}{R^2} r\, d r$$

$$\vec{E} = \hat{z}\, k_e \int \frac{z_0\, d q}{\left(r^2 + z_0^2\right)^{3/2}} = \hat{z}\, k_e\, z_0 \frac{2\,Q}{R^2} \int_0^R \frac{r\, d r}{\left(r^2 + z_0^2\right)^{3/2}}$$

This is an integral of the form: $\int u^{-3/2}\, d u$

$$u = r^2 + z_0^2 \implies d u = 2\, r\, d r \quad \text{and} \int u^{-3/2}\, d u = \frac{1}{-1/2} u^{-1/2} + C = -2\, u^{-1/2} + C$$

We can then evaluate the integral

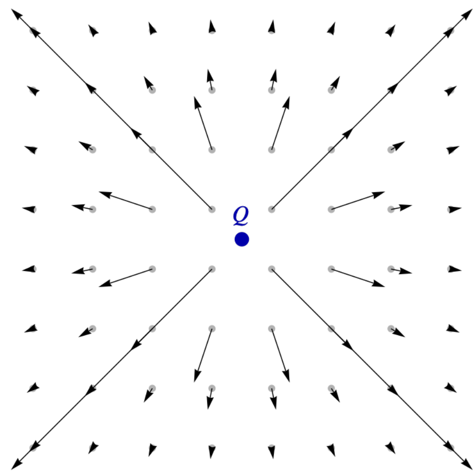
$$\vec{E} = -\hat{z}\, k_e\, z_0 \frac{2\,Q}{R^2} \left.\frac{1}{\sqrt{r^2 + z_0^2}}\right|_0^R = -\hat{z}\, k_e\, z_0 \frac{2\,Q}{R^2} \left(\frac{1}{\sqrt{R^2 + z_0^2}} - \frac{1}{z_0}\right)$$

and then simplify it.

$$\vec{E} = \hat{z}\, k_e \frac{2\,Q}{R^2} \left(1 - \frac{z_0}{\sqrt{R^2 + z_0^2}}\right)$$

Field Diagrams

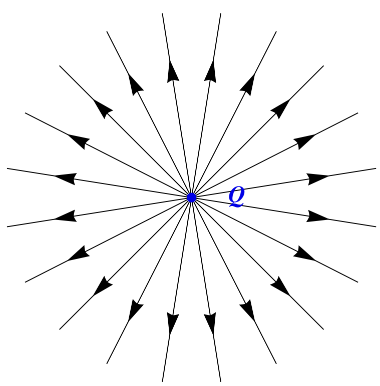
The electric field is a vector field. This means that at each position in space there is a vector. The usual way of representing a general vector field is to draw a grid with a vector at each point in the grid.

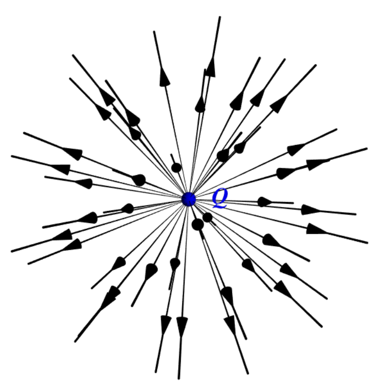


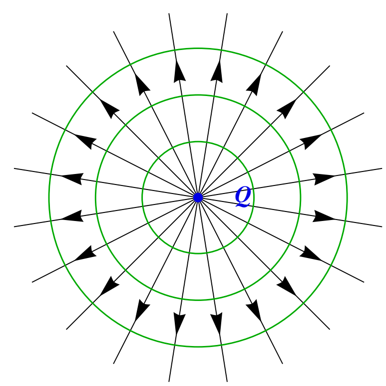
Point Charge Field - Vectors on a Grid

Such diagrams, however, can get quite complicated in the case of electric fields; one gets a tangled mess of overlapping arrows. The convention that is used is to draw continuous curves showing only the direction of the field at some position. The field lines begin at positive charges and end at negative charges. We will see that the density of lines is a measure of the strength (magnitude) of the field.

Consider the field of a positive point charge; the field lines point radially away from the charge.







Left: Point Charge Field in 2D - continuous curves   Center: 3D version of same   Right: Concentric sphere around the charge

Different concentric spheres with the charge at the center will have the same number of lines passing through them. The area of a sphere is $A = 4\pi r^2$, so if we take the number of lines per area we get:

$$\frac{\text{\# of lines}}{\text{Area}} \propto \frac{1}{r^2}$$

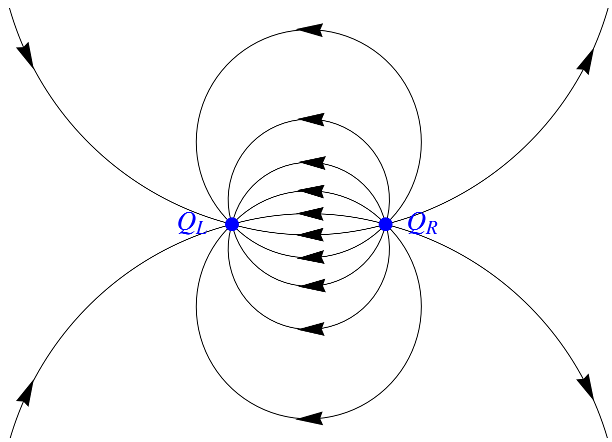
where "$\propto$ " is the proportional symbol. The electric field magnitude is also proportional to $\frac{1}{r^2}$. It follows that

$$\text{mag of field} = E \propto \frac{\text{\# of lines}}{\text{Area}}.$$

This gives the graphical interpretation of the magnitude of the field. When the lines are close together the field is strong and when they are far apart it is weak.

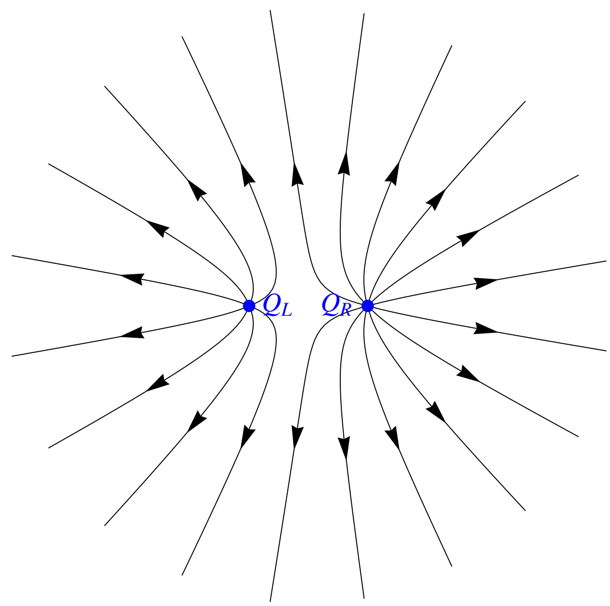
Electric Dipole

$Q_L = -Q$ is the charge on the left and $Q_R = +Q$ is the charge on the right.



Two Positive Charges

$Q_L > 0$ is the charge on the left and $Q_R > 0$ is the charge on the right. Also $Q_L < Q_R$.



One Positive, One Negative, Net Charge Negative

$Q_L < 0$ is the charge on the left and $Q_R > 0$ is the charge on the right. Also $Q_L + Q_R < 0$.