Chapter C

Potential and Potential Energy

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C.1 - Definition of Electric Potential

Review of Potential Energy

Recall from the first semester course that a force is defined to be *conservative* when its work, $W = \int \vec{F} \cdot d\vec{r}$, is zero for any closed path.

$$0 = \oint \vec{F} \cdot d\vec{r} \quad \left(\vec{F} \text{ is conservative.}\right)$$

The circle in the above integral symbol denotes the path is closed. An equivalent definition of a conservative force is that work is independent of path, meaning that the work is the same for all paths between the same two endpoints.

In mechanics, energy is introduced through the work-energy theorem: $W_{\text{net}} = \Delta K$, where $K = \frac{1}{2} m\, v^2$ is defined as the kinetic energy. When a force is conservative we can write its work in terms of the endpoints of the path only. We define potential energy for a conservative force by

$$\Delta U = -W = -\int \vec{F} \cdot d\vec{r}\,.$$

The reason for the negative sign is to get a term to add to the kinetic energy in the work-energy theorem, moving the term from the work side to the energy side of the equation gives the minus sign.

Examples of conservative forces from the first semester course are gravity, both for a uniform field and for a varying field, and the elastic force of a spring. We will now see that the electrostatic force is conservative; this will allow the definition of electric potential energy.

Basic Definitions

First some terminology: We will define electric potential V as the potential energy per test charge, $V = U/q_0$. The similarity of the letters U and V combined with the similar names, potential versus potential energy, creates a confusion between these related but distinct notions. Voltage is potential difference ΔV; that is, differences in potential. When the sign of ΔV is ignored, we will sometimes use the standard, although somewhat ambiguous, notation where voltage is written as just V, instead of the more precise $|\Delta V|$.

The diagram shows the underlying relationships between the force, field, potential energy and potential. The potential V, at the lower right, is what we are introducing here.

$$\begin{array}{ccccc}
\boldsymbol{\vec{F}} & \longleftarrow & \vec{E} = \frac{\vec{F}}{q_0} \text{ and } \vec{F} = Q\,\vec{E} & \longrightarrow & \boldsymbol{\vec{E}} \\
\uparrow & & & & \uparrow \\
\Delta U = -\int \vec{F} \cdot d\vec{r} & & & & \Delta V = -\int \vec{E} \cdot d\vec{r} \\
\downarrow & & & & \downarrow \\
\boldsymbol{U} & \longleftarrow & V = \frac{U}{q_0} \text{ and } \Delta U = Q\,\Delta V & \longrightarrow & \boldsymbol{V}
\end{array}$$

Recall that we define the electric field by considering the force on a test charge and by dividing the charge into the force, $\vec{E} = \vec{F}/q_0$. The potential is defined from the potential energy similarly. The potential is defined as the potential energy per test charge.

$$V = \frac{U}{q_0}$$

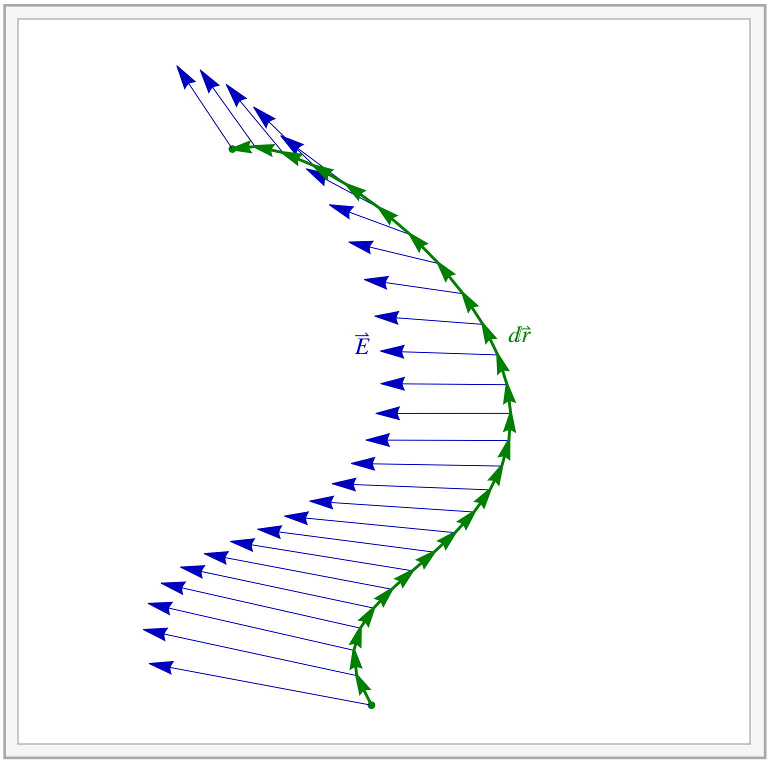
The zero of potential energy is arbitrary and so is the zero of potential. When a charge Q is moved across a potential difference ΔV we get a change in potential energy given by

$$\Delta U = Q\,\Delta V.$$

If the above expressions are taken as the definition of V then dividing both sides of the ΔU formula gives an expression for the voltage (potential difference) as an integral of the electric field over a contour

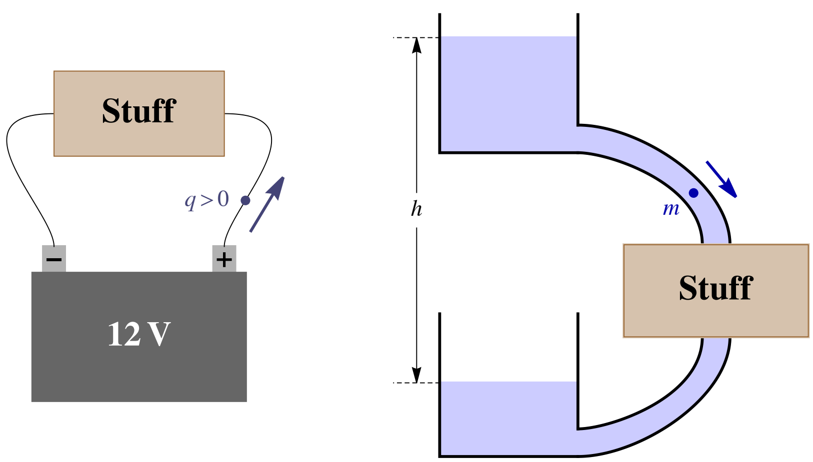
$$\Delta V = -\int \vec{E} \cdot d\vec{r}.$$

This is the fundamental expression showing how potential can be found from electric fields.



Interactive Figure

A Physical Example and Gravitational Analog



Charge flowing from the positive terminal to the negative terminal of a car battery (left) and its gravitational analog with a mass *m* of water flowing from an elevated tank to a lower one.

Consider the 12 V battery in a car connected across some load labeled as "Stuff": the starter, headlights, car stereo, etc. Charge flows from the positive terminals to negative terminals, where when we discuss the flow of charge we mean the flow of positive charge. When a positive charge q flows from the positive terminal to the negative terminal the potential difference (voltage) is negative

$$\Delta V = V_f - V_i = V_- - V_+ = -12\ \text{V}.$$

It follows that since the charge q is positive, the change in the potential energy ΔU is negative.

$$q > 0 \implies \Delta U = q\,\Delta V = q \times (-12\text{ V}) < 0$$

Here U is the electrical energy stored in the battery. This is decreasing and that is providing energy to the load; it turns the starter, powers the headlights or powers the car stereo.

There is a very good gravitational analogy for this. Consider water in an elevated tank, where the energy of flowing water is utilized before flowing into a lower tank. This could be water flowing over a paddle wheel before spilling into a lower tank or some more efficient means of harnessing this energy. If h is the (positive) height difference between the water levels of the two tanks, then when a mass m flows from the top to the bottom tank, where $\Delta y = -h$, there is a drop in potential energy.

$$\Delta U = m\,g\,\Delta y = -m\,g\,h < 0$$

A (theoretical) maximum of mgh of energy can then be harnessed for any application.

In both cases we are losing potential energy; the battery and its gravitational analog are being depleted. To replenish the energy of the gravitational setup we may mechanically pump water from the bottom to the top, where the external work of the pump adds to the potential energy. Moving a mass m of water from the bottom to the top gives

$$W_{\text{external}} = \Delta U = m\,g\,\Delta y = +m\,g\,h > 0.$$

The alternator of a car is a type of a generator; it converts external work provided by the car's engine into electrical energy by pushing (positive) charge to higher potential.

$$W_{\text{external}} = \Delta U = q\,\Delta V = q \times (+12\text{ V}) > 0$$

Units: The SI unit for Electric Potential and Voltage is: volt = V = J/C

Example C.1 - Accelerating Charge

(a) What is the speed of an electron after accelerating from rest across a 12 V potential difference?

Solution

The potential difference is the voltage. The relevant constants are the elementary charge and the electron mass.

$$V = |\Delta V| = 12\text{ V}$$
$$e = 1.602 \times 10^{-19}\text{ C}$$
$$m = m_{\text{electron}} = 9.11 \times 10^{-31}\text{ kg}$$

We use conservation of energy to find the speed. In part (b) we will pay attention to signs but to simplify this problem, let us ignore signs by taking absolute values.

$$K_i + U_i = K_f + U_f \implies 0 = \Delta K + \Delta U \implies |\Delta K| = |\Delta U|$$

Since $K = \frac{1}{2}\,m\,v^2$ and $K_i = 0$ we have $|\Delta K| = \frac{1}{2}\,m\,v_f^2$. When a charge $q = -e$ is moved across a ΔV we have $\Delta U = q\,\Delta V = -e\,\Delta\text{V}$. Taking absolute values gives $|\Delta U| = e\,|\Delta V| = e\,V$. This gives our result.

$$|\Delta K| = |\Delta U| \implies \frac{1}{2}\,m\,v_f^2 = e\,V \implies v_f = \sqrt{\frac{2\,e\,V}{m}} = 2.05 \times 10^6\ \frac{\text{m}}{\text{s}}$$

(b) What is the sign of the potential difference ΔV?

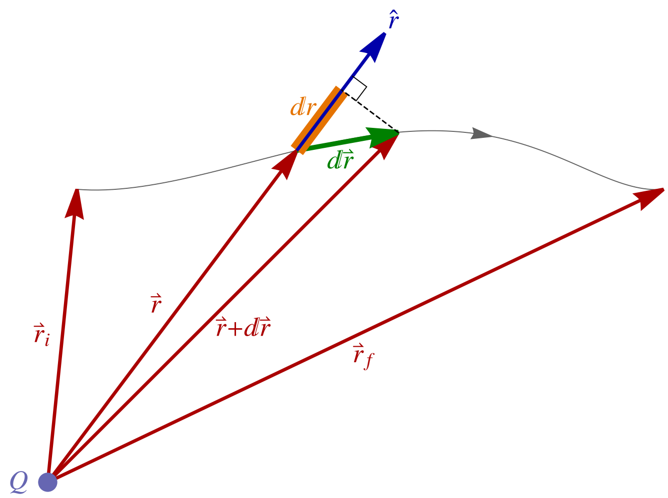
Solution

$\Delta K > 0$ os it follows that $\Delta U < 0$. Since $\Delta U = q\,\Delta V = -e\,\Delta\text{V}$ then $\Delta V > 0$.

C.2 - Potential for Charge Distributions

We have found expressions for electric fields due to charge distributions, first considering the point charge and then the cases of discrete and continuous distributions. We will now do the same for potentials. Since potential is a scalar we will see that the potential calculations are simpler because they lack the vector complications in the field calculations.

Point Charge



Moving along a path in the field of a point charge Q.

We know that the field of a point charge is $\vec{E} = k_e\, Q\, \frac{\hat{r}}{r^2}$ and the general expression for the potential difference from a field. Combining the two gives an integral for the potential difference when moving from $\vec{r}_i$ to $\vec{r}_f$ along the path shown above.

$$\Delta V = -\int \vec{E}\cdot d\vec{r} = -k_e\, Q \int_{\vec{r}_i}^{\vec{r}_f} \frac{\hat{r}}{r^2}\cdot d\vec{r}$$

Multiplying a vector by a unit vector gives the vector component in the unit vector's direction, so $\hat{r}\cdot d\vec{r}$ is just the radial component of $d\vec{r}$ which is just dr. This is the change in the radial distance from the charge taken to be at the origin $dr = \left\|\vec{r} + d\vec{r}\right\| - \|\vec{r}\|$.

$$\hat{r}\cdot d\vec{r} = dr.$$

The limits of the integral then only depend only on the radial distances of the endpoints.

$$\Delta V = -k_e\, Q \int_{r_i}^{r_f} \frac{dr}{r^2} = k_e\, Q\left(\frac{1}{r_f} - \frac{1}{r_i}\right),$$

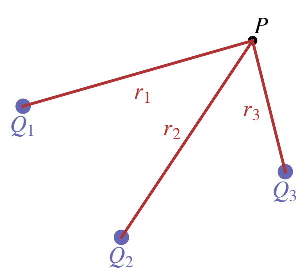
We are looking for some function $V(r)$, describing the potential as a function of position. Since it must satisfy $\Delta V = V(r_f) - V(r_i)$, it is unique up to an arbitrary constant. The simplest choice is

$$V(r) = k_e\, \frac{Q}{r}$$

where the arbitrary constant is chosen to make the potential zero at infinity

$$V(\infty) = 0 \quad \text{or more precisely} \quad \lim_{r\to\infty} V(r) = 0.$$

Discrete Distribution

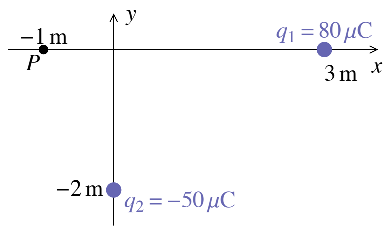


To find the electric potential at a point P due to charges Q_i we will take r_i to be the distance from Q_i to P. The potential is

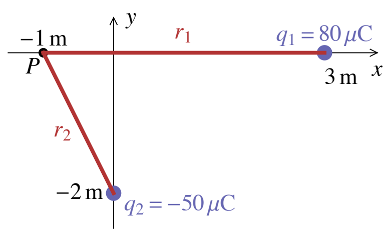
$$V = k_e \sum_i \frac{Q_i}{r_i}$$

Example C.2 - Two Point Charges (Continued)

An $80\mu\mathrm{C}$ is at $(3\,\mathrm{m},\ 0)$ and a $-50\mu\mathrm{C}$ is at $(0,\ -2\,\mathrm{m})$. What is the electric potential at $(-1\,\mathrm{m},\ 0)$?



Solution



P is the point where we evaluate the field, so here: $P = (-1\text{ m},\ 0)$. The distance from $Q_1 = 80\,\mu$C to P is $r_1 = 4$ m and from $Q_2 = -50\,\mu$C to P we have $r_2 = \sqrt{1^2 + 2^2}$ m. where we choose the vector form the other charge to the charge we are calculating the force on. The equivalent forms of the inverse square law for $\vec{E}_i$ are

$$V = k_e \frac{Q_1}{r_1} + k_e \frac{Q_2}{r_2}$$

We can easily find the potential for each charge

$$V_1 = k_e \frac{Q_1}{r_1} = 9.0 \times 10^9 \frac{\text{N}\cdot\text{m}^2}{\text{C}^2} \cdot \frac{80 \times 10^{-6}\text{ C}}{4\text{ m}} = 180\,000\text{ V}$$

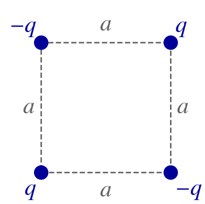
$$V_2 = k_e \frac{Q_2}{r_2} = 9.0 \times 10^9 \frac{\text{N}\cdot\text{m}^2}{\text{C}^2} \cdot \frac{-50 \times 10^{-6}\text{ C}}{\sqrt{1^2 + 2^2}\text{ m}} = -201\,250\text{ V}$$

Summing these gives the total potential.

$$V = V_1 + V_2 = -21\,200\text{ V}$$

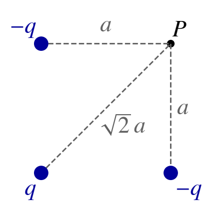
Example C.3 - Work to Move a Charge - Electric Potential Method

Four charges with the same magnitude charges q, $-q$, q and $-q$ are arranged around the corners of a square, as shown. What work is needed to move one of the positive charges to infinity?



Solution

Recall that the potential energy is defined by the formula $\Delta U = -W$ where W is the work done by the force; in the electric case it is the work done by the electric field. The work here is different; we are looking for the done by some agent moving the charge against the field and not the work done by the field. A gravitational analog is lifting something: If asked how must work does it take to left something it is the work done against gravity $W = \Delta U > 0$ and not the work done by gravity. It follows that here we want.



Here we are moving a charge q from P to infinity.

$$W = \Delta U = q\,\Delta V$$

where ΔV is

$$\Delta V = V_f - V_i = V_\infty - V_P = 0 - V_P$$

To find the potential at P we use the expression for a discrete distribution.

$$V = k_e \sum_i \frac{Q_i}{r_i} = k_e \frac{-q}{a} + k_e \frac{q}{\sqrt{2}\,a} + k_e \frac{-q}{a} = -k_e \frac{q}{a}\left(2 - \frac{1}{\sqrt{2}}\right) = -1.293\ k_e \frac{q}{a}$$

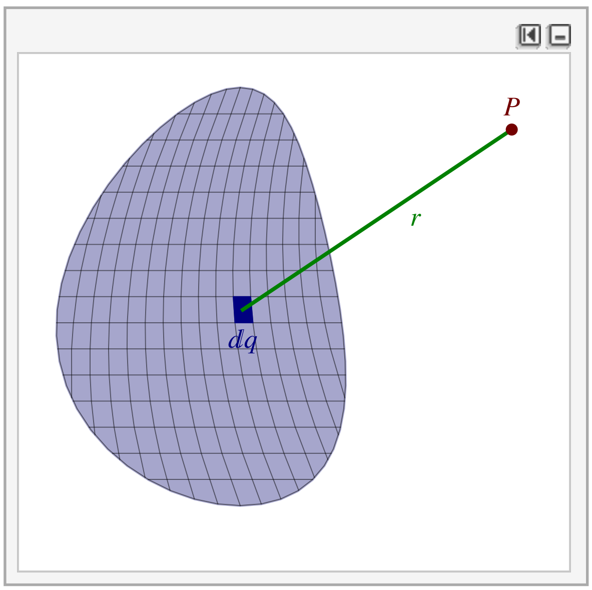
Put the pieces together to solve for the work.

$$W = \Delta U = q\,\Delta V = -q\,V_P = k_e \frac{q^2}{a}\left(2 - \frac{1}{\sqrt{2}}\right) = 1.293\ k_e \frac{q^2}{a}$$

Continuous Distribution

As was done in the case of the electric field we will take r to be the distance from dq to P. The potential is

$$V = k_e \int \frac{dq}{r}.$$

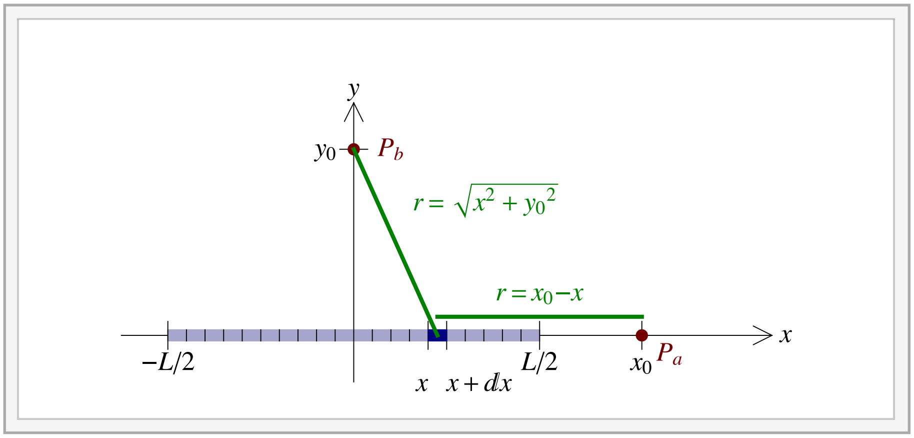


Interactive Figure

Using Integrals to Calculate Electric Potentials

1. Mathematically break up the distribution into an infinite number of infinitesimal pieces. This requires choosing an integration variable and its limits.
2. Find dq, the infinitesimal charge of each infinitesimal piece. You must write the dq in terms of the differential d of the integration variable.
3. Write an expression for r, the distance from dq to the point P where the value of the field is wanted. Note that r is the magnitude of the vector $\vec{r}$ needed to find the electric field.
4. Put the pieces together to form a properly-defined definite integral and evaluate the integral.

Example C.4 - Uniform Line of Charge



Interactive Figure

Consider a line of uniform charge Q along the x-axis from $-L/2$ to $L/2$.

(a) What is the electric potential along the x-axis at x_0, where $x_0 > L/2$?

Solution

Choose x as the integration variable with limits: $-\frac{L}{2} \leq x \leq \frac{L}{2}$.
The charge between x and $x + dx$ is: $dq = \frac{Q}{L}\, dx$.
The distance from dq (at x) to x_0 is: $r = (x_0 - x)$.

$$V = k_e \int \frac{dq}{r} = k_e \int_{-L/2}^{L/2} \frac{1}{x_0 - x} \frac{Q}{L}\, dx = k_e \frac{Q}{L} \int_{-L/2}^{L/2} \frac{dx}{x_0 - x}$$

Use the substitution $u = x_0 - x$ with $du = -dx$. Absorb the minus sign into the expression by swapping the limit and use $\ln a - \ln b = \ln(a/b)$.

$$V = -k_e \frac{Q}{L} \ln(x_0 - x)\Big|_{-L/2}^{L/2} = k_e \frac{Q}{L} \ln(x_0 - x)\Big|_{L/2}^{-L/2} = k_e \frac{Q}{L} (\ln(x_0 + L/2) - \ln(x_0 - L/2)) = k_e \frac{Q}{L} \ln\left(\frac{x_0 + L/2}{x_0 - L/2}\right)$$

(b) What is the electric potential along the positive y-axis at y_0?

Solution

It is the same charge distribution so the first two steps are the same: $-\frac{L}{2} \leq x \leq \frac{L}{2}$ and $dq = \frac{Q}{L}\, dx$.
The distance from dq (at x) to y_0 is: $r = \sqrt{x^2 + y_0^2}$.

$$V = k_e \int \frac{dq}{r} = k_e \frac{Q}{L} \int_{-L/2}^{L/2} \frac{1}{\sqrt{x^2 + y_0^2}}\, dx$$

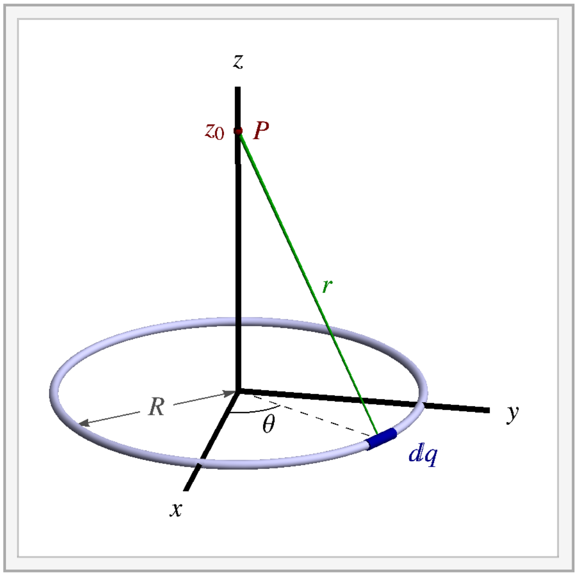
Using $\int du \big/ \sqrt{u^2 + a^2} = \ln\left(u + \sqrt{u^2 + a^2}\right) + C$ we get

$$V = k_e \frac{Q}{L} \ln\left(x + \sqrt{x^2 + y_0^2}\right)\Big|_{-L/2}^{L/2} = k_e \frac{Q}{L} \left(\ln\left(L/2 + \sqrt{(L/2)^2 + y_0^2}\right) - \ln\left(L/2 + \sqrt{(L/2)^2 + y_0^2}\right)\right)$$

Using $\ln a - \ln b = \ln(a/b)$ we get.

$$V = k_e \frac{Q}{L} \ln\left(\frac{L/2 + \sqrt{(L/2)^2 + y_0^2}}{-L/2 + \sqrt{(L/2)^2 + y_0^2}}\right)$$

Example C.5 - Uniform Ring of Charge



Interactive Figure

Consider a thin ring of radius R with uniform charge Q in the xy-plane with the center at the origin. What is the electric potential along the positive y-axis at y_0?

Solution

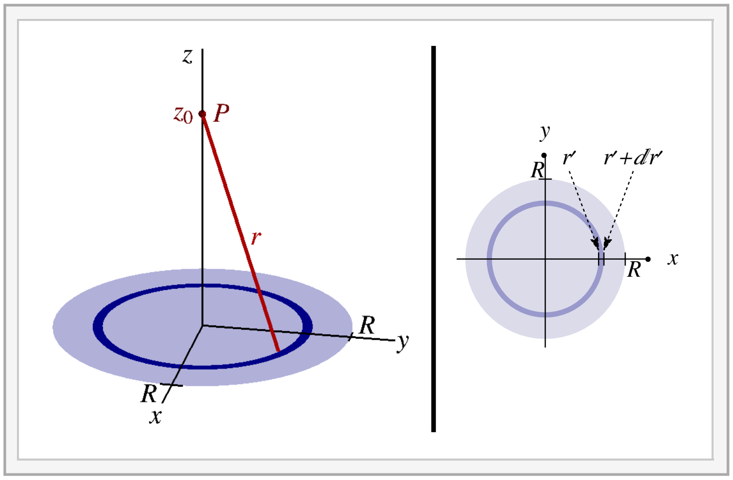
The distance r can be found using the Pythagorean theorem: $r = \sqrt{R^2 + z_0^2}$. Since this is constant as we integrate around the ring it can be pulled out of the integral.

$$V = k_e \int \frac{dq}{r} = k_e \int \frac{dq}{\sqrt{R^2 + z_0^2}} = k_e \frac{1}{\sqrt{R^2 + z_0^2}} \int dq = k_e \frac{Q}{\sqrt{R^2 + z_0^2}}$$

Example C.6 - Disk of Uniform Charge

Consider a thin disk of radius R with uniform charge Q in the xy-plane with the center at the origin. What is the electric field along the positive z-axis at z_0?

Solution



Interactive Figure

As we did in this example in Chapter A we need to break up the disk into concentric rings or infinitesimal thickness. In that example we integrated over r. Here there is a notational issue since the r in the integral, which is shown in red in the figure, is not the same as the integration variable.

$$V = k_e \int \frac{dq}{r}$$

We will choose our integration variable to be r' to distinguish it from the other r, remember that the integration variable is a dummy variable and we can call it anything. This r' varies from 0 to R: $0 \le r' \le R$. We can then write the r in terms of the r' using the Pythagorean theorem.

$$r = \sqrt{r'^2 + z_0^2}$$

The charge between r' and $r' + dr'$ is dq. With a uniform charge, the fraction of the charge is the same as the fraction of the area $dq/Q = dA/A_{\text{tot}}$. The (infinitesimal) area is the length of the curve, the circumference $2\pi r$ times dr, the thickness: $dA = 2\pi r' \, dr'$. The total area is $A_{\text{tot}} = \pi R^2$. This gives the same dq as in the chapter A example

$$dq = \frac{Q}{A_{\text{tot}}} dA = \frac{Q}{\pi R^2} 2\pi r' \, dr' = \frac{2Q}{R^2} r' \, dr'$$

Putting the pieces together we get a well-defined definite integral.

$$V = k_e \int \frac{dq}{r} = k_e \frac{2Q}{R^2} \int_0^R \frac{r' \, dr'}{\sqrt{r'^2 + z_0^2}}$$

This is an integral of the form: $\int u^{-1/2} \, du$

$$u = r'^2 + z_0^2 \implies du = 2r' \, dr' \quad \text{and} \quad \int u^{-1/2} \, du = \frac{1}{1/2} u^{1/2} + C = 2u^{1/2} + C$$

We can then evaluate the integral

$$V = k_e \frac{2Q}{R^2} \sqrt{r'^2 + z_0^2} \Big|_0^R = k_e \frac{2Q}{R^2} \left(\sqrt{R^2 + z_0^2} - z_0 \right)$$

C.3 - Potentials and Electric Fields

Potential from Electric Field

We have seen that the fundamental expression that gives the potential difference from the electric field is

$$\Delta V = -\int \vec{E} \cdot d\vec{r}.$$

This is the change in the potential between the endpoints of the contour of the integral. A simple special case of this is a uniform field. If $\vec{E}$ is uniform (meaning spatially constant) then it comes out of the integral.

$$\Delta V = -\int \vec{E} \cdot d\vec{r} = -\vec{E} \cdot \int d\vec{r} = -\vec{E} \cdot \Delta\vec{r},$$

where $\Delta\vec{r}$ is the vector from the starting position to the final position.

Another important special case is the infinitesimal form; if one moves along a small displacement $d\vec{r}$, the infinitesimal change in the potential is given by

$$dV = -\vec{E} \cdot d\vec{r}.$$

Example C.7 - Work and Potential from Field

Given the electric field

$$\vec{E} = \langle -25, 0, 30 \rangle \times 10^3 \text{ V/m}$$

Find the work needed to move an electron from (3 m, −2 m, 0) to the origin.

Solution

The electron's displacement is $\Delta\vec{r} = \langle -3,\, 2,\, 0\rangle$ m. First we must find the change in the potential.

$$\Delta V = -\vec{E}\cdot\Delta\vec{r} = -\langle 25,\, 0,\, -30\rangle\cdot\langle -3,\, 2,\, 0\rangle\times 10^3\ \mathrm{V} = -\big((-25)(-3) + 0\,(2) + (30)\,0\big)\times 10^3\ \mathrm{V} = -75\times 10^3\ \mathrm{V}$$

The work done by the field is $W = -\Delta U$ but we want the work done moving against the field, which is $W = \Delta U$.

$$q = -e \quad \text{where } e = 1.60\times 10^{-19}\ \mathrm{C}$$

We find the change in potential energy using: $\Delta U = q\,\Delta V$.

$$\Delta U = q\,\Delta V = -e\,\Delta V = 1.2\times 10^{-14}\ \mathrm{J}$$

Equipotentials

Equipotentials are surfaces of constant potential.

Potential is a scalar field, meaning that at each position in space there is a scalar function. A way of representing scalar fields in a two dimensional graph is to draw contour of constant values. On weather maps, the scalars of temperature and pressure are represented by isotherms and isobars, which are contours of constant temperature and pressure. In three dimensions we have surfaces of constant values instead of contours. A surface of constant potential is called an equipotential and these will appear as contours in two dimensional diagrams.

Field lines are perpendicular to equipotentials.

To understand how equipotentials are related to electric field lines consider the infinitesimal form of the potential expression $dV = -\vec{E}\cdot d\vec{r}$. Take $d\vec{r}$ to be some infinitesimal displacement along the equipotential. Along the equipotential $dV = 0$, this is because the potential is constant along an equipotential. We then get $0 = \vec{E}\cdot d\vec{r}$ which implies that the field is perpendicular to the $d\vec{r}$. Since this is true for any direction along the equipotential it follows that the field is always perpendicular to the equipotential.

Electric field lines point toward lower potential.

Begin with the expression $dV = -\vec{E}\cdot d\vec{r}$ and take $d\vec{r}$ to be some infinitesimal displacement in the direction of an electric field line. Since the dot product of two vectors in the same direction is positive $\vec{E}\cdot d\vec{r} > 0$ which implies that $dV = -\vec{E}\cdot d\vec{r} < 0$. This tells us that field lines always point toward lower electric potential.

Electric Field from Potential

To go from the potential to the electric field, begin with the infinitesimal expression $dV = -\vec{E}\cdot d\vec{r}$. To get E_x consider some infinitesimal displacement in the x-direction $d\vec{r} = \hat{x}\,dx$.

$$dV = -\vec{E}\cdot d\vec{r} \quad \text{and} \quad d\vec{r} = \hat{x}\,dx \implies dV = -E_x\,dx$$

Solving for E_x gives $E_x = -\frac{dV}{dx}$. This expression is not strictly correct, because the potential may not only be a function of just x. By moving in the x-direction we are really taking the derivative with respect to x keeping the other variables (y and z) constant. This is what is meant by partial derivatives and the expressions for the components of the field are

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}.$$

The radial part of the field can also be found in a similar manner and for problems with spherical symmetry, the above partial derivative becomes the ordinary derivative

$$E_r = -\frac{\partial V}{\partial r} \quad \text{and} \quad E_r = -\frac{dV}{dr} \ \text{(for spherical symmetry)}$$

It should be mentioned as an *aside* that the above relations can be written as a single vector expression using the gradient operator of vector calculus.

$$\vec{E} = -\vec{\nabla} V$$

Example C.8 - Partial Derivatives and Electric Fields from Potential

(a) Given the function of two variables

$$f(x,\, y) = a\, x^3\, y - b\, y^2$$

Evaluate the partial derivatives: $\partial f / \partial x$ and $\partial f / \partial y$.

Solution

To evaluate the partial derivative with respect to one variable, take the ordinary derivative with respect to that variable treating the other variables as constant.

$$\frac{\partial f}{\partial x} = 3\, a\, x^2\, y \quad \text{and} \quad \frac{\partial f}{\partial y} = a\, x^3 - 2\, b\, y$$

(b) Now use the same function as the electric potential and find the electric field vector.

$$V(x,\, y) = a\, x^3\, y - b\, y^2$$

Solution

To evaluate the partial derivative with respect to one variable, take the ordinary derivative with respect to that variable treating the other variables as constant.

$$E_x = -\frac{\partial V}{\partial x} = -3\, a\, x^2\, y \quad \text{and} \quad E_y = -\frac{\partial V}{\partial y} = -a\, x^3 + 2\, b\, y$$

Write this as a vector.

$$\vec{E} = \langle E_x,\, E_y \rangle = \left\langle -3\, a\, x^2\, y\,,\, -a\, x^3 + 2\, b\, y \right\rangle$$

Conductors in Electrostatics - II

Potential is constant throughout a conductor in electrostatics.

The condition that the electric field is zero implies that the potential is constant. Consider some contour entirely inside a conductor. Integrating the field to get the potential difference gives

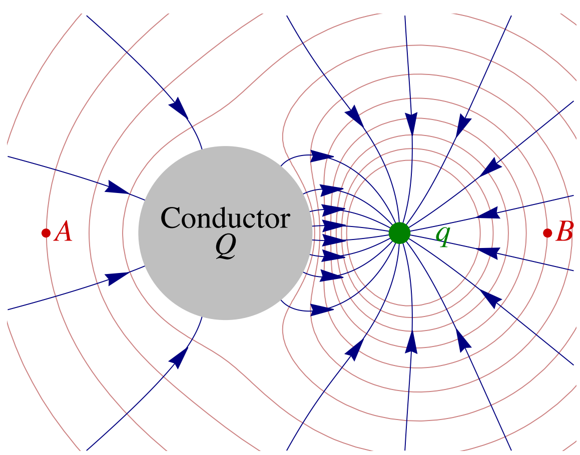
$$\Delta V = -\int \vec{E} \cdot d\vec{r} = 0.$$

Note that this argument does not apply to disconnected conductors. If there are several disconnected conductors each one will have its own potential.

It follows that the surface of a conductor is always an equipotential. The condition that the field is perpendicular to a conductor may now be seen as a consequence of fields being perpendicular to equipotentials.

Example C.9 - Field Lines, Equipotentials and Conductors

The figure shows a point charge q and a conductor with a net charge Q. The field lines are dark blue and the equipotentials are dark red. Two points, A and B are also labeled.



(a) What are the signs of q, Q and $Q + q$?

Solution

For q the field lines point toward it, so it is negative. For the conductor's charge Q, there are more field lines leaving than entering so $Q > 0$. For the total charge the field lines come in from infinity so the total charge is negative $Q + q < 0$.

(b) Where is the electric field larger, at point A or point B?

Solution

At point B the field lines are closer together to the field is stronger there, $E_A < E_B$. (This refers to the last chapter's material.)

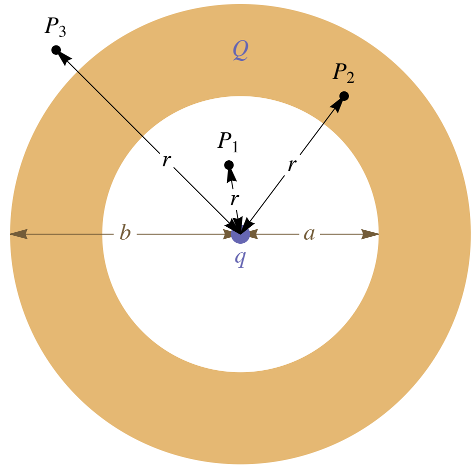
(c) Where is the electric potential larger, at point A or point B?

Solution

This is more subtle. Both points A and B are on equipotentials. Follow the equipotential of point A around to the other side of the conductor and point B. Since electric field lines pass from the B equipotential to the A equipotential and since field lines point toward lower potential we have $V_A > V_B$.

Example C.10 - A Point Charge Inside a Hollow Conducting Sphere (Continued)

A hollow spherical conductor has concentric spherical surfaces with an inside radius of a and outside radius b. The conductor is given a net charge Q. At the conductor's center is a point charge q.



Given the radial components of the electric fields found in chapter B, what is the electric potential as a function of r? Give answers for $r < a$, $a < r < b$ and $r > b$.

Solution

First we recap the first part of this problem in Chapter B. Here we are only interested in E_r, the radial component of $\vec{E}$.

$$\textbf{For } r < a: \quad E_r = k_e \frac{q}{r^2}$$

$$\textbf{For } a < r < b: \quad E_r = 0$$

$$\textbf{For } r > b: \quad E_r = k_e \frac{q+Q}{r^2}$$

To find the potential from the electric field, since we have spherical symmetry,

$$E_r = -\frac{dV}{dr}.$$

To find V from E_r we must anti-differentiate. Each time we do so there is an arbitrary constant. We choose these constants to make $V(r)$ continuous and go to zero at infinity. To simplify the evaluation of the constants we will for from outside in, since we have the fixed reference of $V(\infty) = 0$.

$$\textbf{For } r > b: \quad \frac{dV}{dr} = -E_r = -k_e \frac{q+Q}{r^2} \implies V(r) = k_e \frac{q+Q}{r} + C_1$$

$$V(\infty) = 0 \implies 0 + C_1 = 0 \implies V(r) = k_e \frac{q+Q}{r}$$

At b, and similarly at a, the function must be continuous so the left-handed and right-handed limits must be equal. There is a common shorthand notation for this: $V(b^-) = V(b^+)$.

$$\lim_{r \to b^-} V(r) = \lim_{r \to b^+} V(r) \iff V(b^-) = V(b^+)$$

This means that the different functions for $V(r)$ for $r > b$ and $a < r < b$ must be equal at b.

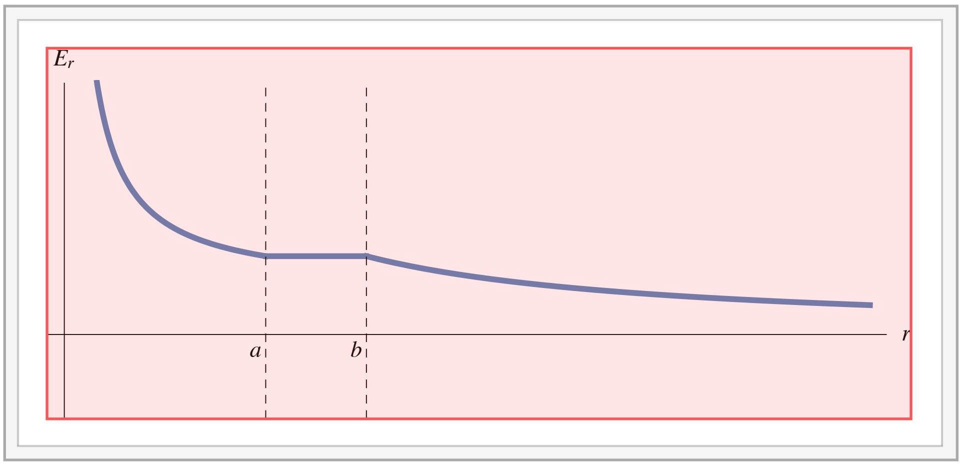
$$\textbf{For } \boldsymbol{a < r < b}\textbf{:} \quad \frac{dV}{dr} = -E_r = 0 \implies V(r) = C_2$$

$$V(b^-) = V(b^+) \implies C_2 = k_e \frac{q+Q}{b} \implies V(r) = k_e \frac{q+Q}{b}$$

The $r < a$ case follows similarly. The constant has a messier form but it is just a constant.

$$\textbf{For } \boldsymbol{r < a}\textbf{:} \quad \frac{dV}{dr} = -E_r = -k_e \frac{q}{r^2} \implies V(r) = k_e \frac{q}{r} + C_3$$

$$V(a^-) = V(a^+) \implies k_e \frac{q}{a} + C_3 = k_e \frac{q+Q}{b} \implies V(r) = k_e \frac{q}{r} + k_e \frac{q+Q}{b} - k_e \frac{q}{a}$$



Interactive Figure

C.4 - Potential Energy of Charge Distributions

We have seen that the potential energy difference when a charge Q is moved across a potential difference ΔV is

$$\Delta U = Q \, \Delta V.$$

To find the potential energy of a configuration we will start with zero potential energy when all the charges are far apart. For a two charge configuration begin with Q_1 in place and move charge Q_2 from infinity to its position a distance r from Q_1. The potential due to Q_1 is varying from 0 to $k_e \frac{Q_1}{r}$.

$$U = \Delta U = Q_2 \left(k_e \frac{Q_1}{r} - 0 \right)$$

This gives the expression for two charges separated by a distance r

$$U = k_e \frac{Q_1 Q_2}{r}.$$

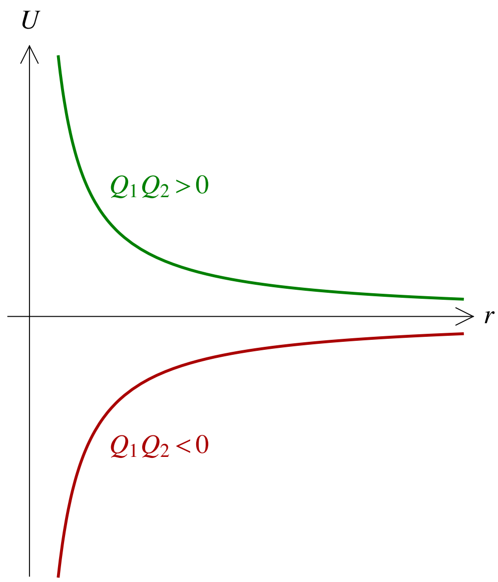


Figure - Potential Energy for Two Point Charges

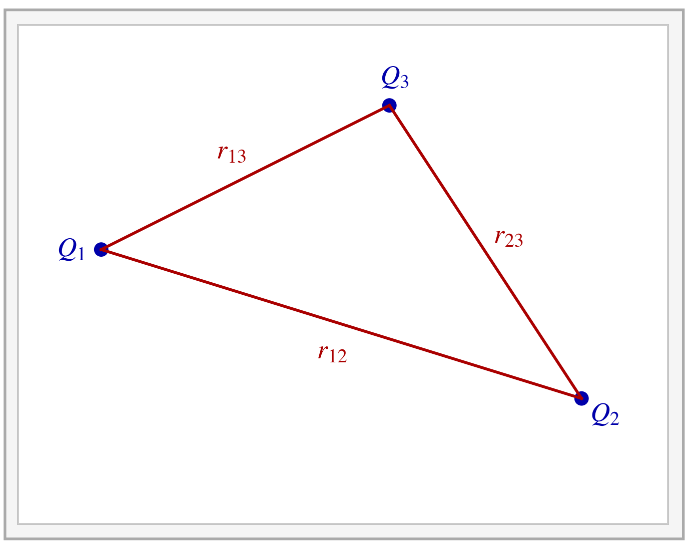
For a three charge configuration we have potential energy between each pair. Taking the distance between charges i and j to be r_{ij} gives the potential energy as

$$U = k_e \frac{Q_1 Q_2}{r_{12}} + k_e \frac{Q_1 Q_3}{r_{13}} + k_e \frac{Q_2 Q_3}{r_{23}}.$$

The general expression is

$$U = k_e \sum_{i<j} \frac{Q_i Q_j}{r_{ij}},$$

where the sum is over all pairs of charges. Note that keeping $i < j$ avoids double counting and excludes $i = j$, which would correspond to the energy between a particle and itself.



Interactive Figure

Example C.11 - Energy and Electrons Released from Rest

Two electrons are initially at rest a distance of 5.0×10^{-10} m. If they are released, then what is their speed when separated by a large distance, i.e. at infinity? (Both electrons will have the same speed.)

Solution

Call r_i the initial distance. The relevant constants are the elementary charge and the electron mass.

$$r_i = 5.0\times 10^{-10}\text{ m}$$
$$e = 1.602\times 10^{-19}\text{ C}$$
$$m = m_{\text{electron}} = 9.11\times 10^{-31}\text{ kg}$$

We use conservation of energy to find the speeds. Since both electrons will have the same speed we have

$$K = 2\times \frac{1}{2} m\, v^2$$

The potential energy is the potential between two point charges.

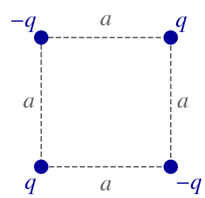
$$U = k_e \frac{q_1\, q_2}{r} = k_e \frac{(-e)\,(-e)}{r} = k_e \frac{e^2}{r}$$

Since the charges are initially at rest $K_i = 0$ and since they are a large distance apart in the end, $U_f = 0$

$$K_i + U_i = K_f + U_f \implies 0 + k_e \frac{e^2}{r_i} = 2\times \frac{1}{2} m\, v_f^2 + 0 \implies v_f = \sqrt{\frac{k_e\, e^2}{m\, r_i}} = 712\,000\ \frac{\text{m}}{\text{s}}$$

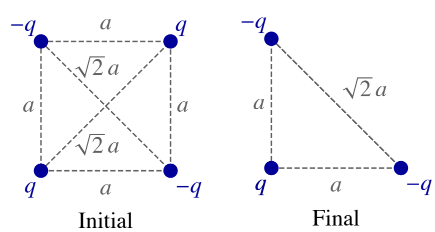
Example C.12 - Work to Move a Charge - Potential Energy Method

Four charges with the same magnitude charges q, -q, q and -q are arranged around the corners of a square, as shown. What work is needed to move one of the positive charges to infinity? This is a repeat of the previous example. Solve it now using potential energy of a configuration.



Solution

We may directly solve this using $W = \Delta U = U_f - U_i$. The initial case has all four charges in place and the final case has only three.



$$U = k_e \sum_{i<j} \frac{Q_i\, Q_j}{r_{ij}},$$

For U_i there are six pairs of the four charges, four sides with distance a and opposite charges and two diagonals with distance $\sqrt{2}\, a$ and like charges:

$$U_i = 4\times k_e \frac{-q^2}{a} + 2\times k_e \frac{q^2}{\sqrt{2}\, a} = -\left(4 - \sqrt{2}\right) k_e \frac{q^2}{a} = -2.5858\, k_e \frac{q^2}{a}$$

The U_f sum consists of three pairs of the remaining three charges, two sides with distance a and opposite charges and one diagonal with distance $\sqrt{2}\, a$ and like charges:

$$U_f = 2\times k_e \frac{-q^2}{a} + k_e \frac{q^2}{\sqrt{2}\, a} = -\left(2 - \frac{1}{\sqrt{2}}\right) \frac{q^2}{a} = -\,1.2929\, \frac{q^2}{a}$$

$$W = \Delta U = U_f - U_i = \left(2 - \frac{1}{\sqrt{2}}\right) \frac{q^2}{a} = 1.2929\, \frac{q^2}{a}$$