Chapter F

Magnetism

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F.1 - Magnetic Dipoles and Magnetic Fields

Electromagnetic Duality

There are two types of "magnetic charge" or poles, North poles **N** and South poles **S**. Playing with bar magnets demonstrates that like poles repel and unlike attract. This is analogous to the situation we had with electric charge. This analogy is a deep one and is called Electromagnetic duality. North and South poles are related to the magnetic field $\vec{B}$ as positive and negative electric charges are to the electric field.

$$\boldsymbol{N} \text{ and } \boldsymbol{S} \text{ are to } \vec{B}$$
$$\text{as}$$
$$+ \text{ and } - \text{ are to } \vec{E}$$

The force of an electric charge q in an electric field gives by analogy the force of a magnetic pole of strength g in a magnetic field.

$$\vec{F} = q\,\vec{E} \text{ and } \vec{F} = g\,\vec{B}$$

If g denotes the "magnetic charge" or pole strength then a North pole corresponds positive g and a South pole to negative. Gauss's law relates the total electric flux though a closed surface to the charge enclosed by the surface. Similarly, the magnetic flux through a closed surface corresponds to the total "magnetic charge" inside.

$$\oint \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_0}\, q_{\text{enclosed}} \text{ and } \oint \vec{B} \cdot d\vec{A} = \mu_0\, g_{\text{enclosed}}$$

where analogous to $1/\varepsilon_0$, the electric constant of proportionality, is the magnetic constant μ_0.

$$\vec{E} = \frac{1}{4\pi\,\varepsilon_0}\, q\, \frac{\hat{r}}{r^2} \text{ and } \vec{B} = \frac{\mu_0}{4\pi}\, g\, \frac{\hat{r}}{r^2}$$

Magnetic Dipoles

A permanent magnet has both North and South poles separated by some distance. When it is placed in a field the **N** pole experiences a force in the direction of the field and the **S** pole has a force opposite the field. If the field is uniform the net force is zero but there is a net torque. This is analogous to an electric dipole and it will be called a magnetic dipole. The strength of a magnet can be described by its magnetic dipole moment $\vec{\mu}$. For an electric dipole $\vec{p}$ the torque and potential energy given by $\vec{\tau} = \vec{p} \times \vec{E}$ and $U = -\vec{p} \cdot \vec{E}$. The corresponding expressions for torque and potential energy of a magnetic dipole in a magnetic field are

$$\vec{\tau} = \vec{\mu} \times \vec{B} \text{ and } U = -\vec{\mu} \cdot \vec{B}.$$

In addition to permanent magnets being dipoles we will see that current loops also are magnetic dipoles. In that case we will write an expression for the magnetic dipole moment.

Gauss's Law for Magnetism and the Absence of Isolated Poles

Isolated magnetic poles could exist but so far none have ever been observed. Gauss's law in the electric case states that electric field lines begin at isolated positive charges and end at isolated negative charges. The absence of isolated magnetic poles implies that magnetic field lines never begin or end; they either form closed loops or go off to infinity

The magnetic analog of Gauss's law is $\oint \vec{B} \cdot d\vec{A} = \mu_0\, g_{\text{inside}}$. The nonexistence of isolated magnetic poles implies that the right hand side is zero and Gauss's law for magnetism becomes

$$\oint \vec{B} \cdot d\vec{A} = 0$$

This is our second of Maxwell's equations. If we apply Gauss's law to a magnet and put a Gaussian surface around the North pole then there is magnetic flux leaving the surface at the end of the magnet. For the flux to be zero through this Gaussian surface the field lines inside the magnet must close back on themselves and form closed loops. Because of this if a bar magnet is cut in half then it doesn't split into a pair of isolated poles; it becomes two smaller dipoles.

If someday isolated poles are discovered, then we may just modify Gauss's law by adding in a magnetic charge term $\mu_0 \, g_{\text{inside}}$ to its right hand side as was shown above. Other modifications to Maxwell's equations associated with magnetic currents will also be needed; these will be discussed later.

F.2 - Force on Moving (Electric) Charges and Currents

Electricity and magnetism are not separate forces where electric fields just exert forces on electric charges and magnetic fields exert forces on magnets. Instead electricity and magnetism are aspects of the same force called electromagnetism. Magnetic fields cause forces on moving (electric) charges and currents.

Magnetic Force on Moving Charges

If a charge Q is moving with a velocity $\vec{v}$ in a magnetic field $\vec{B}$ then the force is given by

$$\vec{F} = Q\,\vec{v} \times \vec{B}.$$

Note that the cross product is a three dimensional thing. The velocity and field vectors define a plane and the force is in the direction perpendicular to the plane. Note also that when a vector is multiplied by a negative scalar its direction changes, so negative charges experience forces opposite that of positive ones.

Force on Currents

Consider the flow of charge carriers of charge q with drift velocity $\vec{v}_d$ through a straight segment of wire of length ℓ with cross-section A. If the density of charge carriers (number/volume) is n then the total number of charge carriers is

$$N = n \times \text{Volume} = n\,A\,\ell.$$

Summing over all the charges gives the total force on the wire in a uniform field

$$\vec{F} = N\;q\,\vec{v}_d \times \vec{B} = n\,A\,\ell\,q\,\vec{v}_d \times \vec{B}$$

Using the expression for current from the previous chapter, $I = |q|\,n\,A\,v_d$, we get

$$\vec{F} = I\,\vec{\ell} \times \vec{B},$$

where the direction of the current is put into the direction of the vector $\vec{\ell}$. Note that the current direction is the same as the drift velocity when q is positive and it is opposite when q is negative. This is built into the above expression.

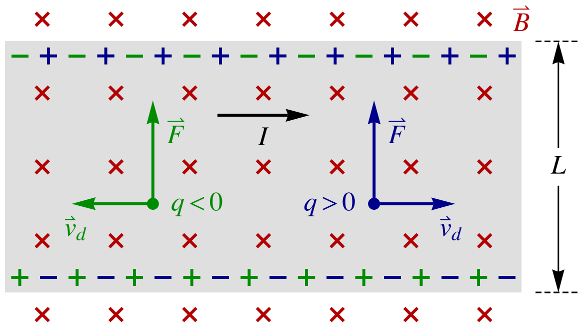
To generalize this expression consider a curved wire with an infinitesimal segment $d\vec{s}$. The force on that segment is $I\,d\vec{s} \times \vec{B}$. Integrating over the length of the wire gives

$$\vec{F} = I \int d\vec{s} \times \vec{B},$$

where the field need not be uniform.

Hall Effect

It is clear from the previous section that we cannot determine the charge of the charge carriers by measuring the magnetic force on a wire; simultaneously changing the signs of q and $\vec{v}_d$ gives the same current. We can, however, use the magnetic force to find the charge of the charge carriers by measuring the voltage across a conducting strip in a magnetic field. Consider a flat conducting strip with a current in a magnetic field. Take the width of the strip, the current and the field to be mutually perpendicular as shown in the diagram.



The magnetic force will push either positive or negative charge carriers toward the top of the wire. This will create a voltage across the strip. The polarity of the voltage depends on the charge of the charge of the charge carriers. If they are positive then the top is positive and otherwise it is negative.

The Hall voltage is the voltage across the strip. If the width of the strip is L then the work on the charge carrier q is $F\,L$. Since the force is $F = |q|\,v\,B$ we have

$$W = |q|\,v_d\,B\,L$$

The work per charge is the induced voltage, since $\Delta U = q\Delta V$.

$$V_{\text{Hall}} = W/q \implies V_{\text{Hall}} = v_d\,B\,L$$

Motion of Charged Particles

Any force that acts perpendicularly to the velocity of a particle doesn't affect the speed of the particle; it only alters its direction. This is the case with the magnetic force $\vec{F} = Q\,\vec{v}\times\vec{B}$. Suppose a particle with speed v is shot into a region of uniform magnetic field with the velocity perpendicular to the field then the magnitude of the force is just $F = |Q|\,v\,B$. Since the speed and the magnitude of the force are constant and the force and velocity are perpendicular, the motion will be uniform circular motion. Using the acceleration for uniform circular motion $a_c = v^2/r$ and Newton's second law we get:

$$F = m\,a \implies |Q|\,v\,B = m\,\frac{v^2}{r} \implies r = \frac{m\,v}{|Q|\,B}.$$

The angular frequency ω is related to the speed and radius by $\omega = v/r$ which gives an expression known as the cyclotron frequency

$$\omega = \frac{|Q|\,B}{m}.$$

If a charged particle moves in a uniform magnetic field with a velocity that is not perpendicular to the field, then the perpendicular component changes as before and the parallel component is unchanged. The resulting motion is a combination of linear and circular motion, giving a helix. The general shape of the path of a charged particle in a uniform magnetic field is helical.

An electromagnetic field is a combination of both electric and magnetic fields. The force of a charged particle in an electromagnetic field is the sum of both electric and magnetic forces and is called the Lorentz force law

$$\vec{F} = Q\left(\vec{E} + \vec{v}\times\vec{B}\right).$$

F.3 - Sources of Magnetic Fields

In our discussion of electric fields we have discussed the force on charges due to fields. Analogously, we have found the magnetic force on moving charges and currents. Our discussion of electric fields is more complete, however, since we have ways to calculate electric fields due to sources, electric charges. We now need to address the sources of magnetic fields. The Biot-Savart Law will be introduced as the analog of the Coulomb's Law integrals over continuous distributions to get electric fields. In cases of symmetry we could use Gauss's Law to find electric fields; as the magnetic analog of this we will introduce Ampere's law.

	Electric Fields	Magnetic Fields
Force on Q (or I)	$\vec{F} = Q\,\vec{E}$	$\vec{F} = Q\,\vec{v}\times\vec{B}$ $\vec{F} = I\int d\vec{s}\times\vec{B}$

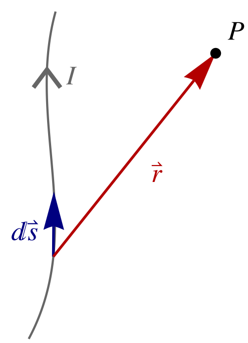
Field due to Q (or I)	$\vec{E} = k_e \int \frac{\hat{r}}{r^2}\, dq$	Biot–Savart Law
	$\oint \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_0} Q_{\text{enclosed}}$	Ampere's Law

Coulomb's Law and Gauss's Law are mathematically equivalent for electrostatics. Similarly, we will see that the Biot-Savart Law is equivalent to Ampere's Law for magnetostatics. Electrostatics allows no movement and thus no currents. Magnetostatics allows currents but requires all currents to be steady. Gauss's Law is fully correct even beyond electrostatics and is one of Maxwell's equations. Ampere's Law, as discussed in this chapter, is only correct in the context of magnetostatics. Next chapter we will introduce Maxwell's addition to Ampere's Law; this will make it generally correct and it will become one of Maxwell's equations.

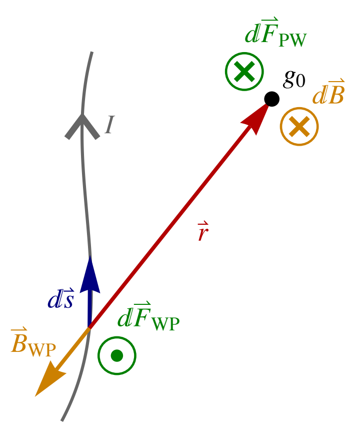
F.4 - Biot-Savart Law

The Biot-Savart law relates the magnetic field at some point P to the current in a wire. The analogous expression for electric fields is $\vec{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{\hat{r}}{r^2}\, dq$. The source is a current I through an infinitesimal segment of wire $d\vec{s}$. Take the vector $\vec{r}$ to be from the source to P. The Biot-Savart law is

$$\vec{B} = \frac{\mu_0}{4\pi} I \int d\vec{s} \times \frac{\hat{r}}{r^2}.$$



Derivation Using a Test Pole



Exploiting duality symmetry we can derive the Biot-Savart Law. To do this introduce a test magnetic pole of strength g_0 at the position P. The fact that these poles have never been observed need not disturb us. The field at the wire due to the pole is

$$\vec{B}_{WP} = \frac{\mu_0}{4\pi} g_0 \frac{-\hat{r}}{r^2}.$$

The negative sign is there because the vector $\vec{r}$ in the diagram is pointing toward the pole where we usually take $\vec{r}$ as pointing away from the charge or pole. The force on the wire due to the pole is then

$$d\vec{F}_{WP} = I\, d\vec{s} \times \vec{B}_{WP} = -\frac{\mu_0}{4\pi} g_0\, I\, d\vec{s} \times \frac{\hat{r}}{r^2}.$$

Using Newton's third law we can relate this to the force on the pole due to the wire

$$d\vec{F}_{PW} = -d\vec{F}_{WP} = \frac{\mu_0}{4\pi} g_0\, I\, d\vec{s} \times \frac{\hat{r}}{r^2}.$$

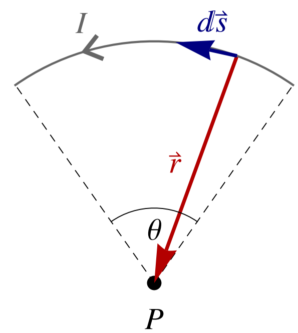
The force on a magnetic pole g_0 in a field $\vec{B}$ is $\vec{F} = g_0 \vec{B}$ so using the pole as a test pole we can write the field at P due to the wire as

$$d\vec{B} = \frac{1}{g_0} d\vec{F}_{PW} = \frac{\mu_0}{4\pi} I\, d\vec{s} \times \frac{\hat{r}}{r^2},$$

which is just the Biot-Savart Law. Note that the result is independent of our test pole.

Examples

Field at the center of a circular arc



Consider a circular arc of radius R and of angle θ in the xy-plane with a counterclockwise current. The vector $\vec{r}$ is from $d\vec{s}$ to the origin, which is the point P.

$$\vec{B} = \frac{\mu_0}{4\pi} I \int \frac{d\vec{s} \times \hat{r}}{r^2}$$

Using $\vec{A} \times \vec{B} = A\,B\,\sin\theta\,\hat{u}$ we get

$$d\vec{s} \times \hat{r} = d\,s \cdot 1 \cdot 1\,\hat{z}.$$

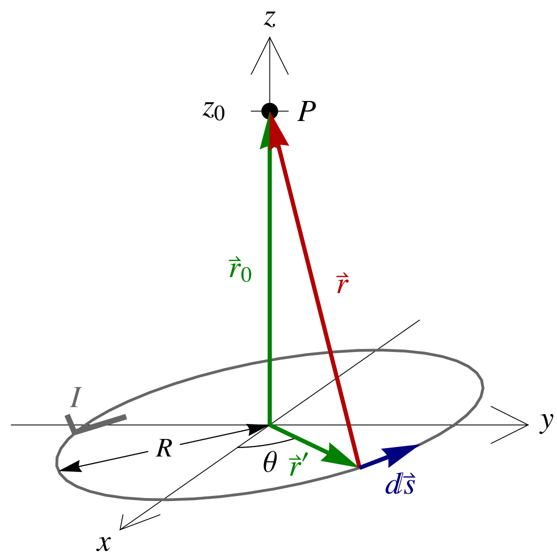
For every point on the arc we have $r = R = \text{const.}$ giving:

$$\vec{B} = \hat{z} \frac{\mu_0 I}{4\pi} \frac{1}{R^2} \int d\,s.$$

The integral is just the total arc length $\int d\,s = R\,\theta$ giving

$$\vec{B} = \hat{z} \frac{\mu_0 I}{4\pi R} \theta.$$

Field at a perpendicular distance z_0 from the center of a circle



Now consider a full circle of radius R in the xy-plane with the center at the origin and a counterclockwise current. Choose the point P to be at z_0 along the positive z-axis.

Integrate around the circle by varying θ from 0 to 2π. The position as a function of θ is given by the vector $\vec{r}'$.

$$\vec{r}' = \langle R\cos\theta,\ R\sin\theta,\ 0\rangle$$

The $d\vec{s}$ is the infinitesimal change in this vector under an infinitesimal change in angle, $d\theta$.

$$d\vec{s} = d\vec{r}' = \langle -R\sin\theta\, d\theta,\ R\cos\theta\, d\theta,\ 0\rangle$$

The vector $\vec{r}$ is the vector from $d\vec{s}$, which is at $\vec{r}'$, to P which is at

$$\vec{r}_0 = \langle 0,\ 0,\ z_0\rangle.$$

This gives

$$\vec{r} = \vec{r}_0 - \vec{r}' = \langle -R\cos\theta,\ -R\sin\theta,\ z_0\rangle.$$

By the Pythagorean theorem, the magnitude of $\vec{r}$ is just

$$r = \sqrt{R^2 + z_0^2}\ .$$

Using $\frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3}$ we get an expression for the field.

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{s} \times \vec{r}}{r^3}$$

The cross product can now be explicitly evaluated using the determinant method

$$\begin{aligned} d\vec{s} \times \vec{r} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -R\sin\theta\, d\theta & R\cos\theta\, d\theta & 0 \\ -R\cos\theta & -R\sin\theta & z_0 \end{vmatrix} \\ &= \hat{x}\,(z_0\, R\cos\theta\, d\theta - 0) \\ &\quad -\hat{y}\,(-z_0\, R\sin\theta\, d\theta - 0) \\ &\quad +\hat{z}\left(R^2\sin^2\theta\, d\theta -- R^2\cos^2\theta\, d\theta\right) \\ &= \left\langle z_0\, R\cos\theta,\ z_0\, R\sin\theta,\ R^2\right\rangle d\theta \end{aligned}$$

Since r is a constant we can bring the $\frac{1}{r^3}$ term out of the integral giving

$$\vec{B} = \frac{\mu_0 I}{4\pi} \frac{1}{(R^2 + z_0^2)^{3/2}} \int_0^{2\pi} \left\langle z_0\, R\cos\theta,\ z_0\, R\sin\theta,\ R^2\right\rangle d\theta.$$

This gives three simple integrals.

$$\int_0^{2\pi} z_0\, R\cos\theta\, d\theta = 0$$

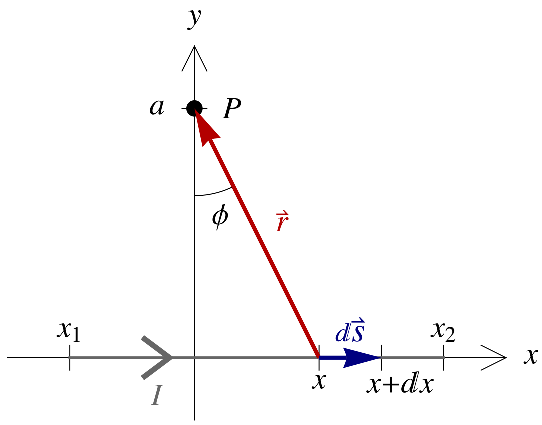
$$\int_0^{2\pi} z_0 \, R \sin\theta \, d\theta = 0$$

$$\int_0^{2\pi} R^2 \, d\theta = 2\pi R^2$$

The final result can, finally, be written

$$\vec{B} = \hat{z} \, \frac{\mu_0 I}{2} \, \frac{R^2}{(R^2 + z_0^2)^{3/2}}.$$

Field due to a line segment



For some straight-line segment choose the x-direction to be the direction of the current and take the segment to be between $x_1$and x_2. The point P is on the y-axis at $y = a$, where $x = 0$ is the point on the line closest to P. The vector $d\vec{s}$ is the vector from x to $x + dx$

$$d\vec{s} = \hat{x} \, dx$$

And the vector $\vec{r}$ is from the $d\vec{s}$ to P.

$$\vec{r} = -x\,\hat{x} + a\,\hat{y}$$

The magnetic field at P is

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{s} \times \vec{r}}{r^3}$$

Evaluating the cross product

$$d\vec{s} \times \vec{r} = \hat{x} \, dx \times (-x\,\hat{x} + a\,\hat{y}) = \hat{z} \, a \, dx$$

and using $r = \sqrt{x^2 + a^2}$ gives

$$\vec{B} = \hat{z} \, a \, \frac{\mu_0 I}{4\pi} \int_{x_1}^{x_2} \frac{dx}{(x^2 + a^2)^{3/2}}.$$

This can be evaluated using a trig substitution. Define the angle ϕ as shown.

The substitution is

$$x = a \tan\phi.$$

The differential becomes

$$dx = \frac{a}{\cos^2\phi} \, d\phi$$

and r becomes

$$r = \sqrt{x^2 + a^2} = \frac{a}{\cos\phi} \implies \frac{1}{(x^2 + a^2)^{3/2}} = \frac{\cos^3\phi}{a^3}$$

Define ϕ_1 and ϕ_2 as the ϕ values corresponding to $x_1$and x_2

$$\vec{B} = \hat{z}\, a \frac{\mu_0 I}{4\pi} \int_{\phi_1}^{\phi_2} \frac{\cos^3\phi}{a^3} \frac{a}{\cos^2\phi} d\phi = \hat{z} \frac{\mu_0 I}{4\pi a} \int_{\phi_1}^{\phi_2} \cos\phi\, d\phi.$$

This gives the final result

$$\vec{B} = \hat{z} \frac{\mu_0 I}{4\pi a} (\sin\phi_2 - \sin\phi_1).$$

This can also be written in terms of the original x variables

$$\vec{B} = \hat{z} \frac{\mu_0 I}{4\pi a} \left(\frac{x_2}{\sqrt{x_2^2 + a^2}} - \frac{x_1}{\sqrt{x_1^2 + a^2}} \right).$$

Note that with the choice of ϕ given here that a negative x value corresponds to a negative ϕ value.

Field of a long straight wire

An important special case of this is that of a long straight wire.

$$x_1 \to -\infty \text{ and } x_2 \to \infty \implies \phi_1 \to -\frac{\pi}{2} = -90° \text{ and } \phi_2 \to \frac{\pi}{2} = 90°$$

$$(\sin\phi_2 - \sin\phi_1) \to 1 - (-1) = 2.$$

The field magnitude a distance r from a long straight wire becomes

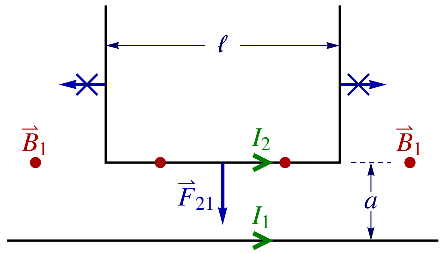
$$B = \frac{\mu_0 I}{2\pi r}.$$

To get the direction of the field for a long straight wire, or for that matter for a segment, put the thumb of your right hand in the direction of the current. The field circulates around the wire in the direction given by your fingers.

F.5 - Magnetic Forces, μ_0 and the Definition of the Coulomb

We have not yet assigned a value to the constant μ_0. We also haven't given a definition of the Coulomb. The Coulomb will be defined in terms of the Ampere, $1\,\text{C} = 1\,\text{A}\cdot\text{s}$, and then the Ampere's definition will be established when we assign a value to the constant μ_0. To gain a physical understanding of these definitions we will consider the magnetic force between parallel wires and derive a magnetic analog to Coulomb's Law.

Forces between Parallel Wires



Consider a long wire with current I_1 and a parallel segment of length ℓ a distance a from the long wire. The parallel segment will have a current I_2; its current will be supplied by perpendicular wires coming from infinity. The magnetic forces on these perpendicular segments will cancel and the net force will just be the force on the segment. To find this force takes two steps: Define $\vec{B}_1$ to be the field due to I_1 at I_2 and then define $\vec{F}_{21}$ as the force on I_2 due to $\vec{B}_1$.

$$B_1 = \frac{\mu_0 I_1}{2\pi a}$$

The direction of $\vec{B}_1$ is out of the page. We can then find the magnetic force.

$$\vec{F}_{21} = I_2 \vec{\ell} \times \vec{B}_1$$

The direction of this force is toward the other wire and its magnitude is

$$F_{21} = I_2 \, \ell \, B_1 = \frac{\mu_0}{2\pi} I_1 I_2 \frac{\ell}{a}.$$

We can make a general statement about magnetic forces between currents. Parallel currents attract and anti-parallel currents repel.

μ_0 and the Ampere

In the expression for the force both ℓ and a are lengths; it follows that the SI units of μ_0 are N/A^2. It should now be clear that redefining an Ampere will change the numerical value of μ_0; assigning a value to it will then provide a definition of the Ampere. The constant μ_0 could be removed completely by defining its value to be 1 but for historical reasons we choose differently. The value of μ_0 is

$$\mu_0 = 4\pi \times 10^{-7} \frac{\mathrm{N}}{\mathrm{A}^2}.$$

If an experiment were set up with the arrangement above, then it could be used to explicitly calibrate an ammeter. If the two currents are forced to be equal $I = I_1 = I_2$, then measuring the force and using the experiment's values for ℓ and a would give the current in Amperes.

F.6 - Ampere's Law

Ampere's law is mathematically equivalent to the Biot-Savart law for magneto-statics, where all currents are steady giving constant fields. This equivalence cannot be demonstrated at this level. We will use Ampere's law similarly to Gauss's law. In cases of symmetry we will use it to find magnetic fields from currents.

Ampere's law is

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 \, I_{\text{enclosed}}.$$

The integral is around a closed contour and I_{enclosed} is the total current enclosed by that contour. If the integral is over some closed contour then there are many different surfaces (an infinite number) that have that contour as its boundary. An example is the Earth's equator that has the northern hemisphere, the southern hemisphere and a disk through the Earth's center as different surfaces that share it as their boundaries. The current I is the current piercing any surface that has the contour as its boundary. We can relate the orientation of the boundary to the orientation of the contour (the direction of integration around the contour.)

Cylindrical Symmetry

In any case of cylindrical symmetry choose a circular contour. By symmetry and using Gauss's law for magnetism we get that the field rotates around the axis. This gives

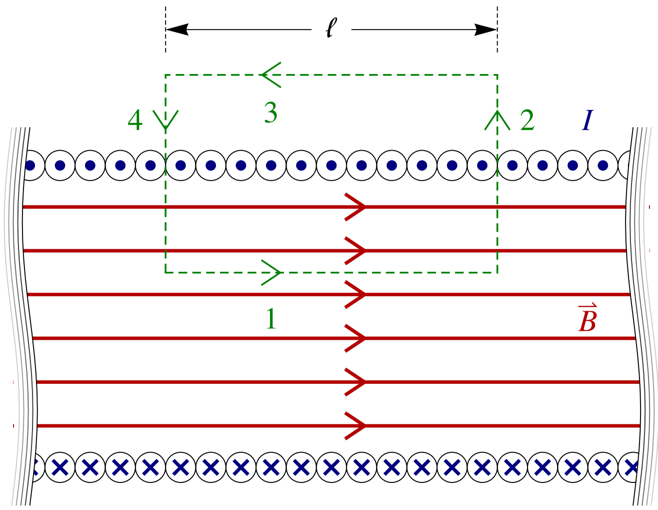
$$\oint \vec{B} \cdot d\vec{r} = 2\pi r B.$$

Inserting this into Ampere's law gives the general expression for cylindrical symmetry

$$B = \frac{\mu_0 \, I_{\text{enclosed}}}{2\pi r}.$$

Note that the field for a long wire is a trivial special case of this.

Long Solenoid



For an (infinitely) long solenoid take the current to be I and the density of turns to be n.

$$n = \frac{\text{\# of turns}}{\text{length}}$$

The field inside the solenoid is uniform and the field outside is zero. (The field outside approaches zero as the length become infinite.) Choose the contour to be four segments as shown

$$\oint \vec{B}\cdot d\vec{r} = \int_1 \vec{B}\cdot d\vec{r} + \int_2 \vec{B}\cdot d\vec{r} + \int_3 \vec{B}\cdot d\vec{r} + \int_4 \vec{B}\cdot d\vec{r}$$
$$= B\,\ell + 0 + 0 + 0$$

There are $n\,\ell$ turns through the contour giving

$$I_{\text{enclosed}} = n\,\ell\,I$$

It follows from Ampere's law that the field anywhere inside a long solenoid is

$$B = \mu_0\,n\,I.$$

F.7 - Current Loops as Magnetic Dipoles

The net force on a current loop in a uniform magnetic field is zero.

$$\vec{F}_{\text{net}} = I\oint d\vec{s}\times\vec{B} = I\left(\oint d\vec{s}\right)\times\vec{B} = \vec{0}$$

The field does affect the loop, though. There is a torque on it.

We saw earlier using electromagnetic duality (the analogy between electric and magnetic fields and charges) that the torque on a magnetic dipole is $\vec{\tau} = \vec{\mu}\times\vec{B}$. In showing there is a torque on a current loop we will demonstrate that a current loop is a magnetic dipole. This is a second source of magnetic dipoles; in addition to permanent magnets being dipoles, we will now see that current loops are dipoles as well.

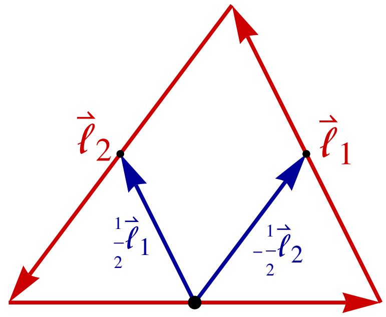
To calculate the dipole moment of a loop or N-turn coil, we will first find the moment of a single triangular loop by calculating the torque on a triangular loop in a uniform field.

A Single Triangular Loop

The definition of torque about an origin due to some force is $\vec{\tau} = \vec{r}\times\vec{F}$, where $\vec{r}$ is the vector from the origin to where the force $\vec{F}$ acts. Consider first a line segment of length ℓ with current I. In finding the force on this segment we took $\vec{\ell}$ to be in the direction of the current and the force is $\vec{F} = I\,\vec{\ell}\times\vec{B}$. The torque on the segment becomes that of the force acting at the midpoint of the segment $\vec{r}_{\text{mid}}$

$$\vec{\tau} = I\,\vec{r}_{\text{mid}}\times\left(\vec{\ell}\times\vec{B}\right).$$

Consider a single triangular loop carrying a current I in a uniform magnetic field $\vec{B}$. Torque depends on one's choice of origin, but whenever the net force vanishes the net torque is independent of the choice of origin. We will choose the origin to be at the center of one side; this removes the contribution of that segment to the torque since $\vec{r}_{\text{mid}} = \vec{0}$. The two sides that do contribute are labeled $\vec{\ell}_1$and $\vec{\ell}_2$, and take their directions to be the direction of the current.



$$\vec{\tau} = I\left(-\frac{1}{2}\vec{\ell}_2\right)\times\left(\vec{\ell}_1\times\vec{B}\right) + I\left(\frac{1}{2}\vec{\ell}_1\right)\times\left(\vec{\ell}_2\times\vec{B}\right) + \vec{0}$$

We can simplify this using an identity satisfied by cross products.

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) + \vec{B}\times\left(\vec{C}\times\vec{A}\right) + \vec{C}\times\left(\vec{A}\times\vec{B}\right) = \vec{0}$$

This identity can be rewritten as

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) - \vec{B}\times\left(\vec{A}\times\vec{C}\right) = \left(\vec{A}\times\vec{B}\right)\times\vec{C}.$$

A bit of algebraic manipulation $\vec{A}\to\vec{\ell}_1$, $\vec{B}\to\vec{\ell}_2$ and $\vec{C}\to\vec{B}$ gives

$$\vec{\tau} = I\left(\frac{1}{2}\vec{\ell}_1\times\vec{\ell}_2\right)\times\vec{B}.$$

Recall that the magnitude of the cross product is the area of a parallelogram. A triangle is half of that. The direction of the cross product is perpendicular to the two vectors and thus to the triangle. It follows that the area vector of the triangle is $\vec{A} = \frac{1}{2}\vec{\ell}_1\times\vec{\ell}_2$. Note that the right-hand rule gives the outward normal as the direction of $\vec{A}$ and the right-hand rule also associates that direction with a counterclockwise current. We can now write torque as $\vec{\tau} = I\,\vec{A}\times\vec{B}$. Since the torque on a magnetic dipole is $\vec{\tau} = \vec{\mu}\times\vec{B}$, we can write the magnetic moment of a single triangular current loop as

$$\vec{\mu} = I\,\vec{A}.$$

Planar Current Loops or Coils with *N* Turns

We can now generalize this result to a general planar loop. Any loop may be broken up into infinitesimal pieces each of area $d\vec{A}$. Summing over all these infinitesimal areas gives $\vec{A} = \int d\vec{A}$. The area vector has a magnitude A and its direction is normal to the loop.

$$\vec{A} = A\,\hat{n}$$

To get the proper direction use a right-hand rule. Wrap your fingers around loop in the direction of the current. The thumb of your right hand points in the direction of the unit normal $\hat{n}$ that gives the direction of $\vec{A}$. The magnetic moment is then

$$\vec{\mu} = I\,\vec{A}. \text{ (Single turn loop)}$$

If there are N turns then each turn contributes $I\,\vec{A}$ to the magnetic moment. The total magnetic dipole moment is

$$\vec{\mu} = N\,I\,\vec{A}. \text{ (N turn coil)}$$

N is the number of turns, I is the current and $\vec{A} = A\,\hat{n}$ is the area vector defined as above.