Chapter E

Current, Resistance and DC Circuits

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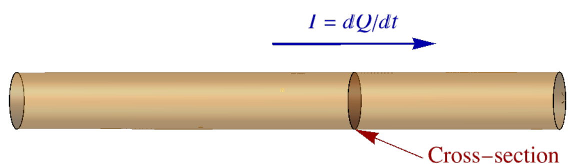
E.1 - Current and Current Density

Basic Definitions



dQ is the charge passing the cross-section in time dt.

If dQ is the charge that passes through some surface, usually a cross-section of a wire, in time dt then the current I is defined by

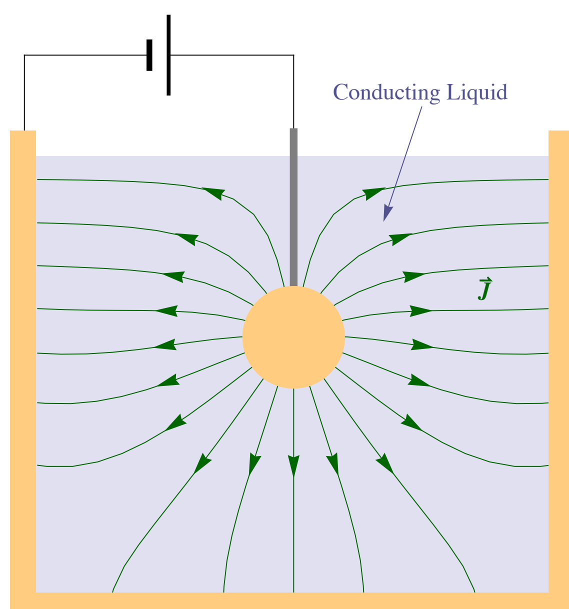
$$I = \frac{dQ}{dt}$$

> **Units:** The SI unit for current is: ampere = A = C/s

The current is related to the surface integral of a vector field, the current density $\vec{J}$, by

$$I = \int \vec{J} \cdot d\vec{A}$$

Note that is is analogous to how the electric field and electric flux are related: $\Phi = \int \vec{E} \cdot d\vec{A}$; $\vec{E}$ and $\vec{J}$ are vector fields and Φ and I are properties of a surface. When the surface is the cross-section of a wire, the above expression becomes simply $I = J A$.



An electrode in a conducting liquid inside a conducting cup. $\vec{J}$ is the current density.

Drift Velocity

We can relate the current and current density to the flow of charge carriers. Let q be the charge of the charge carriers.

$$q = \text{charge of charge carriers}$$

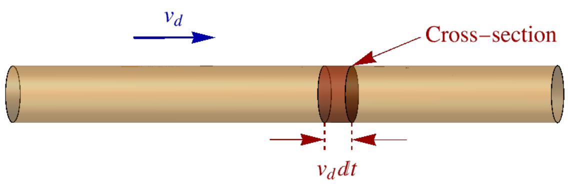
The typical case is a metal, where the charge carriers are electrons and thus q is negative. Semiconductors can have positive or negative charge carriers. Define n as the density of charge carriers.

$$n = \frac{\#\text{ of charge carriers}}{\text{Volume}}$$

The drift velocity is the bias in the motion of the charge carriers. Without a current the charge carriers are moving but there is no bias in the motion; their average velocity (vector) is zero. If there is an electric field in the conductor the charge carriers will move with a bias and the drift velocity is the average velocity of the charge carriers.

$$\vec{v}_d = \vec{v}_{\text{average}} = \text{drift velocity}$$

A useful analogy is the motion of gas molecules. The average velocity of gas molecules is zero, unless there is a wind, and then the average velocity is the wind velocity. The wind velocity is the analog of drift velocity.



Consider the simple case of current in a wire with cross-section A. In a time dt all the charge in the right cylinder with base A and width $v_d\,dt$ will pass the surface. This charge is

$$dQ = \frac{\text{charge}}{\text{Volume}} \times \text{Volume} = |q|\, n \times A\, v_d\, dt$$

The current is dQ/dt giving

$$I = |q|\, n\, A\, v_d.$$

This can be generalized to a vector expression for the current density

$$\vec{J} = q\, n\, \vec{v}_d.$$

For negative charge carriers the drift velocity is opposite to the current density.

E.2 - Resistance

Ohm's Law

If there is an electric field in a conductor then there will be a current. We can define the conductivity and resistivity by the microscopic form of Ohm's law,

$$\vec{J} = \sigma\,\vec{E} \quad \text{(microscopic form)}$$

$$\sigma = \text{Conductivity}$$

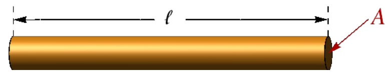
$$\rho = \frac{1}{\sigma} = \text{Resistivity}$$

The conductivity and resistivity are properties of a material. For an object, like a wire, we can define a quantity called the resistance R by the macroscopic form of Ohm's law

$$V = I\,R \quad \text{(macroscopic form)}$$

$$R = \text{Resistance}$$

Units: The SI unit for Resistance is: ohm = Ω = V / A



The resistance, which is a property of a wire, is related to the resistivity, which is a property of the wire's material. We want an expression for the resistance of a wire of length ℓ with cross-sectional area A. The electric field is related to the voltage and the length.

$$\Delta V = -\int \vec{E} \cdot d\vec{r} \implies V = |\Delta V| = E\,\ell$$

It follows that:

$$J = \sigma E \implies \frac{I}{A} = \frac{1}{\rho}\frac{V}{\ell}.$$

Ohm's law then gives

$$R = \frac{\rho\,\ell}{A}.$$

$V = I\,R$ relates the voltage *across* a resistor to the current *through* it. When passing through the resistor in the direction of the current, it is a voltage drop, a decrease in potential. To measure the voltage across a resistor connect the leads of the voltmeter to either side of the resistor. To measure the current through a resistor connect the ammeter in line with the resistor.

Material	Resistivity – ρ (Ω m)
Copper	1.68×10^{-8}
Aluminum	2.65×10^{-8}
Silver	1.47×10^{-8}
Gold	2.22×10^{-8}
Glass	$10^5 - 10^8$

Resistivities for Different Materials at 20°C

Example E.1 - Resistance of a Copper Wire

A 500-m length of copper wire with diameter of 3.5 mm is connected across a 1.5 V battery. What is the current through the wire? For copper: $\rho = 1.68 \times 10^{-8}\ \Omega \cdot \text{m}$.

Solution

Using ρ, the length and diameter we can find the resistance.

$$\ell = 500\text{ m}\,,\quad A = \pi\,r^2 = \pi\left(\frac{d}{2}\right)^2 = \pi\left(\frac{0.0035\text{ m}}{2}\right)^2 = 9.6211 \times 10^{-6}\text{ m}^2 \implies R = \frac{\rho\,\ell}{A} = 0.83708\ \Omega$$

Since we are given the voltage the current can easily be found using Ohm's law.

$$V = 1.5\text{ V} \implies I = V/R = 1.72\text{ A}$$

Variation of Resistance with Temperature

Resistance in a metal is caused by collisions between the moving electrons and the vibrating atoms. If there were no vibration in the atoms there would be no collisions and the resistance would be zero. As the temperature is increased the vibrational motion of the atoms increases and the collisions increase. This is why resistance increases with temperature. The increase of resistivity with temperature can be described by

$$\Delta\rho = \alpha\,\rho_0\,\Delta T \text{ or } \rho = \rho_0\,(1 + \alpha\,\Delta T).$$

Here α is defined as the temperature coefficient, which is a property of a material. ρ_0 is the resistivity at temperature T_0 and ρ is the resistivity at T. $\Delta T = T - T_0$ and $\Delta\rho = \rho - \rho_0$. Multiplying by ℓ/A gives expressions for the resistance, where R_0 is the resistance at temperature T_0 and R is the resistance at T and $\Delta R = R - R_0$.

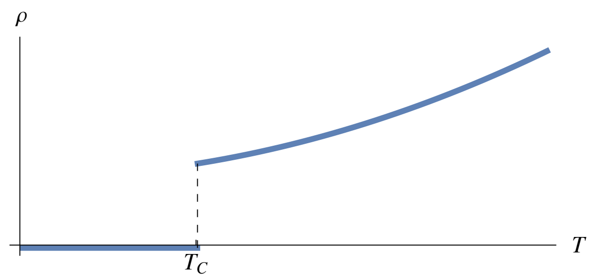
$$\Delta R = \alpha\, R_0\, \Delta T \text{ or } R = R_0\,(1 + \alpha\, \Delta T).$$

In (pure) semiconductors, the conduction mechanism is different and the resistance decreases with temperature; it turns out that at higher temperatures there are more available charge carriers.

Material	Temperature Coefficient – α (K^{-1})
Silver	0.0038
Copper	0.00386
Aluminum	0.00429
Gold	0.0081
Platinum	0.00393
Iron	0.00651
Mercury	0.0009
Carbon (graphite)	–0.0005
Silicon (pure)	–0.07
Germanium (pure)	–0.05

Temperature coefficients near 20°C

One would expect that as the temperature approaches absolute zero, the resistivity would approach zero but what is observed in many materials is much more dramatic; below some critical temperature, the resistivity abruptly drops to exactly zero. This phenomenon is known as superconductivity.



Resistivity as a function of absolute temperature for a superconductor. Below a critical temperature T_C the resistivity abruptly drops to zero.

Example E.2 - Temperature Variation of Resistance of a Copper Wire

At what temperature will the resistance of a copper wire be 5% less than at 20 °C? For copper $\alpha = 3.4\times10^{-3}/\text{K}$. (Temperature differences are the same in celsius as kelvin so we will use kelvin for temperature differences.)

Solution

We do not know the resistance of the wire but we do know that $\Delta R = -0.05\, R_0$, so the resistance R_0 will cancel.

$$\Delta R = \alpha R_0\, \Delta T \implies -0.05\, R_0 = \alpha\, R_0\, \Delta T \implies \Delta T = -0.05/\alpha = -14.7\text{ K}$$

Using $T_0 = 20$ °C we get the final temperature.

$$T = T_0 + \Delta T = 5.3\text{ °C}$$

E.3 - Power and DC Voltage Sources

Power in General

Power is generally defined as the time derivative of some energy or work

$$\mathcal{P} = \frac{d}{dt}\text{Energy}.$$

When a charge Q is moved across a potential difference ΔV the potential energy difference is $\Delta U = Q\,\Delta V$. It follows that when an infinitesimal charge $d\,Q$ moves across a voltage of V the infinitesimal energy change is $d\,U = V\,d\,Q$. Writing $\mathcal{P} = d\,U/d\,t$ and using $I = d\,Q/d\,t$ gives

$$\mathcal{P} = V\,I.$$

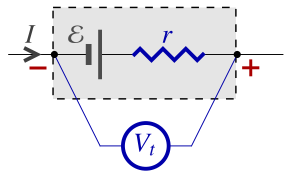
Power Dissipated in a Resistor

Ohm's law $V = I\,R$ relates the voltage drop across a resistor to the current through it. Using it we can write equivalent expressions for the power dissipated in a resistor.

$$\mathcal{P} = V\,I = I^2\,R = \frac{V^2}{R}$$

The energy lost to resistance is dissipated as heat. This is called *Joule heating*.

Terminal Voltage



Treat every DC voltage source as an ideal voltage source with EMF (electromotive force) $\mathcal{E}$ in series with its internal resistance r. The voltage across the terminals V_t of the source is then

$$V_t = \mathcal{E} - I\,r.$$

When there is no load, $I = 0$, the terminal voltage V_t is the same as the EMF $\mathcal{E}$. With a load the terminal voltage drops.

Example E.3 - Terminal Voltage

The measured voltage across a D-cell battery is 1.486 V when no current is drawn. When the battery produces a 250 mA current, its measured voltage is 1.454 V. What is the internal resistance of the battery?

Solution

The voltage with no load (current) is the EMF.

$$\mathcal{E} = 1.486\text{ V}$$

The other voltage, with the current, is the terminal voltage.

$$I = 0.250\text{ A} \quad \text{and} \quad V_t = 1.454\text{ V}$$

We then solve for the internal resistance.

$$V_t = \mathcal{E} - I\,r \quad \Longrightarrow \quad r = \frac{\mathcal{E} - V_t}{I} = 0.128\,\Omega$$

Circuit Diagrams and Nodes

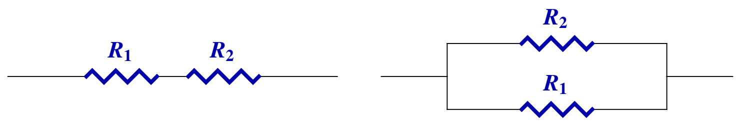
A real-world wire has resistance. When we draw circuit diagrams we always consider the wires to be perfect conductors. Since $R = 0$, the voltage drop across a wire is zero. A wire in a circuit diagram is a point of constant voltage; this is what we call a node. The most effective way to analyze complex circuit diagrams is in terms of nodes and the circuit elements (voltage sources, resistors, capacitors, etc.) connected between nodes.

When it is necessary to consider the real-world resistance of a wire one can simply view it as an ideal conductor with a resistor with $\rho\,\ell/A$ of resistance placed in line.

Voltages in circuits are always differences. If we choose some node to be zero voltage then we can assign a voltage to each node in a circuit. A point of zero voltage in a circuit is called a *ground*.

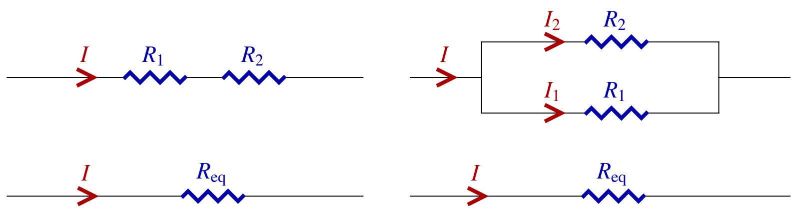
E.4 - Combinations of Resistors

Series and Parallel



Series Parallel

Any combination of resistors with one wire in and one wire out can be reduced to its equivalent resistance. If the combination were placed inside some black box then outside the box the combination would look like a single resistor, which we call its equivalent resistance. For series and parallel resistor combinations, there are simple formulas for finding these equivalent resistances.



Series Parallel

- Series

Resistors are in series when all the current through one passes through the others; there is no branching between them. The total voltage is the sum of the voltages.

$$I = I_1 = I_2 = \ldots \quad \text{and} \quad V = V_1 + V_2 + \ldots$$

Using $V = I\,R$ gives $I\,R_{eq} = I\,R_1 + I\,R_2 + \ldots$. The equivalent resistance of series resistors is given by

$$R_{eq} = R_1 + R_2 + \ldots.$$

- Parallel

Resistors are in parallel when the voltage across the one is the same as the voltage across the others. Resistors are in parallel when they are connected between the same two nodes, where a node is a point of constant voltage in a circuit. The current branches and the total current is the sum of the currents.

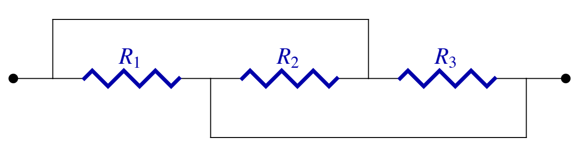
$$V = V_1 = V_2 = \ldots \quad \text{and} \quad I = I_1 + I_2 + \ldots$$

Using $I = V / R$ gives $V \big/ R_{eq} = V / R_1 + V / R_2 + \ldots$. The equivalent resistance of series resistors is given by

$$R_{eq} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \ldots\right)^{-1}.$$

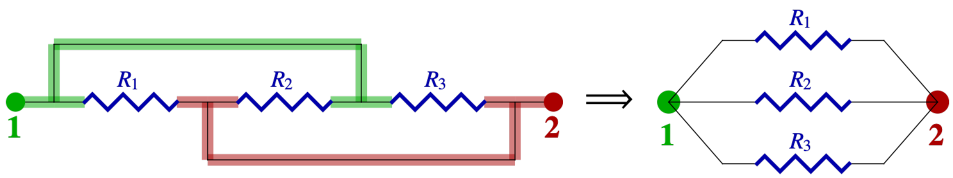
Example E.4 - Identifying Nodes

What is the equivalent resistance of this network of resistors?



Solution

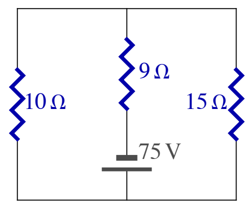
Recall that a wire in a circuit diagram is a perfect conductor, so there is no voltage drop across wires in circuit diagrams. A node is a point of constant potential in a circuit. To identify nodes follow along a wire as far as possible without hitting some circuit element, like a resistor, voltage source or capacitor. There are two nodes, the points drawn in the diagram; label the node on the left as 1 and the node on the right as 2. It is useful, conceptually, to mark nodes with colors. Using green for node 1 and red for node 2, then trace as far as you can go along perfect conductors until you hit a resistor. This gives the picture on the left below.



The next step is to rewrite the nodes as points and draw the resistors as they are connected between the nodes. This is the picture on the right, above. All three resistors are connected across the same two nodes so they are in parallel. We can then write down the equivalent resistance.

$$R_{\text{eq}} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1}$$

Example E.5 - Resistor Circuit with Voltage Source

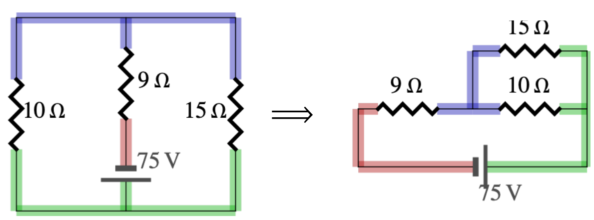


The diagram shows three resistors connected in a circuit with a 75 V battery. Complete the table with the voltage across, the current through and the power dissipated in each of the three resistors.

	10 Ω	**9 Ω**	**15 Ω**
V			
I			
$\mathcal{P}$			

Solution

First identify the three nodes. When we isolate the source from the resistors, it is then clear that the circuit is equivalent to the one shown below on the right.



The 10 Ω and 15 Ω resistors are in parallel and their equivalent is in series with the 9 Ω. We can then find the equivalent resistance that the course is connected across.

$$R_{\text{eq}} = 9\,\Omega + \left(\frac{1}{10\,\Omega} + \frac{1}{15\,\Omega}\right)^{-1} = 9\,\Omega + 6\,\Omega = 15\,\Omega$$

This equivalent resistance determines I_{battery} the current that the battery will provide. The 9 Ω resistor is in series with the battery so the current provided by the battery will pass through it. This allows us to begin filling in the table with the current through the 9 Ω resistor.

$$I_{9\,\Omega} = I_{\text{battery}} = \frac{V_{\text{battery}}}{R_{\text{eq}}} = \frac{75\text{ V}}{15\,\Omega} = 5\text{ A}$$

With a table like this, once one thing in a column is known, we can find the rest of that column using $V = I\,R$ and $\mathcal{P} = I\,V$

$$V_{9\,\Omega} = I_{9\,\Omega}\,9\,\Omega = 45\text{ V}$$

We can also find the power.

$$\mathcal{P}_{9\,\Omega} = I_{9\,\Omega}\,V_{9\,\Omega} = 225\text{ W}$$

The voltages across the $10\,\Omega$ and $15\,\Omega$ resistors are equal, because they are in parallel and that voltage added to $V_{9\,\Omega}$ will give the total voltage V_{battery}.

$$V_{10\,\Omega} = V_{15\,\Omega} = V_{\text{battery}} - V_{9\,\Omega} = 75\text{ V} - 45\text{ V} = 30\text{ V}$$

$I = V/R$ allows us to find the currents through the $12\,\Omega$ and $24\,\Omega$ resistors.

$$I_{10\,\Omega} = \frac{V_{10\,\Omega}}{10\,\Omega} = \frac{30\text{ V}}{10\,\Omega} = 3\text{ A} \quad \text{and} \quad I_{15\,\Omega} = \frac{V_{15\,\Omega}}{15\,\Omega} = \frac{30\text{ V}}{15\,\Omega} = 2\text{ A}$$

We can also use $\mathcal{P} = I\,V$ to finish the table with the last power values.

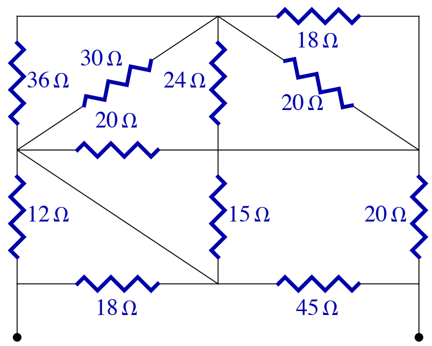
$$\mathcal{P}_{10\,\Omega} = I_{10\,\Omega}\,V_{10\,\Omega} = 90\text{ W} \quad \text{and} \quad \mathcal{P}_{15\,\Omega} = I_{15\,\Omega}\,V_{15\,\Omega} = 60\text{ W}$$

	10 Ω	**9 Ω**	**15 Ω**
V	30 V	45 V	30 V
I	3 A	5 A	2 A
$\mathcal{P}$	90 W	225 W	60 W

In solving such problems there is an important consistency check. The current through the $9\,\Omega$ resistor branches into the $10\,\Omega$ and $15\,\Omega$, so we must have $I_{9\,\Omega} = I_{10\,\Omega} + I_{15\,\Omega}$, which is clearly satisfied.

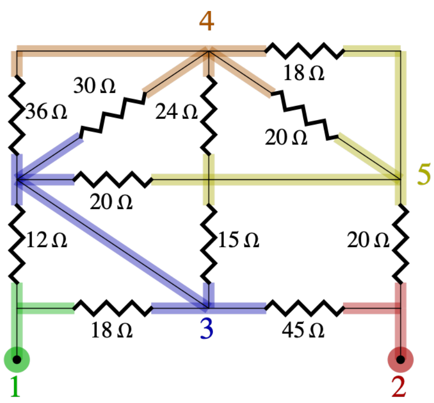
Example E.6 - An Ugly Example

What is the equivalent resistance of this network of resistors?

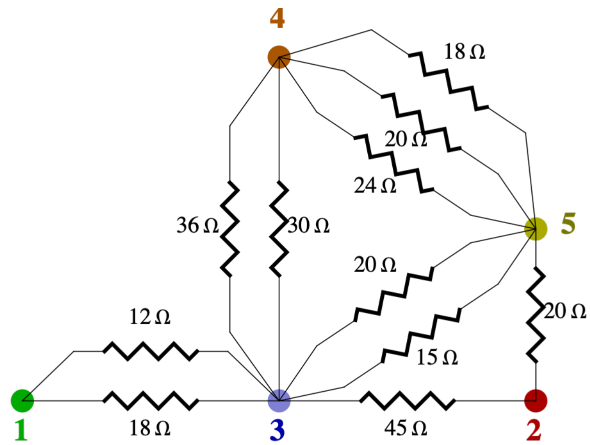


Solution

First identify the nodes; there are five.



We now look to see what is connected between each node.

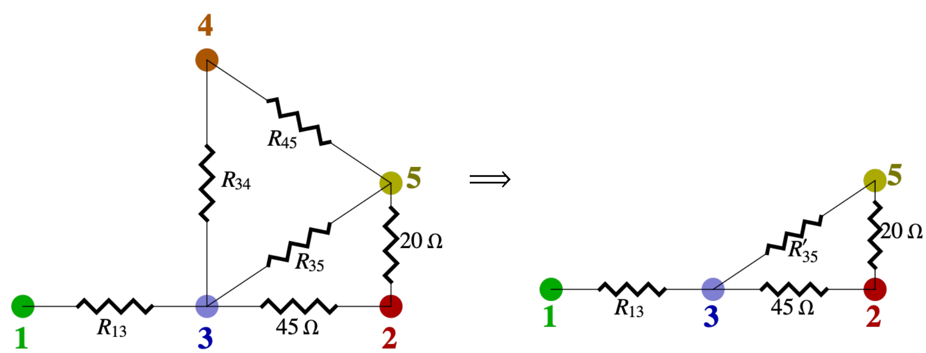


Between node 1 (green) and node 3 (blue) are the 18 Ω and 12 Ω resistors; two resistors between the same two nodes are in parallel. Call their equivalent R'_{13}.

$$R_{13} = \left(\frac{1}{18\,\Omega} + \frac{1}{12\,\Omega}\right)^{-1} = 7.2\,\Omega$$

Similarly, we see that the 30 Ω and 36 Ω are connected between node 3 (blue) and node 4 (orange); combine these to get R_{34}. Between node 4 (red) and node 5 (yellow) we have the 24 Ω, the 20 Ω and the 18 Ω which we replace with R_{45}. Between node 3 (blue) and node 5 (yellow) we have the 15 Ω andthe 20 Ω which we replace with R_{35}. Between node 2 (red) and node 3 (blue) is just the 45 Ω resistor and between node 2 (red) and node 5 (yellow) we have just the 20 Ω.

$$R_{34} = \left(\frac{1}{30\,\Omega} + \frac{1}{36\,\Omega}\right)^{-1} = 16.364\,\Omega\,,\quad R_{45} = \left(\frac{1}{24\,\Omega} + \frac{1}{20\,\Omega} + \frac{1}{18\,\Omega}\right)^{-1} = 6.7925\,\Omega \quad \text{and} \quad R_{35} = \left(\frac{1}{15\,\Omega} + \frac{1}{20\,\Omega}\right)^{-1} = 8.5714\,\Omega$$



Now we have reduced the resistor network to the diagram above on the left. R_{34} and R_{45} are in series and that is in parallel with the R_{35}. Combine these to get R'_{35} as shown in the diagram to the right.

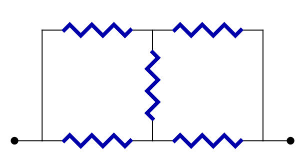
$$R'_{35} = \left(\frac{1}{R_{35}} + \frac{1}{R_{34} + R_{45}}\right)^{-1} = 6.2558\,\Omega$$

This resistance is then combined in series with the 20 Ω, which in turn is parallel with the 45 Ω. We can then find the equivalent resistance between nodes 1 and 2 by adding the R_{13} which is in series.

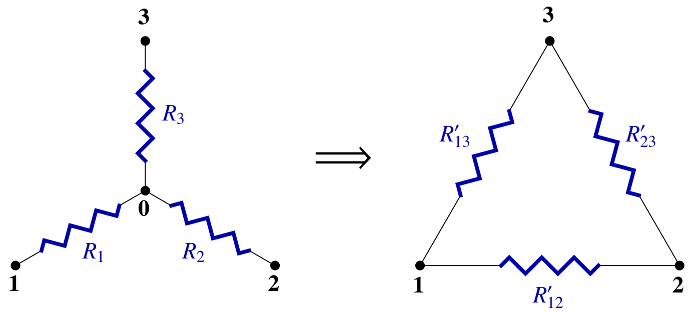
$$R_{\text{eq}} = R_{13} + \left(\frac{1}{45\,\Omega} + \frac{1}{R'_{35} + 20\,\Omega}\right)^{-1} = 23.78\,\Omega$$

Node Reduction

Not all resistor networks can be reduced to their equivalent resistance using the series and parallel rules mentioned above. As an example, consider the following network of resistors



After staring at this for a while, one should become convinced that no pair of resistors is either in series or parallel. For these networks another approach is needed. The diagram below shows the idea of this method. Where three resistors R_1, R_2 and R_3 diverge from a central node, marked node 0 in the diagram, we can replace these three with three other resistors R'_{12}, R'_{13} and R'_{23}.

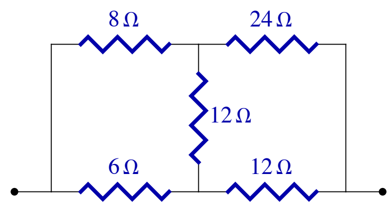


The new resistances can be related to the old ones by the simple expression

$$R'_{ij} = \frac{R_i\, R_j}{R_{\shortparallel}} \text{ where } R_{\shortparallel} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1}$$

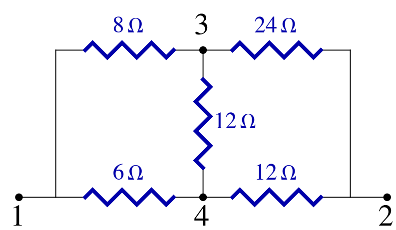
Example E.7 - Node Reduction

Find the equivalent resistance of the following network of resistors.



Solution

Labeling the nodes we have two internal nodes 3 and 4. The other two nodes 1 adn 2 are external, meaning that these are what we are finding the equivalent resistance across.



The node reduction procedure removes an internal node (3 or 4 here) and replaces the three resistors connected to the removed node with three new resistors. Here we will choose to remove node 4. The three resistors connected to node 4 are the 6 Ω and the two 12 Ω resistors. First we calculate $R_{\shortparallel}$ using these three resistances.

$$R_{\shortparallel} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1} = \left(\frac{1}{6\,\Omega} + \frac{1}{12\,\Omega} + \frac{1}{12\,\Omega}\right)^{-1} = 3\,\Omega$$

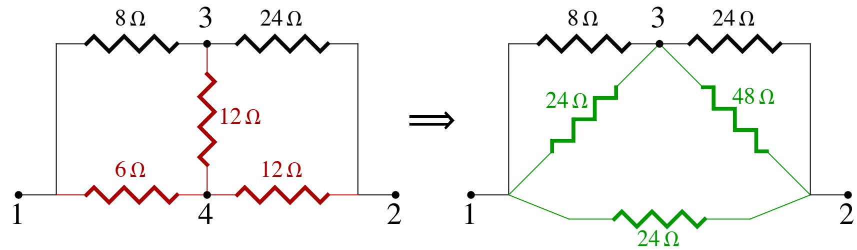
The new resistances are calculated using

$$R'_{ij} = \frac{R_i\, R_j}{R_{\shortparallel}}$$

For our three removed resistance we find the new resistance values.

$$\frac{6\,\Omega \times 12\,\Omega}{3\,\Omega} = 24\,\Omega\,, \quad \frac{6\,\Omega \times 12\,\Omega}{3\,\Omega} = 24\,\Omega \text{ and } \frac{12\,\Omega \times 12\,\Omega}{3\,\Omega} = 48\,\Omega$$

These resistors connect the far tip of the two resistors used in calculation. For example, the 48 Ω resistor connects the far tip of the two 12 Ω.



On the left are the three resistors we remove are shown in red. At the right are the three new resistances show in green.

We now have the network of resistors to the right. Although we have not reduced the number of resistors we have removed a node and the circuit is now simpler; there are parallel resistors now. Between nodes 1 and 3 we have the 8 Ω and 24 Ω in parallel and between 2 and 3 we have the 48 Ω and 24 Ω resistors. Call these new resistances R_{13} and R_{23}.

$$R_{13} = \left(\frac{1}{8\,\Omega} + \frac{1}{24\,\Omega}\right)^{-1} = 6\,\Omega \quad \text{and} \quad R_{23} = \left(\frac{1}{24\,\Omega} + \frac{1}{48\,\Omega}\right)^{-1} = 16\,\Omega$$

These two resistors are then in series and their equivalent is in parallel with the 24 Ω resistor between nodes 1 and 2. The overall equivalent is then found.

$$R_{\text{eq}} = \left(\frac{1}{24\,\Omega} + \frac{1}{R_{13} + R_{23}}\right)^{-1} = 11.48\,\Omega$$

Students are encouraged to repeat this calculation with node 3 removed instead of 4. The procedure is the same and, of course, the answer is also the same.

Node Reduction - General *N* Resistor Case

The formula above generalizes to N resistors leaving a point. Call the resistors R_i with $i = 1, \ldots, N$. They connect a common node labeled by 0 with an external node labeled by i. Now we replace the N resistors with ones connecting all possible pairs of the N external nodes. The new resistances have the values

$$R'_{ij} = \frac{R_i\, R_j}{R_{\parallel}} \text{ where } R_{\parallel} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \ldots\right)^{-1}.$$

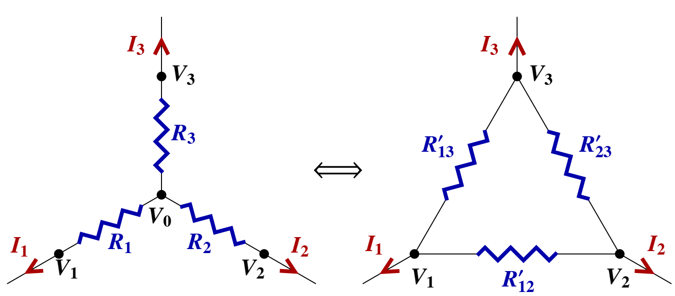
The $N = 2$ case is just two resistors in series and the series formula is a special case of this.

$$N = 2 \implies R_{\parallel} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1} = \frac{R_1\, R_2}{R_1\; + \; R_2} \implies R'_{12} = \frac{R_1\, R_2}{R_{\parallel}} = R_1 + R_2$$

Summarizing the number of new resistances we have this table.

Number of R_i	Number of R'_{ij}
2	1
3	3
4	6
N	$N\,(N-1)/2$

Proof of Node Reduction Formula



For clarity, we will consider the proof of the formula for the three resistor case but the following derivation can easily be modified to prove the formula in the general case. Take the voltages at each node to be V_1, V_2, V_3 and V_0. Take the outward (away from 0) current through R_i as I_i. It follows that

$$I_1 = \frac{V_0 - V_1}{R_1}, \quad I_2 = \frac{V_0 - V_2}{R_2} \text{ and } I_3 = \frac{V_0 - V_3}{R_3}.$$

The condition that the currents sum to zero gives V_0.

$$I_1 + I_2 + I_3 = 0 \implies V_0\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) = \frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3}$$

$$\implies V_0 = R_{\shortparallel}\left(\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3}\right) \text{ where } R_{\shortparallel} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1}$$

Using this value for V_0 we can find the current as a function of voltage

$$I_1 = \frac{V_0}{R_1} - \frac{V_1}{R_1} = \frac{R_{\shortparallel}}{R_1}\left(\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3}\right) - \frac{V_1}{R_1}.$$

Multiply the term on the right by one, in the form $R_{\shortparallel}/R_{\shortparallel}$, then cancel terms and regroup.

$$I_1 = \frac{R_{\shortparallel}}{R_1}\left(\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3}\right) - \frac{V_1}{R_1} \times R_{\shortparallel}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$= \frac{R_{\shortparallel}}{R_1}\left(\frac{V_2 - V_1}{R_2} + \frac{V_3 - V_1}{R_3}\right) = \frac{V_2 - V_1}{R'_{12}} + \frac{V_3 - V_1}{R'_{13}}$$

where we have used

$$R'_{ij} = \frac{R_i R_j}{R_{\shortparallel}}.$$

This gives the simple result

$$I_1 = \frac{V_2 - V_1}{R'_{12}} + \frac{V_3 - V_1}{R'_{13}}.$$

Clearly, there is nothing special about I_1 in the above derivation and we can derive similar results for I_2 and I_3.

$$I_2 = \frac{V_1 - V_2}{R'_{12}} + \frac{V_3 - V_2}{R'_{23}} \text{ and } I_3 = \frac{V_1 - V_3}{R'_{13}} + \frac{V_2 - V_3}{R'_{23}}$$

The above relations prove our result. It shows the current to voltage relation is the same for the 3 resistances R_1, R_2 and R_3 leaving the 0 node as for the replacement resistances R'_{12}, R'_{13} and R'_{23} connecting the external nodes.

The Equivalent Resistance Theorem

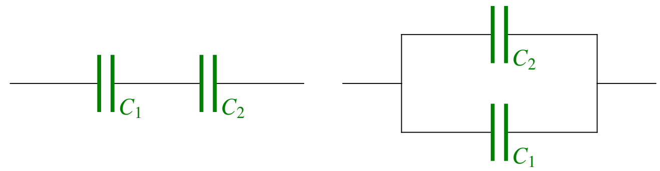
Any network of resistors with two external nodes may be reduced to a single equivalent resistance between the external nodes. An algorithm for finding the equivalent resistance of a network follows.

Specify the resistor network with a set of nodes, two external and the rest internal, and with a set of resistors, where each resistor is labeled by a resistance *and* by the pair of nodes it connects.

The first step is to remove all resistances in parallel; these are resistors between the same two nodes. The second step is to apply the node reduction procedure to any of the internal nodes. Continue iterating these two steps until all internal nodes are removed and there is a single resistor.

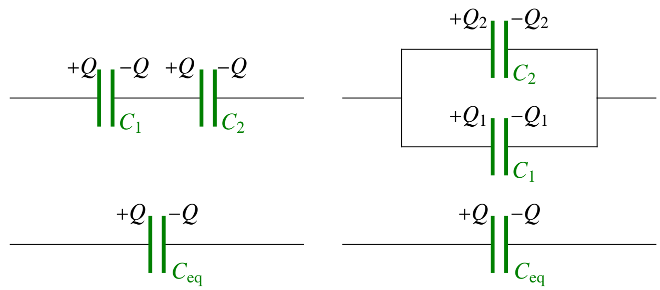
To make this most calculationally efficient, remove an internal node with the smallest number of resistances. First look for a pair in series (or $N = 2$.) If none are in series then look for $N = 3$, then $N = 4$, etc.

E.5 - Combinations of Capacitors



Series Parallel

As we saw for resistors, any network of capacitors can be reduced to an equivalent capacitance. For capacitors its charge plays the role the current played in resistors. (Recall that $I = dQ/dt$.)



Series Parallel

The voltage to charge relation for a capacitor is

$$V = \frac{Q}{C}.$$

- Series

In the case of series resistors the charge on each capacitor is the same and both are the same as the charge on the equivalent. The voltages add.

$$Q = Q_1 = Q_2 = \ldots \quad \text{and} \quad V = V_1 + V_2 + \ldots$$

Using the voltage to charge relation gives $Q/C_{\text{eq}} = Q/C_1 + Q/C_2 + \ldots$ which gives the expression for equivalent capacitance

$$C_{\text{eq}} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \ldots\right)^{-1}.$$

- Parallel

For parallel resistors the voltages are equal and the charges add.

$$V = V_1 = V_2 = \ldots \quad \text{and} \quad Q = Q_1 + Q_2 + \ldots$$

Using $Q = C\,V$ gives $C_{\text{eq}}\,V = C_1\,V + C_2\,V + \ldots$ giving

$$C_{\text{eq}} = C_1 + C_2 + \ldots.$$

Note that the series and parallel formulas for capacitors are reversed relative to their resistor counterparts.

- Node Reduction

The node reduction formula also applies to capacitors as well. The new capacitors have the values

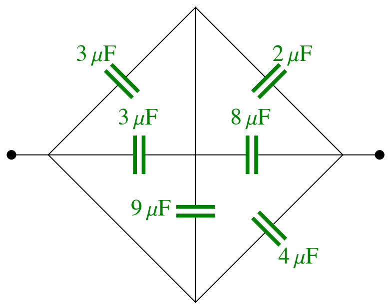
$$C'_{ij} = \frac{C_i\,C_j}{C_{\shortparallel}}$$

but the $C_{\shortparallel}$ has a different form.

$$C_{\shortparallel} = C_1 + C_2 + \ldots$$

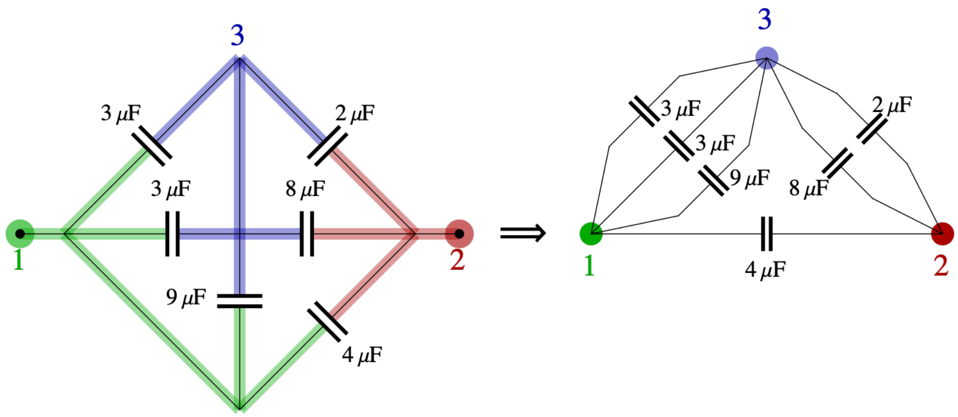
Example E.8 - Equivalent Capacitance

What is the equivalent capacitance of the capacitor network shown?



Solution

First do a nodal analysis on this capacitor network. It is then clear how they are connected.



The two $3\,\mu\text{F}$ and the $9\,\mu\text{F}$ capacitors are in parallel, as are the $2\,\mu\text{F}$ and $8\,\mu\text{F}$ capacitors. We add to find the equivalent capacitances.

$$3\,\mu\text{F} + 3\,\mu\text{F} + 9\,\mu\text{F} = 15\,\mu\text{F} \quad \text{and} \quad 2\,\mu\text{F} + 8\,\mu\text{F} = 10\,\mu\text{F}$$

These two resulting capacitances are in series

$$\left(\frac{1}{15\,\mu\text{F}} + \frac{1}{10\,\mu\text{F}}\right)^{-1} = 6\,\mu\text{F}$$

and this $6\,\mu\text{F}$ is then in parallel with the final $4\,\mu\text{F}$ giving the overall capacitance.

$$C_{\text{eq}} = 6\,\mu\text{F} + 4\,\mu\text{F} = 10\,\mu\text{F}$$

E.6 - Kirchhoff's Rules

Kirchhoff's rules are used to solve for the currents in the case of a circuit involving many resistors and DC voltage sources. A junction is a point in the circuit where three or more wires meet; if there are just two wires it is just a bend in the wire and not a junction. For every branch in the circuit we can define a current. Kirchhoff's rules gives a set of linear equations in the currents. It is not essential to choose the proper direction for the currents, and in fact one typically doesn't know the current directions until a solution is found. If the chosen current direction is wrong then that current will be negative when the solution is found.

Junction Rule

At every junction in a circuit the total current in is equal to the total current out.

$$\sum I_{\text{in}} = \sum I_{\text{out}}.$$

In every case (at least where the circuit is one connected piece) the junction rule equations will not be independent; there will always be one equation more than is needed. Summing all the equations gives $\sum I = \sum I$ which is equivalent to $0 = 0$. (This is because every current leaves one junction and enters another.) Because of this any one junction rule equation is the negative of the sum of the others. To get an independent set of equations one must delete one (any one) of the equations.

Loop Rule

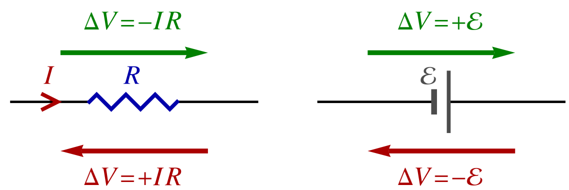
Around every closed loop in a circuit the sum of all the voltage gains is zero.

$$\sum \Delta V = 0$$

The sign conventions are:

When moving through a resistor in the direction of the current: $\Delta V = -I\,R$.
When moving through a resistor opposite the current: $\Delta V = +I\,R$.

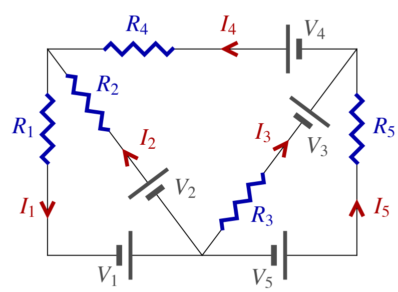
When moving through a DC source from - to + terminals: $\Delta V = +\mathcal{E}$.
When moving through a DC source from + to - terminals: $\Delta V = -\mathcal{E}$.



To avoid non-independent equations consider only the smallest loops.

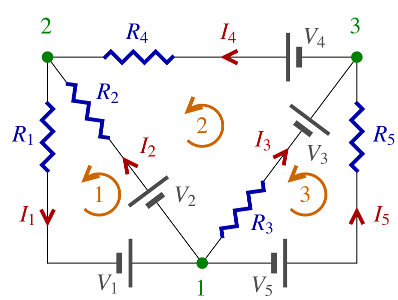
Example E.9 - Kirchhoff's Rules 1

Find an independent set of equations that could be solved for the five currents: I_1, I_2, I_3, I_4 and I_5.



Solution

Below, the three junctions are labeled 1, 2 and 3 and the the loops are shown as the counterclockwise arrows.



Junction Rule

At each junction, the total current in equals the total current out. Omit any one of the junction rule equations.

Junction 1: $I_1 = I_2 + I_3 + I_5$ (Omit this)

Junction 2: $I_2 + I_4 = I_1$

Junction 3: $I_3 + I_5 = I_4$

Any one junction rule equation is equivalent to the sum of the others, so one should be omitted. Here the first was removed since it had the most variables but either of the other two could also have been removed. Note that the number of wires meeting at a junction must be the same as the number of currents in that equation.

Loop Rule

For each loop the sum of the voltage gains is zero. Following the sign conventions described above we get three more independent equations.

$$\text{Loop 1: } 0 = V_2 - I_2\,R_2 - I_1\,R_1 + V_1$$

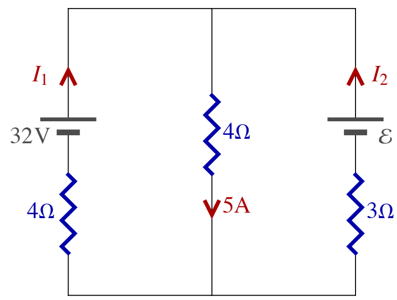
$$\text{Loop 2: } 0 = -I_3\,R_3 + V_3 + V_4 - I_4\,R_4 + I_2\,R_2 - V_2$$

$$\text{Loop 3: } 0 = V_5 - I_5\,R_5 - V_3 + I_3\,R_3$$

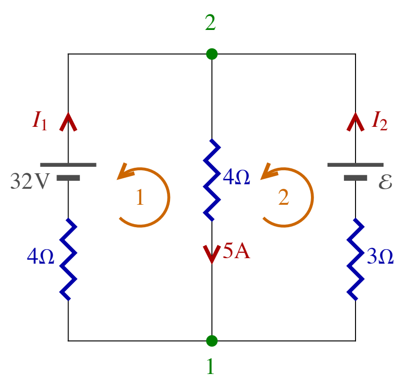
After omitting the one of the junction rule equations we have a set of five independent linear equations that can be solved for the five unknown currents.

Example E.10 - Kirchhoff's Rules 2

Find the two unknown currents and the unknown EMF: I_1, I_2 and $\mathcal{E}$.



Solution



Using the junction rule for junction 1 gives the same expression as for junction 2.

$$\text{Junction 1:} \quad 5 = I_1 + I_2$$

The two loop equations are.

$$\text{Loop 1:} \quad 0 = +5\times 4 - 32 + 4\,I_1$$

$$\text{Loop 2:} \quad 0 = -3\,I_2 + \mathcal{E} - 5\times 4$$

Loop 1 equation allows us to find I_1.

$$0 = +5\times 4 - 32 + 4\,I_1 \implies I_1 = 3\text{ A}$$

The junction equation gives us I_2

$$5 = I_1 + I_2 \implies I_2 = 5 - I_1 = 2\text{ A}$$

and the loop 2 equation then gives us the unknown $\mathcal{E}$.

$$0 = -3\,I_2 + \mathcal{E} - 5\times 4 \implies \mathcal{E} = 26\text{ V}$$