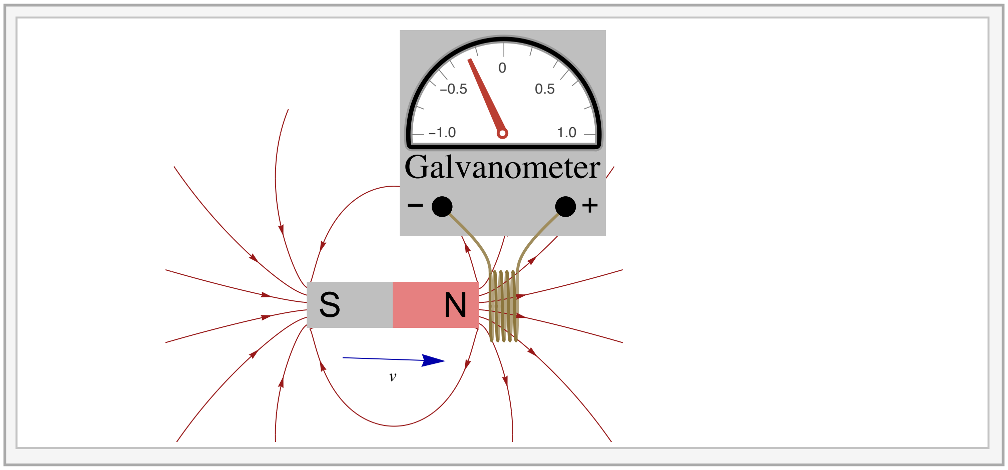
Chapter G

Faraday's Law and Electrodynamics

Blinn College - Physics 2326 - Terry Honan

Faraday's law describes how a generator works. A changing magnetic flux through a conducting loop induces an EMF (Electromotive Force) around the loop. Faraday's law will become our fourth of Maxwell's equations.

G.1 - Faraday's Law



We can demonstrate magnetic induction by moving a magnet toward a coil connected to a galvanometer, where a deflection of the galvanometer's needle indicates a current. Moving the magnet toward the coil induces an EMF in the solenoid, which creates a current in the coil-galvanometer circuit. Pulling the magnet away from the coil induces a current in the opposite direction. If the magnet is at rest in the coil no current is induced. Moreover, increasing the number of magnets or using a stronger magnet increases these effects. It is clear that the induced EMF depends on the change in the flux. If $\mathcal{E}_{\text{ave}}$ is the average induced EMF and Φ is the magnetic flux through each loop of the coil then we get the proportionality

$$\mathcal{E}_{\text{ave}} \propto \Delta\Phi.$$

If the speed of the magnet is increased the effect is enhanced and slowing it diminishes it. This suggests an inverse proportionality with the time.

$$\mathcal{E}_{\text{ave}} \propto \frac{1}{\Delta t}.$$

Since in a coil all the loops are connected in series it follows that $\overline{\mathcal{E}}$ is proportional to the number of loops N.

Combining these proportionalities gives

$$\mathcal{E}_{\text{ave}} \propto N\,\frac{\Delta\Phi}{\Delta t}$$

and it turns out that the proportionality becomes an equality when we consider the magnitude of the average induced EMF $|\overline{\mathcal{E}}|$ and the absolute value of the change in flux with time

$$|\mathcal{E}_{\text{ave}}| = N\left|\frac{\Delta\Phi}{\Delta t}\right|.$$

If we consider the polarity (sign) of the induced EMF then this adds a sign, which is known as Lenz's law.

$$\mathcal{E}_{\text{ave}} = -N\,\frac{\Delta\Phi}{\Delta t}.$$

We will discuss Lenz's law in detail later.

Taking the limit as Δt goes to zero these expressions become

$$|\mathcal{E}| = N\left|\frac{d\Phi}{dt}\right| \text{ and } \mathcal{E} = -N\frac{d\Phi}{dt}.$$

This is analogous to the definitions of average and instantaneous velocity in kinematics where $v_{\text{ave}} = \frac{\Delta x}{\Delta t}$ and $v = \frac{dx}{dt}$.

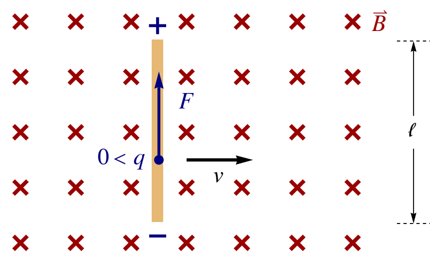
G.2 - Motional EMF

If a conductor moves in a magnetic field there is a magnetic force on the charge carriers. This magnetic force does work on the charge carriers. The EMF is the work per charge.

$$\mathcal{E} = \frac{W}{q}.$$

Translation of a Conducting Rod in a Uniform Field

Consider a conducting rod of length ℓ translating in a uniform magnetic field. Take the rod, its velocity and the field to be mutually perpendicular.



If q is some charge carrier then the force on it is

$$\vec{F} = q\,\vec{v}\times\vec{B}. \implies F = q\,v\,B$$

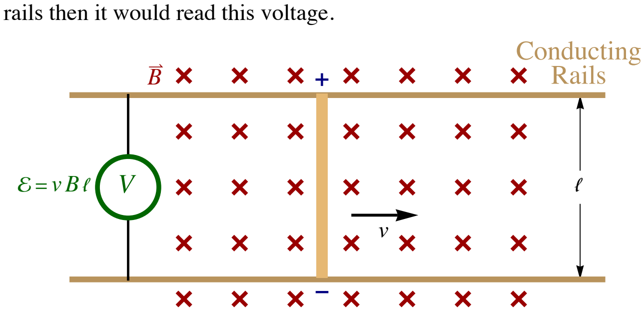
The work is

$$W = \int \vec{F}\cdot d\vec{r} = F\,\ell = q\,v\,B\,\ell.$$

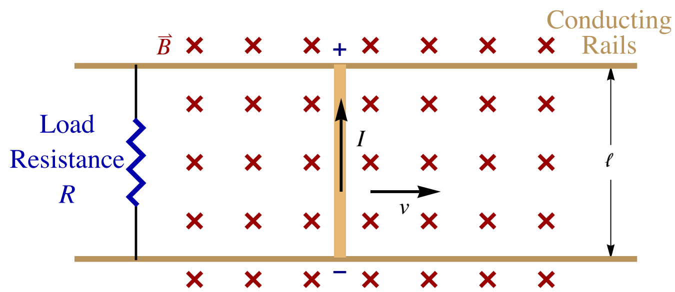
Using $\mathcal{E} = W/q$ gives the induced EMF across the rod

$$\mathcal{E} = v\,B\,\ell$$

How can one measure this EMF? If a voltmeter is connected across the ends of the rod and it is moved with it then the same EMF is induced in the leads to the meter and it reads zero. To avoid this, imagine the rod moving with its ends sliding along a conducting rail. If the voltmeter is connected between the rails then it would read this voltage.



A DC Generator and Conservation of Energy



In fact, what we have here is a simple DC generator. Instead of connecting a voltmeter between the rails we could connect anything and it would be given a steady DC voltage. If a DC motor is connected then this motor could do work. We must address the question of conservation of energy. Where does this energy come from?

To see this place a load resistor R across the conducting rails. Ohm's law gives the current through the load

$$I = \frac{\mathcal{E}}{R}.$$

The rate of power dissipation in the load, which is the power output of the generator, is

$$\mathcal{P}_{\text{out}} = I\,\mathcal{E} = I\,v\,B\,\ell,$$

where we are making the idealizing assumption that all the resistance in the circuit is in the load.

This is a complete circuit, so all the current through the load passes through the rod. A current through a conductor in a magnetic field creates a magnetic force, $\vec{F}_{\text{mag}}$.

$$\vec{F}_{\text{mag}} = I\,\vec{\ell}\times\vec{B} \implies F_{\text{mag}} = I\,\ell\,B$$

To keep the rod moving at a constant speed there must be zero net force, so there must be some external applied force $\vec{F}_{\text{app}}$. Making another idealizing assumption of no friction we get

$$\vec{F}_{\text{app}} = -\vec{F}_{\text{mag}} \implies F_{\text{app}} = F_{\text{mag}} = I\,\ell\,B$$

The applied force does work and this is the source of the energy. The rate that it does work is the power $\mathcal{P}_{\text{in}}$. Power is related to force by

$$\mathcal{P} = \frac{d\,\text{Work}}{d\,t} = \frac{\vec{F}\cdot d\vec{r}}{d\,t} = \vec{F}\cdot\vec{v}$$

so we get in this case

$$\mathcal{P}_{\text{in}} = F_{\text{app}}\,v = I\,\ell\,B\,v.$$

Comparing this with the power output gives

$$\mathcal{P}_{\text{in}} = \mathcal{P}_{\text{out}}.$$

If we relax our idealizing assumptions and include mechanical friction and resistance in the circuit elsewhere than the load then we get

$$\mathcal{P}_{\text{in}} > \mathcal{P}_{\text{out}}.$$

Example G.1 - Motional EMF

A conducting rod slides with a speed of 18 m/s along parallel horizontal conducting rails separated by 1.2 m. Suppose there is a downward magnetic field of 15 mT.

(a) What is the voltage across the rod?

Solution

Using the values $v = 18$ m/s, $\ell = 1.2$ m and $B = 15\times10^{-3}$ T we get

$$\mathcal{E} = v\,B\,\ell = 0.324\text{ V}$$

(b) Which side of the rod, left or right (relative to the velocity) of the rod, is at higher potential?

Solution

We need to find the force on a positive charge moving with the rod. With your fingers down and thumb forward, your palm points to the left. If positive charges are pushed to the left then that is the side at higher potential.

(c) Suppose a load resistance placed across the rails to complete the circuit as a DC generator. If there is a 5.5 mA current flowing, then what is the load resistance.

Solution

When you complete the circuit with a resistance then we have, using the EMF as a voltage and using the current as $I = 5.5 \times 10^{-3}$ A.

$$V = \mathcal{E} = I\,R \implies R = \frac{\mathcal{E}}{I} = 58.9\,\Omega$$

(d) What is the backward magnetic force on the rod after the load resistance is added?

Solution

The magnetic force on the rod, now that there is a current, is: $F = I\,\ell\,B = 9.9 \times 10^{-5}$ N

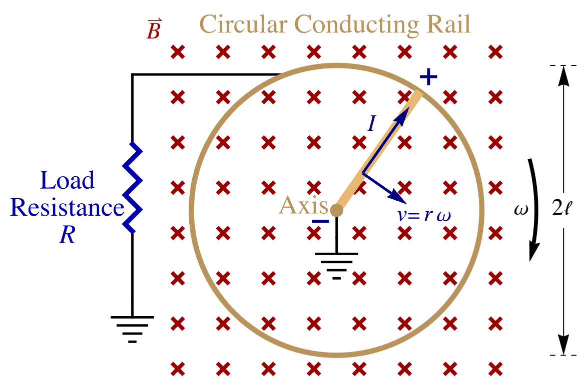
(e) What is the rate of power dissipation in this circuit?

Solution

$$\mathcal{P} = \mathcal{P}_{\text{out}} = I\,\mathcal{E} = 1.78 \times 10^{-3}\ \text{W}$$

Rotation of a Conducting Rod in a Uniform Field - The DC Generator

The above describes a crude DC generator, but it is not a practical device because it requires an infinite track and an infinitely long region of uniform field. To make a more practical generator consider a rod rotating in a uniform field. Take the rod to be of length ℓ rotating with an angular velocity ω in a uniform field B.



A rod rotates while sliding along a horizontal conducting rail. It is connected across a load resistance The axis and the far end of the load resistance are connected to a ground; this means that they are internally connected and are at the same electric potential, which we usually take to be zero.

The velocity varies with r, the distance from the center by

$$v = r\,\omega.$$

The force on a charge carrier becomes

$$\vec{F} = q\,\vec{v} \times \vec{B}. \implies F = q\,v\,B = q\,r\,\omega\,B$$

and the work is

$$W = \int \vec{F} \cdot d\vec{r} = \int_0^{\ell} F\,d\,r = q\,\omega\,B \int_0^{\ell} r\,d\,r = \frac{1}{2}\,q\,\omega\,B\,\ell^2.$$

$\mathcal{E} = W / q$ gives the EMF

$$\mathcal{E} = \frac{1}{2}\,\omega\,B\,\ell^2.$$

This is a DC generator. If a bicycle wheel with N spokes rotates similarly in a magnetic field the same formula applies since voltage sources are connected in parallel. A rotating disk is also a DC generator where ℓ becomes the disk radius; this may be viewed as the limit of an infinite number of spokes.

Example G.2 - Rotating Conducting Disk

A horizontal conducting disk of radius R rotates clockwise (when viewed from above) with an angular velocity ω in a uniform upward magnetic field of magnitude B.

(a) What is the magnitude of the potential difference between the center of the disk, its axis, and its rim?

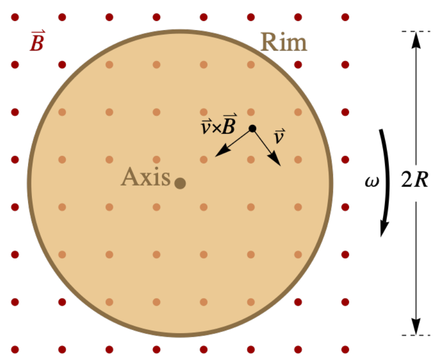
Solution

As mentioned above, the disk can be viewed as a wheel with many spokes where the number of spokes goes to infinity. We then use the formula for a rod replacing ℓ with the radius R. The voltage, the magnitude of the potential difference, is the EMF $\mathcal{E}$.

$$V = \frac{1}{2}\,\omega\,B\,R^2.$$

(b) Which is at higher potential the rim of the center?

Solution



The figure above represents a top view of the arrangement. Choose some point on the disk and draw the velocity vector. Using the right-hand rule find the direction of $\vec{v}\times\vec{B}$; this is inward. Since the force on a charge Q is $\vec{F} = Q\,\vec{v}\times\vec{B}$, then positive charges are pushed toward the center. The center becomes positively charged and thus is at higher potential.

G.3 - Faraday's Law Examples

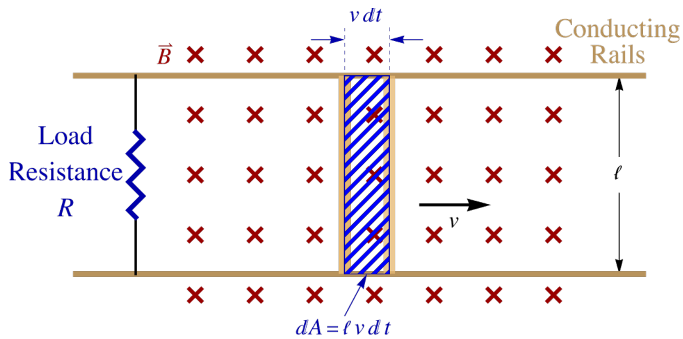
Usually when we use Faraday's law we will only consider the magnitude of the induced EMF or current. We will see how to find the polarity in the Lenz's law discussion that follows. We can neglect polarity information by inserting absolute values into Faraday's law.

$$|\mathcal{E}_{\text{ave}}| = N\left|\frac{\Delta\Phi}{\Delta t}\right|. \text{ and } |\mathcal{E}| = N\left|\frac{d\Phi}{dt}\right|$$

If one is given some problem involving induced currents then this is related to the induced EMF by Ohm's law

$$\mathcal{E} = I\,R$$

The Translating Rod



There is an equivalence between motional EMF and Faraday's law. For the case of the translating rod that we analyzed above using motional EMF, we ought to be able to derive the same results using Faraday's law directly. If a load resistor is placed in the circuit then the complete circuit consists of the rod, rails and load. As the rod moves the area inside the circuit loop is increasing and so is the flux. In the time dt the rod moves by $v\,dt$ and the area increases by

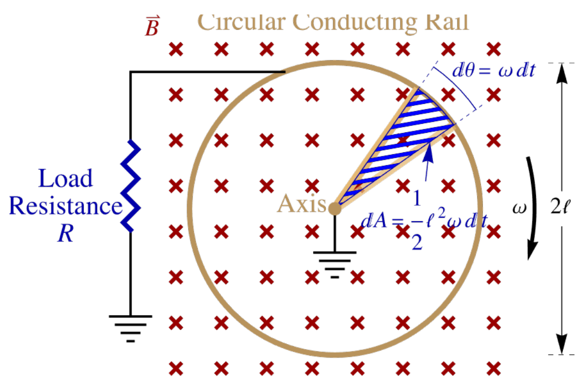
$$dA = \ell\, v\, dt$$

The flux, since the field is uniform and perpendicular to the surface, is $\Phi = B\,A$ and since the field is constant we get $\frac{d\Phi}{dt} = B\,\frac{dA}{dt}$. The induced EMF becomes

$$|\mathcal{E}| = \left|\frac{d\Phi}{dt}\right| = B\,\frac{dA}{dt} = B\,\ell\,v$$

which is the same as the expression derived using motional EMF considerations.

The Rotating Rod



We can similarly derive the expression for a rotating rod. This is the same as the translating rod except the expression for the dA becomes

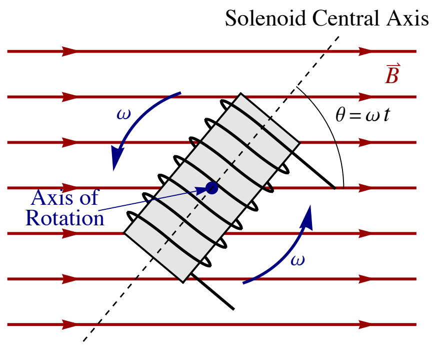
$$dA = \frac{1}{2}\,\ell^2\,\omega\,dt.$$

To get this we have used the area formula for an arc $A = \pi R^2\,\frac{\theta}{2\pi} = \frac{1}{2}\,R^2\,\theta$ and that the angle subtended by the rod $d\theta$ in time dt is given by $d\theta = \omega\,dt$. The induced EMF can be found

$$|\mathcal{E}| = \left|\frac{d\Phi}{dt}\right| = B\,\frac{dA}{dt} = \frac{1}{2}\,B\,\ell^2\,\omega$$

which is the same as the previously derived expression.

The AC Generator



Consider a solenoid or coil with its central axis rotating with angular velocity ω in a uniform magnetic field of magnitude B. Take the central axis to be perpendicular to the rotational axis and the rotational axis to be perpendicular to the field. The coil has N turns and each loop has a cross-sectional area A. Take the angle between the axis of the coil, the normal to each loop, and the field to be θ. This varies by

$$\theta = \omega\, t$$

The flux through a loop is

$$\Phi = \vec{B} \cdot \vec{A} = B\, A \cos\theta = B\, A \cos \omega\, t.$$

By Faraday's law $\mathcal{E} = -N \frac{d\Phi}{dt}$ we get the induced EMF as a function of time to be

$$\mathcal{E}(t) = N\, B\, A\, \omega \sin \omega\, t.$$

This is an AC voltage. We will discuss AC in detail in the AC circuit chapter. We will state for future reference that the peak EMF is

$$\mathcal{E}_{\text{max}} = N\, B\, A\, \omega$$

The frequency f of AC is the same as the rotational frequency which is related to the angular velocity by

$$\omega = 2\, \pi\, f$$

Example G.3 - Faraday's Law with a Rotating Coil

A flat circular coil has 20 turns, a radius of 12 cm and sits in a vertical magnetic field of magnitude 30 mT. The coil is initially in a horizontal plane and is rotated to a vertical plane in 2.3 s.

(a) What is the average induced EMF in the coil during the rotation.

Solution

The radius allows us to find the cross-sectional area of each loop. We also know the number of turns N, the field and Δt, the time.

$$N = 20,\ B = 0.030\ \text{T},\ \Delta t = 2.3\ \text{s and } r = 0.12\ \text{m} \implies A = \pi\, r^2 = 0.04544\ \text{m}^2$$

The flux is $\Phi = \vec{B} \cdot \vec{A} = B\, A \cos\theta$, where initially the normal is aligned with the field, $\theta_i = 0$, and at the end the normal is perpendicular to the field, $\theta_f = 90\,°$.

$$\Phi_i = B\, A \cos 0 = B\, A \ \text{ and } \ \Phi_f = B\, A \cos 90\,° = 0$$

We can now find the magnitude of the average induced EMF using Faraday's law.

$$|\mathcal{E}_{\text{ave}}| = N \left| \frac{\Delta\Phi}{\Delta t} \right| = N \frac{|\Delta\Phi|}{\Delta t} = N \frac{|\Phi_f - \Phi_i|}{\Delta t} = N \frac{B\, A}{\Delta t} = 0.0118\ \text{V}$$

(b) If the two ends of the coil are shorted (connected) and the coil has a total resistance of 85 mΩ, then what is the average induced current during the rotation.

Solution

Given the resistance then Ohm's law gives the current. The EMF is just a voltage.

$$R = 0.085\ \Omega \ \text{ and } \ \mathcal{E} = I\, R \implies I = \mathcal{E}/R = 0.138\ \text{A}$$

Example G.4 - Faraday's Law with a Varying Field

A solenoid with a vertical central axis has 90 turns, a length of 30 cm and a radius of 5 cm. It sits in an upward vertical magnetic field that decreases at a rate of 35 T/s. (That is: $dB_z/dt = -35\,\mathrm{T/s}$) What is the magnitude of the induced EMF between the terminals of the solenoid.

Solution

The length of the solenoid is unimportant. We know the rate of change of the field, the number of turns and the radius gives us the cross-sectional area of each loop.

$$|dB_z/dt| = 35\,\mathrm{T/s} \quad N = 90 \quad \text{and} \quad r = 0.05\,\mathrm{m} \implies A = \pi r^2 = 0.007854\,\mathrm{m}^2$$

Faraday's Law gives us the magnitude of the field.

$$|\mathcal{E}| = N\left|\frac{d\Phi}{dt}\right| = N\left|\frac{dB}{dt}\right|A = 24.7\,\mathrm{V}$$

G.4 - Lenz's Law

The minus sign in Faraday's law is known as Lenz's law. It gives the polarity of the induced EMF. If the circuit is completed with a load resistance the induced EMF causes an induced current. The induced current induces a magnetic field $\vec{B}_{\text{ind}}$ which in turn induces a flux Φ_{ind} through the loop.

- The induced flux tends to cancel the change in the flux.
- This cancellation is exact for a superconductor. The magnetic flux through a superconducting loop is a constant.

Using Lenz's Law

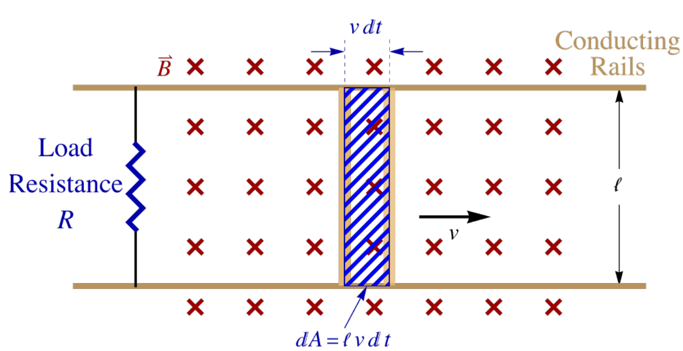
A simple procedure for applying Lenz's law follows. Identify a sign convention and then make a table consisting of the appropriate signs for Φ, $\frac{d\Phi}{dt}$, Φ_{ind} and give the sense of circulation for $\mathcal{E}$ or I. Flux is a scalar but it has a sign; we can give flux a sign relative to the direction of the positive unit normal. The first step is to identify the circuit loop and choose a positive normal direction relative to the loop. Make a table with the first three columns having signs and the fourth having a sense of circulation.

- The sign of Φ is the direction of the field through the loop. If the field is in the same direction as the normal it is positive and negative otherwise.
- The sign of $d\Phi/dt$ is the same as Φ when the flux is increasing, and opposite to Φ when the flux is decreasing.
- By Lenz's law, the sign of Φ_{ind} is always opposite to that of $d\Phi/dt$.
- To get the direction of the induced emf and current, $\mathcal{E}$ and I, use the right-hand rule to determine which sense current gives the induced flux.

Example G.5 - Polarity of Motional EMF

When we used Faraday's law to find the EMF of the translating rod, we only found its magnitude. In the motional EMF analysis of the same problem we saw we induced a counterclockwise voltage and current. Use Lenz's law to get this same result.

Solution

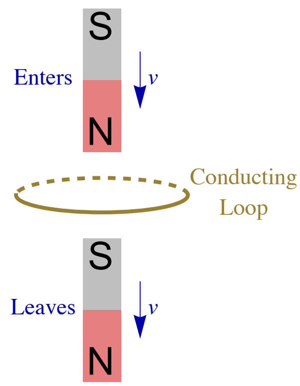


Follow the procedure outlined above. Here we will choose the positive direction to be out of the page and this corresponds to a counterclockwise sense of rotation. Since the field is opposite the normal we have the flux as negative. The flux is increasing (more field lines pass through with time) so the sign of the derivative of the flux is the same as the flux, negative. (it is becoming increasingly negative.) The induced flux is always opposite the change in the flux, so it is positive. That corresponds to a counterclockwise current and EMF.

Positive normal	Sign of Φ	Sign of $d\Phi/dt$	Sign of Φ_{ind}	Sense of $\mathcal{E}$ or I
$\bullet$	$-$	$-$	$+$	

Example G.6 - Magnets Falling through Conducting Loops

(a) A magnet with its north pole at the bottom is dropped through a horizontal conducting loop. What is the sense of the induced current as the magnet enters and then leaves? Answer clockwise or counterclockwise as viewed from above.



Solution

The field of the magnet points away from the north pole and toward the south pole. Choose the positive normal to be upwards.

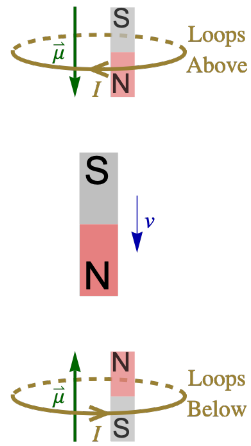
As the magnet enters the field is downward (away from the north pole) so the flux will have a negative sign. The flux is increasing as the magnet enters, since more lines pass through with time; this means that the sign of $d\Phi/dt$ is the same and negative. (The flux is becoming increasingly negative.) The induced flux, by Lenz's law, is always opposite $d\Phi/dt$, so it is positive and this corresponds to a counterclockwise induced current.

As the magnet leaves the field is downward (toward the south pole) as well, pointing toward the south pole. Here the number of field lines is decreasing so $d\Phi/dt$ is opposite Φ and positive. This gives a negative induced flux and a clockwise induced current.

Positive normal		Sign of Φ	Sign of $d\Phi/dt$	Sign of Φ_{ind}	Sense of $\mathcal{E}$ or I
$\uparrow$	Enters	$-$	$-$	$+$	counterclockwise
	Leaves	$-$	$+$	$-$	clockwise

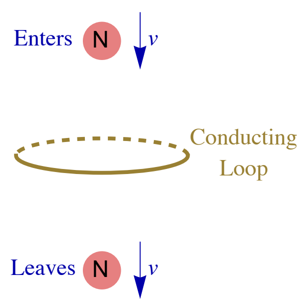
(b) A magnet with its north pole at the bottom is dropped through a long vertical aluminum (conducting but not magnetic) pipe. The falling magnets induce circular eddy currents in the pipe and energy is lost to the Joule-heating of the currents, slowing the magnet's descent through the aluminum tube. Identify the induced magnetic moments from the loops above and below the falling magnetic when inside the tube.

Solution



From the analysis in part (a) we can identify the induced currents and then the induced magnetic moments due to the falling magnet. Loops below the magnet correspond to the case where the magnet enters the loop in part (a). The induced current is counterclockwise and the corresponding magnetic moment is, by the right-hand rule, upwards; since an upward magnetic moment is equivalent to a magnet with the north pole on the top, we can see that induced magnetic moments of the loops below push upward on the falling magnet. For the loops above, this now corresponds to the case where the magnet leaves the loop. Now the induced current is clockwise giving a downward magnetic moment which is equivalent to a bar magnet with the north pole on the bottom; this induced magnetic moment will then pull upwards on the falling magnet below. It is now clear that all the induced currents slow the falling magnet. This must be the case; energy is being lost to Joule heating.

(c) Suppose instead that an isolated north pole were discovered. How would the results in part (a) change is an isolated north pole passed through.



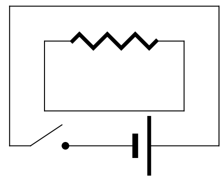
Solution

The case of the pole entering the loop is the same as when the bar magnet enters. As the pole leaves the field is now upwards and decreasing; this gives a positive induced flux and a counterclockwise induced current.

Positive normal		Sign of Φ	Sign of $d\Phi/dt$	Sign of Φ_{ind}	Sense of $\mathcal{E}$ or I
↑	Enters	−	−	+	counterclockwise
	Leaves	+	−	+	counterclockwise

This shows an experiment to look for isolated magnetic poles. A superconducting loop is monitored for currents. If a spontaneous current is set up in the loop then that is evidence that an isolate poles passed through the loop. When a dipole passes through the induced currents cancel giving no net effect.

Example G.7 - Induced Current around an Inner Loop Due to a Changing Current in an Outer Loop



While the switch is closed there is a steady counterclockwise current in the outer loop and this creates a magnetic field, by the right-hand rule, that is out of the page. This steady current creates a steady flux through the inner loop and thus, doesn't induce a current in the inner loop. But when the switch is closed, at that instant, there is an abrupt increase in the current and when the switch is opened there is an abrupt drop in the current at that instant.

What is the direction of the induced current through the resistor when the switch is closed and then when the switch is opened?

Solution

The clockwise current in the outer loop creates an outward field in the inner loop, so choosing the positive normal to be outward then the flux is positive in both cases. When the switch is closed, the current increases so the flux increases and when the switch is opened the current decreases and the flux decreases.

Positive normal		Sign of Φ	Sign of $d\Phi/dt$	Sign of Φ_{ind}	Sense of $\mathcal{E}$ or I
$\bullet$	Closed	+	+	−	clockwise $\Longrightarrow$ Current to the right through R
	Opened	+	−	+	counterclockwise $\Longrightarrow$ Current to the left through R

G.5 - Maxwell's Equations

EMF and Induced Electric Fields

The induced EMF can be related to an induced electric field. The electric field pushes the charge carriers in the conductor around the circuit. The work done on a charge carrier of charge q is

$$W = \int \vec{F} \cdot d\vec{r} = q \int \vec{E} \cdot d\vec{r}.$$

This is the energy gain for each charge carrier. The EMF is defined as the energy gain per charge carrier $\mathcal{E} = W/q$, allowing us to write the induced EMF in terms of the induced electric field.

$$\mathcal{E} = \int \vec{E} \cdot d\vec{r}.$$

Faraday's law can now be written as

$$\oint \vec{E} \cdot d\vec{r} = -\frac{d\Phi_m}{dt}.$$

This is the fourth of Maxwell's equations. The m subscript on the flux has been included to emphasize that we are dealing with the magnetic flux. The other form of Faraday's applied to a conducting loop, where this applies whether or not there is a conductor. A changing magnetic flux generally induces an electric field, if there is a conducting wire then this electric field causes an EMF in the wire. The integrals over electric fields along a line are analogous to the integrals over magnetic fields in Ampere's law. We choose a closed contour for integration and Φ_m is the magnetic flux through any surface that has the contour as its boundary.

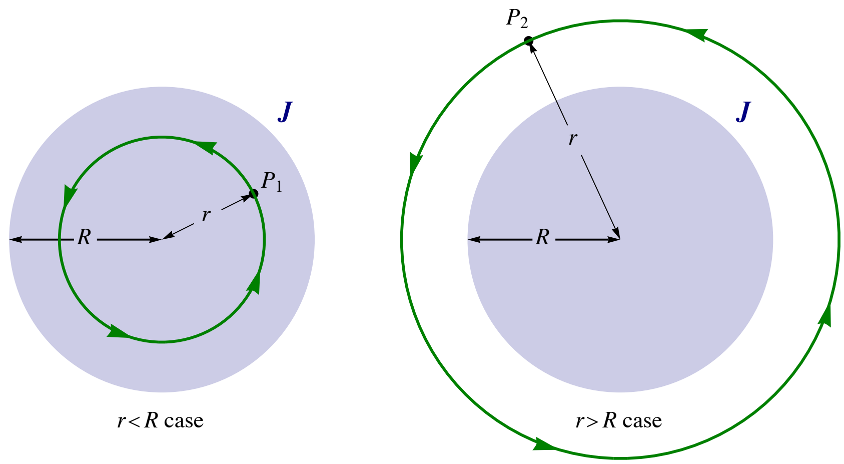
Note that the N has been omitted in the above expression. It is implied that for a wire we integrate along the entire wire; thus if the wire circulates N times, we integrate around the loop N times.

Example G.8 - Induced Electric Field from Faraday's Law (and Ampere's Law Analogy)

The left-hand side of our rewritten Faraday's law is the integral of the electric field around a closed loop. In Ampere's law the left-hand side was the integral of a magnetic field around a closed loop. Before stating the Faraday's law problem let us begin with an analogous Ampere's law problem.

(a) A long wire with a circular cross-section of radius R has a uniform outward current density J. What is the magnetic field as a function of r? Include answers for $r < R$ and $r > R$.

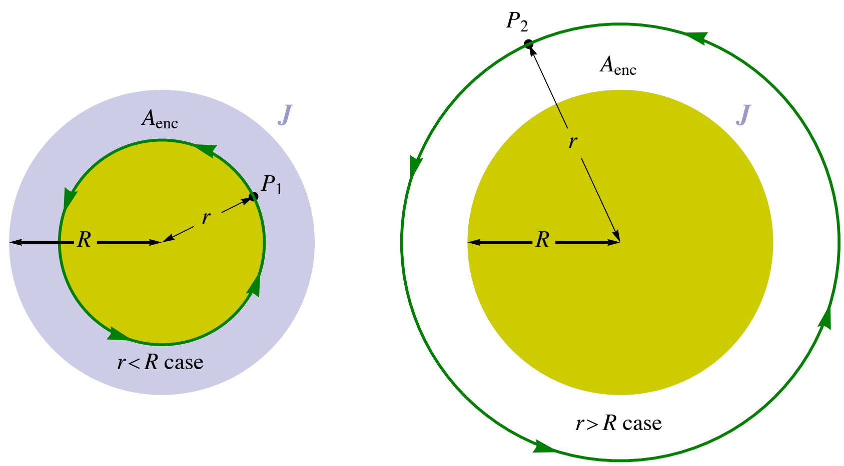
Solution



Contours for the two cases: $r < R$ and $r > R$.

Using Ampere's law and the cylindrical symmetry we can find an expression for the magnetic field in terms of I_{enclosed}.

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 \, I_{\text{enclosed}} \implies B \, 2\,\pi\, r = \mu_0 \, I_{\text{enclosed}} \implies B = \frac{\mu_0 \, I_{\text{enclosed}}}{2\,\pi\, r}$$



For both cases $I_{\text{enclosed}} = J\,A_{\text{enclosed}}$

For both cases we can write $I_{\text{enclosed}} = J\,A_{\text{enclosed}}$.

$$I_{\text{enclosed}} = J\,A_{\text{enclosed}} \implies B = \frac{\mu_0 \, J\, A_{\text{enclosed}}}{2\,\pi\, r}$$

For $r < R$:

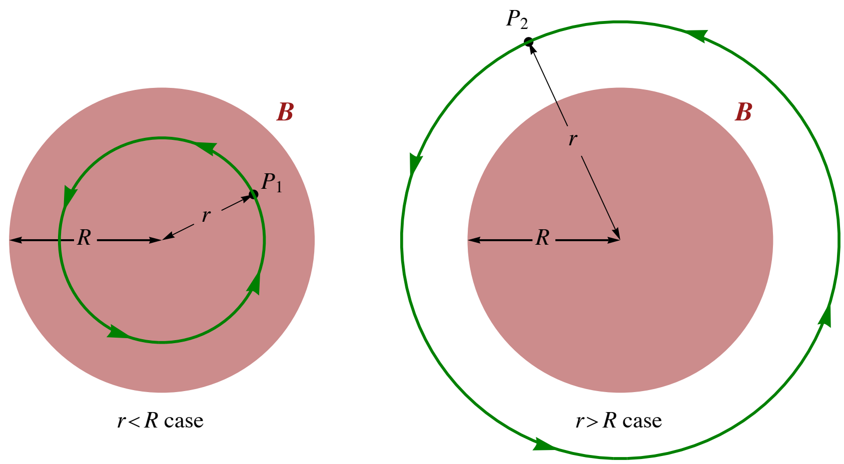
$$A_{\text{enclosed}} = \pi\, r^2 \implies B = \frac{\mu_0 \, J\, r}{2}$$

For $r > R$:

$$A_{\text{enclosed}} = \pi\, R^2 \implies B = \frac{\mu_0 \, J\, R^2}{2\, r}$$

(b) A long solenoid with a circular cross-section of radius R has a uniform outward magnetic field B, that varies as dB/dt. What is the electric field as a function of r? Include answers for $r < R$ and $r > R$.

Solution



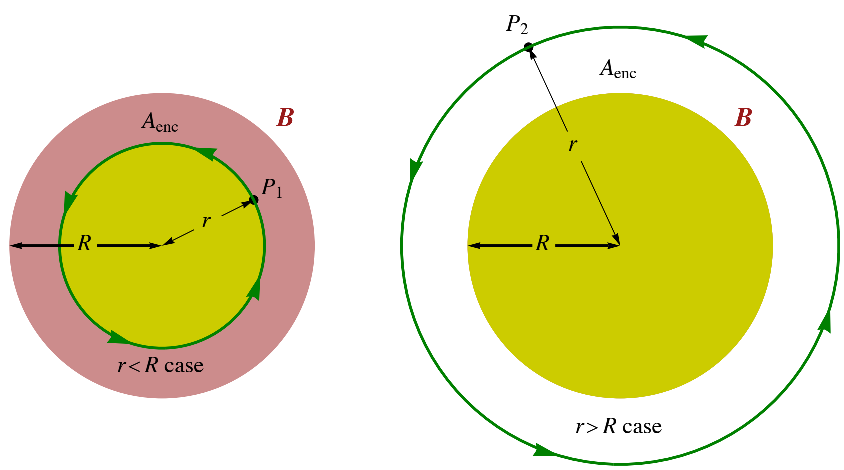
Contours for the two cases: $r < R$ and $r > R$.

Now we also have cylindrical symmetry but with Faraday's law and similarly we can find the electric field. The magnetic flux through the contour is

$$\Phi_m = A_{\text{enclosed}}\, B$$

where A_{enclosed} is the area of the magnetic field inside the contour. Exploiting the cylindrical symmetry we can find an expression for the electric field in terms of the enclosed area, A_{enclosed}.

$$\oint \vec{E} \cdot d\vec{r} = -\frac{d\Phi_m}{dt} \implies E\, 2\pi r = -A_{\text{enclosed}} \frac{dB}{dt} \implies E = -\frac{A_{\text{enclosed}}}{2\pi r} \frac{dB}{dt}$$



For $r < R$:

$$A_{\text{enclosed}} = \pi r^2 \implies E = -\frac{r}{2} \frac{dB}{dt}$$

For $r > R$:

$$A_{\text{enclosed}} = \pi R^2 \implies E = -\frac{R^2}{2r} \frac{dB}{dt}$$

Summary of Electromagnetism

We have been gradually building up our list of Maxwell's equations. We started with electrostatics, where charges were not allowed to move. Gauss's law

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

was introduced as being mathematically equivalent to Coulomb's law, where Coulomb's law refers to the inverse square law for electric fields. Coulomb's law is not correct beyond electrostatics but Gauss's law is. If charges are allowed to move then Coulomb's law (the inverse square law) implies that the effect of moving a charge is felt instantaneously at large distances. This is known as action at a distance and is not present in electrodynamics.

Next we considered magnetostatics, where we allow for electric currents but assume all currents are constant so that all magnetic fields are constant. Gauss's law for magnetism was introduced as a mathematical statement of the absence of isolated magnetic poles.

$$\oint \vec{B} \cdot d\vec{A} = 0$$

Just as with its electrical counterpart, the assumption of statics is not needed. Ampere's law

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 \, I_{\text{enclosed}}$$

was introduced in the context of magnetostatics. Maxwell realized that this was not correct if the assumption of statics is relaxed and showed that this deficiency can be remedied by the introduction of an extra term to Ampere's law. This will be discussed in the next section.

Gauss's law involves only electric fields. Gauss's law for magnetism and Ampere's law involve only magnetic fields. Faraday's law

$$\oint \vec{E} \cdot d\vec{r} = -\frac{d\Phi_m}{dt}$$

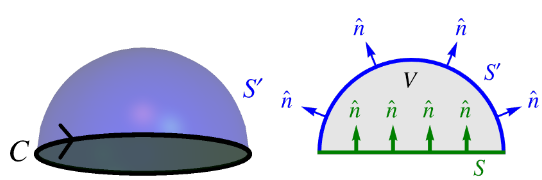
is genuinely electromagnetic, meaning that it involves both fields. It is also electrodynamic; this means that no assumptions of statics are made and, in fact, the induction effect requires dynamical (changing) fields.

Maxwell's Displacement Current Addition to Ampere's Law

If C is any closed contour and S is some surface with C as its boundary $C = \partial S$, then Ampere's law can be written as

$$\int_C \vec{B} \cdot d\vec{r} = \mu_0 \, I(S).$$

where the integral is over the contour and $I(S)$ is the current through S. Different surfaces may share C as their boundary; for example, the equator may be viewed as the boundary of the northern hemisphere, of the southern hemisphere or of the disk through the center of the Earth.



Let S and S' be any two surfaces that share C as their boundary. In magnetostatics the currents through S and S' must be equal. Beyond magnetostatics, the currents can be different; this occurs when the charge enclosed between the surfaces varies. These two surfaces may be viewed as the boundary of some three dimensional volume V

$$\partial V = S' - S$$

and $I(S') - I(S)$ is the total current leaving V. It follows that if $Q(V)$ is the total charge of V then the current leaving V is the rate that $Q(V)$ is decreasing

$$I(S') - I(S) = -\frac{d}{dt} Q(V).$$

If the electric flux through some surface S is written as $\Phi(S)$ then Gauss's law states that

$$Q(V) = \varepsilon_0 \int_{\partial V} \vec{E} \cdot d\vec{A} = \varepsilon_0 \, \Phi_e(\partial V) = \varepsilon_0 \, \Phi_e(S') - \varepsilon_0 \, \Phi_e(S).$$

Combining the above two expressions gives

$$I(S') - I(S) = -\varepsilon_0 \frac{d}{dt} \Phi_e(S') + \varepsilon_0 \frac{d}{dt} \Phi_e(S).$$

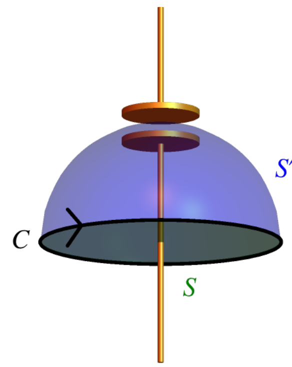
Rearranging this, we can see that although the current is different through S and S', if we add the displacement current $\varepsilon_0 \frac{d}{dt} \Phi_e$ to the current we get a result that is the same for both surfaces.

$$I(S') + \varepsilon_0 \frac{d}{dt} \Phi_e(S') = I(S) + \varepsilon_0 \frac{d}{dt} \Phi_e(S).$$

This new added term, the displacement current $\varepsilon_0 \frac{d}{dt} \Phi_e = \varepsilon_0 \frac{d}{dt} \int \vec{E} \cdot d\vec{A}$, is Maxwell's addition to Ampere's law. The modification of Ampere's law, sometimes referred to the Ampere-Maxwell law, becomes

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 I_{\text{enclosed}} + \mu_0 \varepsilon_0 \frac{d}{dt} \Phi_e$$

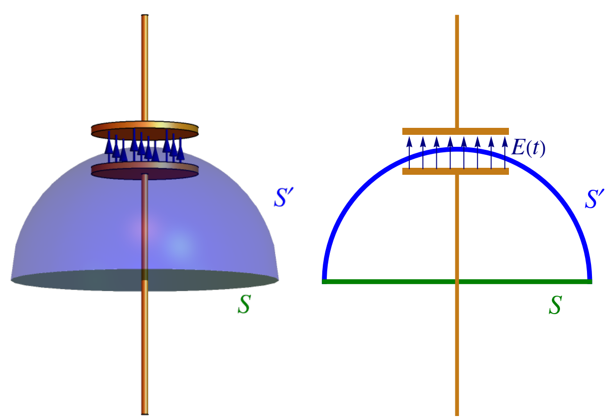
Another way of viewing this addition is in terms of Electromagnetic duality, the symmetry between electric and magnetic fields. Faraday's law says that a changing magnetic flux induces an electric field so we should expect that similarly, a changing electric flux induces a magnetic field.



The diagram above shows a wire and a capacitor. The surface S is pierced by the wire and the surface S' passes through the gap between the capacitor's plates. If there is a current passing through the wire then that current passes through S but not S'. The current increases the charge on the plates of the capacitor and increases the flux through the surface S'; the increasing flux means that there is a displacement current through S'.

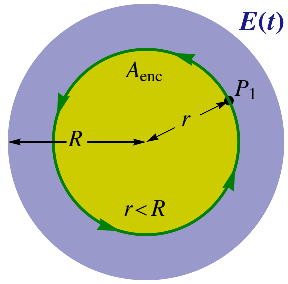
Example G.9 - The Displacement Current and the Ampere-Maxwell Law

An empty parallel-plate capacitor has a circular cross-section of radius R with a plate separation of d. Take the separation d to be small compared to R, so we may safely assume that the varying electric field $E(t)$ is uniform inside and zero outside. There are two surfaces shown: S' passes through the vacuum gap between the plates of the capacitor and S passes through the wire carrying the current that feeds charge to the capacitor.



(a) Inside the capacitor (for $r < R$) what is $B(t)$, the circulating magnetic field?

Solution



We use the Ampere-Maxwell law where the current vanishes: $I_{enclosed} = 0$. The electric flux is $\Phi_e = E\,A_{enclosed}$ where $A_{enclosed} = \pi\,r^2$.

$$\oint \vec{B} \cdot d\vec{r} = \mu_0\,\varepsilon_0\,\frac{d}{dt}\,\Phi_e \implies B(t)\,2\,\pi\,r = \mu_0\,\varepsilon_0\,\frac{d}{dt}\left(E(t)\,\pi\,r^2\right) \implies B(t) = \mu_0\,\varepsilon_0\,\frac{r}{2}\,\frac{d\,E(t)}{d\,t}$$

(b) What is the charge on the capacitor as a function of time?

Solution

The surface charge density is related to the electric field

$$E = \frac{\sigma}{\varepsilon_0} \implies \sigma(t) = \varepsilon_0\,E(t)$$

and the surface charge density is the total charge divided by the total area.

$$\sigma = \frac{Q}{A} = \frac{Q}{\pi\,R^2} \implies Q(t) = \pi\,R^2\,\sigma(t) = \varepsilon_0\,\pi\,R^2\,E(t)$$

(c) What are the current and displacement current through the surface S'?

Solution

There is a vacuum between the plates so there is no current. The displacement current can be found in terms of the total electric flux, which is $\Phi_e = E\,A_{tot} = E\,\pi\,R^2$.

$$I_{enclosed} = 0 \quad \text{and} \quad I_{displacement} = \varepsilon_0\,\frac{d}{d\,t}\int \vec{E} \cdot d\vec{A} = \varepsilon_0\,\frac{d}{d\,t}\left(E(t)\,\pi\,R^2\right) = \varepsilon_0\,\pi\,R^2\,\frac{d\,E(t)}{d\,t}$$

(d) What are the current and displacement current through the surface S?

Solution

Here there is no displacement current but there is an actual current, $I = d\,Q/d\,t$.

$$I_{displacement} = 0 \quad \text{and} \quad I_{enclosed} = \frac{d\,Q(t)}{d\,t} = \varepsilon_0\,\pi\,R^2\,\frac{d\,E(t)}{d\,t}$$

Maxwell's Equations

With the addition of the new term to Ampere's law, Maxwell had a set of four equations that was far more than the sum of its parts; the equations gave a complete and consistent dynamical theory of the electromagnetic field.

$$\oint \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_0}\,Q_{enclosed}$$

$$\oint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{B} \cdot d\vec{r} = \mu_0\,I_{enclosed} + \mu_0\,\varepsilon_0\,\frac{d}{d\,t}\,\Phi_e$$

$$\oint \vec{E}\cdot d\vec{r} = -\frac{d}{dt}\Phi_m$$

Newton's second law specifies the dynamics of particles and systems of particles. Maxwell's equations do the same for electric and magnetic fields; they describe their dynamics. Changing magnetic fields induce electric fields by Faraday's law and changing electric fields induce magnetic fields through the Ampere-Maxwell law.

Maxwell's equations form the basis of electrodynamics; it gives the dynamics of the electromagnetic field. With Coulomb's law, and similarly with Newton's law of gravity, we have action-at-a-distance, which is a direct interaction between distant charges (or masses.) With a field theory like electrodynamics we avoid action-at-a-distance. The basic idea is this: particles create fields, fields propagate through space at the speed of light or slower and then fields exert forces on particles. The force between particles is not direct but is mediated by a field. The propagation of the fields is determined by Maxwell's equations. Maxwell's equations also describe how to find the fields from the charges. To find the force on a charge due to a field we must use one extra thing, the Lorentz force law, which combines the electric force law with the magnetic one giving

$$\vec{F} = Q\left(\vec{E} + \vec{v}\times\vec{B}\right).$$

G.6 - Aside on Vector Calculus

S is some 2D surface in 3D. ∂S is its bounding contour.

Recall that the orientation of a surface is a consistent choice of normals, where any orientable surface has two orientations. The orientation of a contour is the direction of integration along the contour. The boundary of a surface is a closed contour. The orientation on the contour ∂S is related to the orientation of S by the right hand rule.

V is some volume, a 3D region. ∂V is its bounding surface.

The boundary of a volume is a closed surface. We choose the outward normal as the orientation of ∂V.

Operations of Vector Calculus

The Gradient Operator - $\vec{\nabla}$

The gradient operator is

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle.$$

As an operator it only makes sense when it acts on something else. Another example is the differential operator $\frac{d}{dx}$ that only makes sense when it acts on some function.

Gradient - $\vec{\nabla} f$

The gradient of a scalar field f is the vector field given by

$$\vec{\nabla} f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

Divergence - $\vec{\nabla}\cdot\vec{F}$

The divergence of a vector field $\vec{F}$ is a scalar field.

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}F_x + \frac{\partial}{\partial y}F_y + \frac{\partial}{\partial z}F_z.$$

Curl - $\vec{\nabla}\times\vec{F}$

The curl of a vector field is a vector field.

$$\vec{\nabla}\times\vec{F} = \hat{x}\left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right)$$

$$+\hat{y}\left(\frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z\right)$$
$$+\hat{z}\left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right)$$

Theorems of Vector Calculus

The theorems of vector calculus are all generalizations of the fundamental theorem of calculus. Generally, they relate the integral of some derivative over a region to an integral over the boundary of the region.

The Fundamental Theorem of Calculus

$$\int_a^b \frac{d f(x)}{d x}\, d x = f(b) - f(a)$$

This says that the integral of the derivative of any function $f(x)$ over the closed interval $[a, b]$ is equal to the function evaluated over the boundary of the interval, which are the two points a and b, with the signs giving the orientation.

Gauss's Theorem or the Divergence Theorem

$$\int_V \vec{\nabla}\cdot\vec{F}\; d\,\mathrm{Vol} = \int_{\partial V} \vec{F}\cdot d\vec{A}$$

Stokes' Theorem

$$\int_S \left(\vec{\nabla}\times\vec{F}\right)\cdot d\vec{A} = \int_{\partial S} \vec{F}\cdot d\vec{r}$$

Differential Form of Maxwell's Equations

We can rewrite Maxwell's equations in terms of the divergence and curl. The first step is to write the four equations in terms of volumes V surfaces S and their boundaries. Starting with Gauss's law we can write the total charge inside a volume as an integral over the charge density (charge/volume) ρ

$$Q_{\text{enclosed}} = Q(V) = \int_V \rho\, d\,\mathrm{Vol}.$$

Next, apply Gauss's theorem to the left hand side. The resulting expression is true for any volume V, so we can neglect the integrals and equate the integrands. This becomes the differential form of Gauss's law:

$$\oint \vec{E}\cdot d\vec{A} = \frac{1}{\varepsilon_0}\, Q_{\text{enclosed}} \implies \int_{\partial V} \vec{E}\cdot d\vec{A} = \frac{1}{\varepsilon_0}\int_V \rho\, d\,\mathrm{Vol}$$

$$\implies \int_V \vec{\nabla}\cdot\vec{E}\; d\,\mathrm{Vol} = \frac{1}{\varepsilon_0}\int_V \rho\, d\,\mathrm{Vol} \implies \vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\,\rho.$$

Similarly, we can write Gauss's law for magnetism in a differential form

$$\int_{\partial V} \vec{B}\cdot d\vec{A} = 0 \implies \vec{\nabla}\cdot\vec{B} = 0.$$

A current through a surface is related to the current density integrated over it

$$I_{\text{enclosed}} = I(S) = \int_S \vec{J}\cdot d\vec{A}.$$

Next apply Stokes' theorem to the left hand side of the expression. Since the resulting expression for any surface S we can equate the integrands.

$$\oint \vec{B}\cdot d\vec{r} = \mu_0\, I_{\text{enclosed}} + \mu_0\, \varepsilon_0\, \frac{d}{d t}\,\Phi_e$$

$$\Longrightarrow \int_{\partial S} \vec{B}\cdot d\vec{r} = \mu_0 \int_S \vec{J}\cdot d\vec{A} + \mu_0\, \varepsilon_0\, \frac{d}{dt} \int_S \vec{E}\cdot d\vec{A}$$

$$\Longrightarrow \int_S \left(\vec{\nabla}\times\vec{B}\right)\cdot d\vec{A} = \mu_0 \int_S \vec{J}\cdot d\vec{A} + \mu_0\, \varepsilon_0 \int_S \frac{\partial}{\partial t}\, \vec{E}\cdot d\vec{A}$$

$$\Longrightarrow \quad \vec{\nabla}\times\vec{B} = \mu_0\, \vec{J} + \mu_0\, \varepsilon_0\, \frac{\partial}{\partial t}\, \vec{E}$$

A similar calculation gives the differential form of Faraday's law

$$\int_{\partial S} \vec{E}\cdot d\vec{r} = -\frac{d}{dt} \int_S \vec{B}\cdot d\vec{A} \quad \Longrightarrow \quad \vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\, \vec{B}$$

Summarizing, we can write Maxwell's equations in terms of divergence and curl; this is known as the differential form of Maxwell’s equations.

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\, \rho$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\, \vec{J} + \mu_0\, \varepsilon_0\, \frac{\partial}{\partial t}\, \vec{E}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\, \vec{B}$$