Chapter H

Inductance and Transient Circuits

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As a consequence of Faraday's law a changing current through one coil induces an EMF in another coil; this is known as mutual inductance. Similarly, a changing flux in a coil induces an EMF in the same coil; this is self inductance and a circuit component with inductance is called an inductor. We will also discuss simple circuits with inductors combined with our other linear circuit elements: resistors, capacitors and DC voltage sources.

H.1 - Mutual and Self Inductance

Mutual Inductance

Consider a pair of coils, one called the primary coil and the other called the secondary. A current I_1 through the primary creates a magnetic field $\vec{B}_1$ which in turn creates a magnetic flux through the secondary coil Φ_{12}.

$$I_1 \implies \vec{B}_1 \implies \Phi_{12}$$

A changing current creates a changing flux which induces an EMF in the secondary coil.

$$\frac{d}{dt} I_1 \implies \frac{d}{dt} \Phi_{12} \implies \mathcal{E}_2$$

The above relationships are proportionalities. Define the constant of proportionality as the mutual inductance M_{12}.

$$\mathcal{E}_2 = -M_{12} \frac{d}{dt} I_1$$

It is beyond the scope of this class to prove that the mutual inductance of coil 1 on coil 2 is the same as that of 2 on 1, and this will just be called M.

$$\mathcal{E}_2 = -M \frac{d}{dt} I_1 \text{ where } M = M_{12} = M_{21}$$

We will revisit mutual inductance in the next chapter in the context of a transformer. A transformer is the case where, ideally, all the flux from one coil passes through the other.

Self Inductance

There is also inductance of a coil on itself. This is called self inductance. When we use the term inductance by itself self inductance is implied. A current through a coil creates a field and that causes a flux through the coil itself.

$$I \implies \vec{B} \implies \Phi$$

A changing current creates a changing flux with induces an EMF in the coil.

$$\frac{dI}{dt} \implies \frac{d}{dt} \Phi \implies \mathcal{E}$$

The above relationships are proportionalities. The constant of proportionality is defined as the inductance L.

$$\mathcal{E} = -L \frac{dI}{dt}$$

The sign in the above expression is due to Lenz's law. Take ΔV to be the change in the voltage when moving through the inductor in the direction of the current. A simple Lenz's law analysis shows that if the current is increasing the voltage change is negative. If we write V as the voltage drop we get

$$\Delta V = -L \frac{dI}{dt} \text{ and } V = L \frac{dI}{dt}.$$

The sign conventions for inductors is the same as that for resistors and capacitors

$$\Delta V = -I R \text{ and } V = I R$$

$$\Delta V = -\frac{Q}{C} \text{ and } V = \frac{Q}{C}$$

Inductance of a Long Solenoid

Consider a long solenoid of length ℓ, with N turns and a cross-sectional area A. As before we define n as the number of turns per length $n = N/\ell$. Passing from the current to the field to the flux gives

$$I \implies B = \mu_0 n I \implies \Phi = B A = \mu_0 n I A$$

and using Faraday's law we get

$$\mathcal{E} = -N \frac{d\Phi}{dt} = -N \mu_0 n A \frac{dI}{dt}.$$

We can read the inductance from this expression. Writing L in terms of both N and n gives

$$L = \mu_0 n^2 A \ell = \mu_0 \frac{N^2}{\ell} A.$$

H.2 - Energy Considerations

Energy in an Inductor

Inductors, like capacitors, store energy, while resistors dissipate energy. Use U to denote the energy in an inductor. The rate that energy is being stored in an inductor is

$$\frac{dU}{dt} = \mathcal{P} = I V = I L \frac{dI}{dt}.$$

Integrating the above expression gives

$$U = \frac{1}{2} L I^2 + \text{constant}.$$

Choosing $U = 0$ when $I = 0$ fixes the constant and gives the desired expression.

$$U = \frac{1}{2} L I^2$$

Energy in a Magnetic Field

In the capacitance chapter we derived an expression for the energy density (Energy/Volume) in an electric field.

$$u = \frac{1}{2} \varepsilon_0 E^2$$

To derive this we used the fact that the electric field is uniform inside a parallel plate capacitor. Combining expressions for the energy in a capacitor and for the capacitance gave the above expression for u. A similar analysis will give the energy density in a magnetic field.

The magnetic field is uniform inside a long solenoid. Combining the expression for the inductance of a long solenoid with the energy in an inductor gives

$$U = \frac{1}{2} L I^2 \text{ and } L = \mu_0 n^2 A \ell \implies U = \frac{1}{2} \mu_0 n^2 A \ell I^2$$

Using $B = \mu_0 n I$ and $u = U/\text{Volume} = U/(A \ell)$ gives the energy density in a magnetic field.

$$u = \frac{1}{2\,\mu_0} B^2$$

H.3 - Transient Circuits and Differential Equations

The voltage drops across a resistor, capacitor and inductor are, respectively,

$$V = I\,R, \quad V = \frac{Q}{C} \text{ and } V = L\,\frac{dI}{dt}.$$

Remember that these are voltage drops, so $\Delta V = -V$. If we build a circuit out of these and dc voltage sources, where $\Delta V = \mathcal{E}$, we then get an equation for I and Q. Since I is the time derivative of Q

$$I = \frac{dQ}{dt}$$

this will give a differential equation for Q or for I.

Comments on ODEs (Ordinary Differential Equations)

A differential equation (DE) is some equation involving a function and its derivatives.

The differential equation is solved to find the function.

The order of a DE is the highest number of derivatives.

If there is at most a second derivative it is a second order equation.

If it is a function of one variable it is an ordinary differential equation (ODE). For functions of several variables there are partial differential equations (PDE).

For functions of more than one variable we take partial derivatives instead of ordinary ones. The differential equations course (taken after Cal. III) is on ODEs.

The general solution of a p^{th} order ODE is any solution involving p independent arbitrary constants.

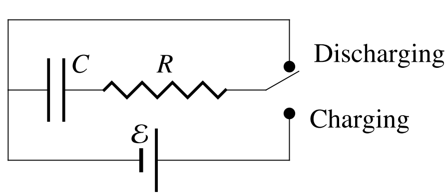
It is easy to verify that something is a solution to a differential equation; it is just a matter of taking derivatives and plugging into an equation. If a solution has the correct number of arbitrary constants then we can conclude that this is the general solution.

H.4 - RC Circuits

When the switch is thrown to the Charging position the current flows from the battery to charge the capacitor. In the Discharging position the charge flows from the capacitor and its energy is dissipated in the resistor. The charge on the capacitor is related to the current in the wire by

$$I = \frac{dQ}{dt}.$$

Note that when the capacitor is discharging the charge is decreasing and thus the current is negative.



Discharging

For the discharging case, applying the loop rule to the circuit gives:

$$0 = I\,R + \frac{Q}{C}.$$

Using the fact that the current is the derivative of the charge we can rewrite this as a first order differential equation.

$$0 = R\,\frac{d\,Q}{d\,t} + \frac{1}{C}\,Q \quad \text{or} \quad \frac{d\,Q}{d\,t} = -\frac{1}{\tau}\,Q$$

where τ is defined as the time constant

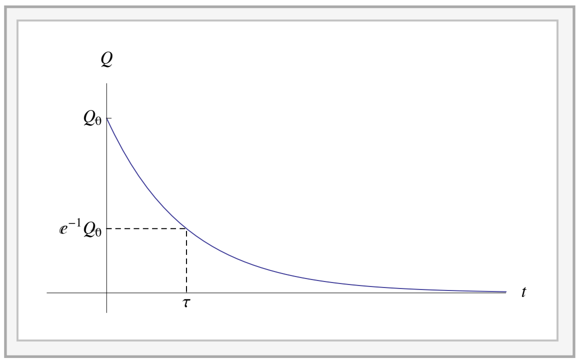
$$\tau = R\,C.$$

Since the derivative of an exponential is itself we can guess a solution of $e^{-t/\tau}$ and it is easy to verify that this is a solution. If we multiply this by a constant α then it still is a solution

$$Q(t) = \alpha\,e^{-t/\tau}.$$

Since α is arbitrary and we are beginning with a first order ODE we can conclude that this is the general solution. If we define the initial charge as Q_0 then $Q_0 = Q(0) = \alpha$ and the solution becomes

$$Q(t) = Q_0\,e^{-t/\tau}.$$



Interactive Figure - Discharging an RC Circuit

Charging

The loop rule for the charging case gives:

$$0 = \mathcal{E} - I\,R - \frac{Q}{C}$$

and we can rewrite this as a first order differential equation.

$$\mathcal{E} = R\,\frac{d\,Q}{d\,t} + \frac{1}{C}\,Q$$

Using the same time constant we can guess a solution of the form

$$Q(t) = \alpha\,e^{-t/\tau} + \beta.$$

Inserting this into our differential equation gives

$$\mathcal{E} = -R\,\frac{\alpha}{\tau}\,e^{-t/\tau} + \frac{1}{C}\left(\alpha\,e^{-t/\tau} + \beta\right)$$

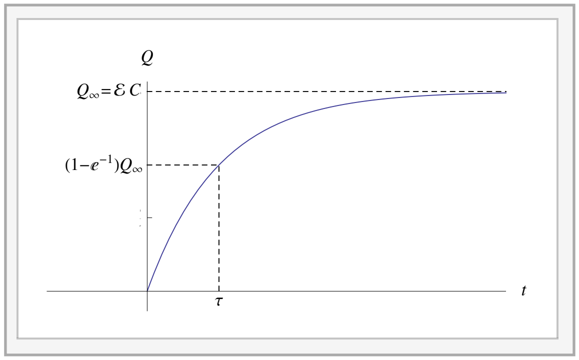
The terms involving α cancel and the equation requires that β has the value $\beta = \mathcal{E}\,C$ for it to be a solution. This means that

$$Q(t) = \alpha\,e^{-t/\tau} + \mathcal{E}\,C$$

is a solution to the differential equation. Since it is a first order equation and α is an arbitrary constant we can conclude that it is the general solution.

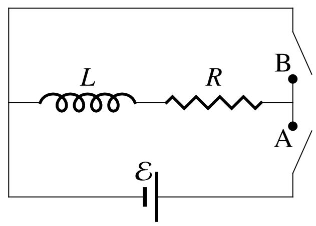
The solution we desire is where at $t = 0$ the capacitor is uncharged $Q(0) = 0$, giving

$$Q(t) = Q_\infty\left(1 - e^{-t/\tau}\right) \text{ where } Q_\infty = \mathcal{E}\,C.$$



Interactive Figure - Charging an RC Circuit

H.5 - RL Circuits



Decaying Current

Begin with switch **A** closed and **B** opened. This creates a current through the inductor and resistor. Close switch **B** and then open **A**. This causes the current to flow through the top branch of the above circuit. Applying the loop rule around the circuit gives

$$0 = L\,\frac{dI}{dt} + R\,I.$$

This is a first order ODE (ordinary differential equation) similar to that of a discharging capacitor

$$\frac{dI}{dt} = -\frac{1}{\tau}\,I,$$

where the time constant τ is defined by

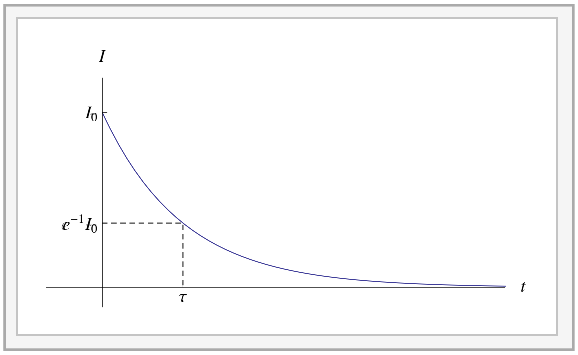
$$\tau = \frac{L}{R}.$$

The general solution of this differential equation for the current $I(t)$ is

$$I(t) = I_0\,e^{-t/\tau},$$

where the arbitrary constant is labelled I_0 because it is the current at time zero. This is a simple exponential decay, analogous to the decay of the charge for a discharging capacitor.

The initial energy in the inductor $U = \frac{1}{2} L\, I_0^2$ is converted to heat in the resistor.



Interactive Figure - Current Decay in an RL Circuit

Growing Current

With both switches opened giving zero current, close switch **A** at $t = 0$. This causes the current to gradually build up to a steady-state value. Apply the loop rule to the circuit gives the first order ODE.

$$\mathcal{E} = L \frac{dI}{dt} + R\,I.$$

Guess a solution of the form

$$I(t) = \alpha\, e^{-t/\tau} + \beta.$$

Inserting this guess into the differential equation gives

$$\mathcal{E} = L\left(-\frac{\alpha}{\tau} e^{-t/\tau}\right) + R\left(\alpha\, e^{-t/\tau} + \beta\right).$$

The definition of the time constant gives a cancellation of the $e^{-t/\tau}$ terms for any value of α, making α our arbitrary constant. Our guess will only be a solution when β has the value

$$\beta = \frac{\mathcal{E}}{R}.$$

Since we have a solution with one arbitrary constant and it is a first order ODE we can conclude that

$$I(t) = \alpha\, e^{-t/\tau} + \frac{\mathcal{E}}{R}$$

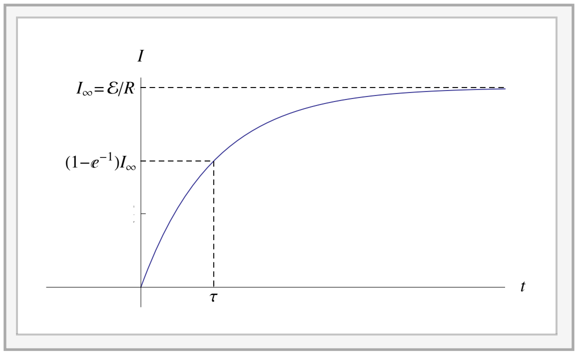
is the general solution. We are looking for a solution where the initial current is zero $I(0) = 0$, giving

$$\alpha = -\frac{\mathcal{E}}{R}.$$

The growth of the current can be written as

$$I(t) = I_\infty \left(1 - e^{-t/\tau}\right) \text{ where } I_\infty = \frac{\mathcal{E}}{R}$$

is the steady-state current.

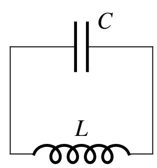


Interactive Figure - Current Growth in an RL Circuit

H.6 - LC Circuits, RLC Circuits and Their Mechanical Equivalents

There is an important analogy between the LC Circuit and the mass-spring system. We will see that the solution to both equations is oscillatory. We can add damping to these cases by inserting a resistor into the electrical system and adding friction to the mechanical system.

The LC Circuit



Consider a simple loop circuit containing just a capacitor C and inductor L. The loop rule gives

$$0 = L\frac{dI}{dt} + \frac{Q}{C}.$$

Write the current I as the time derivative the charge on the capacitor

$$I = \frac{dQ}{dt}$$

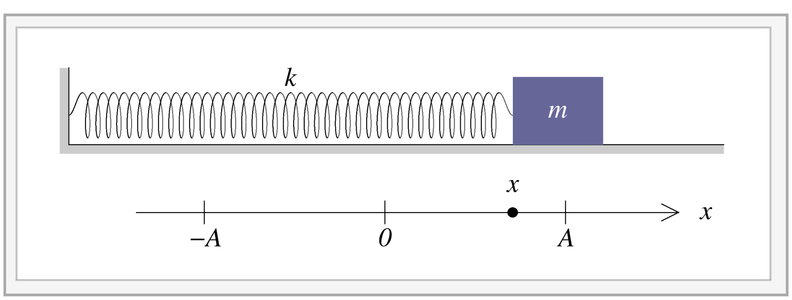
and define the angular frequency by

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

This gives the second order ODE

$$0 = \frac{d^2 Q}{dt^2} + \omega_0^2 Q.$$

The Mass-Spring System



Interactive Figure

The mechanical analog of this is a mass-spring system. The force of a spring is given by Hooke's law $F = -k\,x$. Applying Newton's second law gives a second order ODE

$$F_{\text{net}} = m\,a \implies -k\,x = m\,\frac{d^2 x}{d\,t^2}$$

Defining the angular frequency by

$$\omega_0 = \sqrt{\frac{k}{m}}\,,$$

gives the second order ODE

$$0 = \frac{d^2 x}{d\,t^2} + \omega_0^2\,x\,.$$

The Analogy without Damping

Summarizing the analogy as stated above: The charge, current and derivative of the current are the analogs of the position, velocity and acceleration. The energy in the inductor $U_L = \frac{1}{2}\,L\,I^2$ and the kinetic energy of the mass $K = \frac{1}{2}\,m\,v^2$ are analogous, as are the energy in the capacitor $U_C = \frac{1}{2\,C}\,Q^2$ and the potential energy of the spring $U = \frac{1}{2}\,k\,x^2$.

LC Circuit	Mass–Spring
Q	x
$I = \frac{dQ}{dt}$	$v = \frac{dx}{dt}$
$\frac{dI}{dt}$	$a = \frac{d^2x}{dt^2}$
L	m
C	$\frac{1}{k}$
$\omega_0 = \frac{1}{\sqrt{L\,C}}$	$\omega_0 = \sqrt{\frac{k}{m}}$
$U_L = \frac{1}{2}\,L\,I^2$	$K = \frac{1}{2}\,m\,v^2$
$U_C = \frac{1}{2\,C}\,Q^2$	$U = \frac{1}{2}\,k\,x^2$

Solution of the Undamped ODE

With the analogy clearly stated, let us solve the second order ordinary differential equation

$$0 = \frac{d^2 Q}{d\,t^2} + \omega_0^2\,Q\,.$$

Since the second derivatives of both sine and cosine are the negatives of themselves, it follows that

$$\cos\omega_0\,t \text{ and } \sin\omega_0\,t$$

are solutions to our differential equation. Because the ODE has the simple properties (a homogeneous linear equation) that:

(*i*) *a* constant times *a* solution is *a* solution and
(ii) the sum of two solutions is *a* solution,

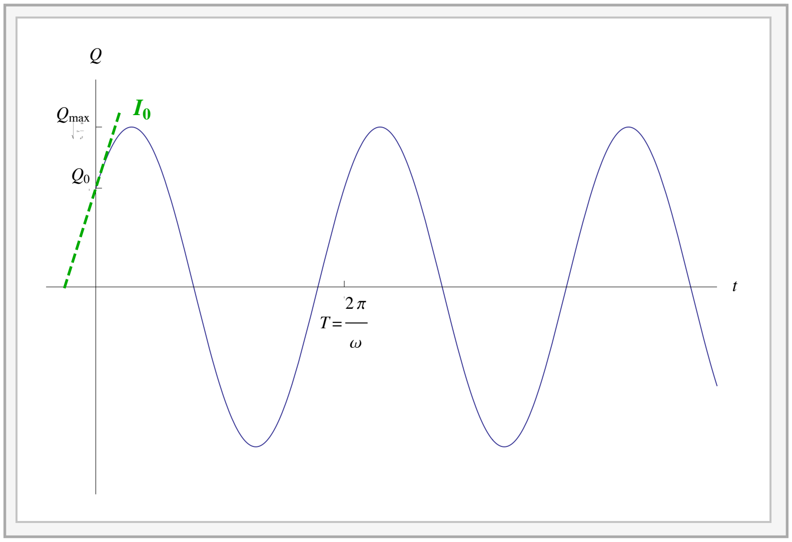
it follows that

$$Q(t) = B\cos\omega_0\,t + C\sin\omega_0\,t$$

is a solution, where B and C are arbitrary constants. Since we have solution to a second order ODE with two arbitrary constants, we can conclude that this is the general solution. Setting $Q(0) = Q_0$ and $I(0) = I_0$ where $I = d\,Q/d\,t$ gives $B = Q_0$ and $C = I_0/\omega$. Another way of presenting this solution is

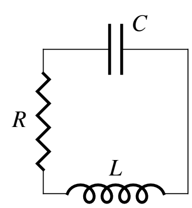
$$Q(t) = Q_{\max}\cos(\omega_0\,t + \phi)$$

where the arbitrary constants are $Q_{\max}$, the amplitude, and ϕ, the phase angle.



Interactive Figure - Charge as a Function of Time for an LC Circuit

LCR Circuit



We can add damping to the LC circuit by adding a resistor R to the series circuit. The loop rule gives

$$0 = L \frac{dI}{dt} + I R + \frac{Q}{C}.$$

This gives us the second order ODE

$$0 = \frac{d^2 Q}{dt^2} + \beta \frac{dQ}{dt} + \omega_0^2 Q$$

where ω_0 is the same as in the undamped case and

$$\beta = \frac{R}{L}.$$

The Damped Mass-Spring System

The mechanical damping is achieved by adding viscous friction, which is the force $-bv$, to the Hooke's law force. Newton's second law gives

$$F_{net} = m a \implies -k x - b v = m \frac{d^2 x}{dt^2}.$$

This becomes the second order ODE

$$0 = \frac{d^2 x}{dt^2} + \beta \frac{dx}{dt} + \omega_0^2 x$$

where ω_0 is the same as in the undamped case and now

$$\beta = \frac{b}{m}.$$

The Analogy with Damping

We can add to the previous table the values of the damping constants β.

LCR Circuit	Damped Mass–Spring
R	b
$\beta = \frac{R}{L}$	$\beta = \frac{b}{m}$

Note the similarities between resistance and friction. Both remove energy from the system, friction removes mechanical energy and resistance removes electrical energy. The energy lost to each goes to heat.

Solution with Damping

$$0 = \frac{d^2 Q}{d t^2} + \beta \frac{d Q}{d t} + \omega_0^2 Q$$

To solve the above differential equation we will guess a solution of the form

$$Q(t) = A\, e^{-\gamma t} \cos(\omega t + \phi).$$

This will be a solution only when the constants γ and ω have specific values, which we will write in terms of the constants in the equation, β and ω_0. The constants A and ϕ will be our arbitrary constants. To verify this is a solution and find γ and ω, we must plug our guess into the differential equation. First we must evaluate the derivatives

$$Q = A\, e^{-\gamma t} \cos(\omega t + \phi).$$

$$\frac{d Q}{d t} = A\, e^{-\gamma t} [-\gamma \cos(\omega t + \phi) - \omega \sin(\omega t + \phi)].$$

$$\frac{d^2 Q}{d t^2} = A\, e^{-\gamma t} \left[\left(\gamma^2 - \omega^2\right) \cos(\omega t + \phi) + 2 \gamma \omega \sin(\omega t + \phi)\right].$$

$$0 = \frac{d^2 Q}{d t^2} + \beta \frac{d Q}{d t} + \omega_0^2 Q \implies$$

$$\begin{aligned} 0 = \; & A\, e^{-\gamma t} \left[\left(\gamma^2 - \omega^2\right) \cos(\omega t + \phi) + 2 \gamma \omega \sin(\omega t + \phi)\right] \\ & + A\, e^{-\gamma t} [\; -\beta \gamma \cos(\omega t + \phi) - \beta \omega \sin(\omega t + \phi)] \\ & + A\, e^{-\gamma t} \left[\; \omega_0^2 \cos(\omega t + \phi) + 0 \;\right] \end{aligned}$$

For this equality to be true at all times, the terms multiplying $\cos(\omega t + \phi)$ and $\sin(\omega t + \phi)$ must separately vanish, giving

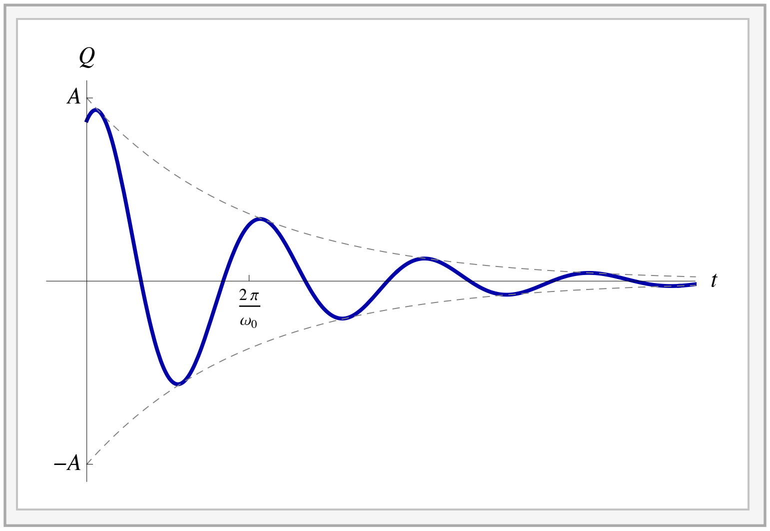
$$0 = \gamma^2 - \omega^2 - \beta \gamma + \omega_0^2 \text{ and } 0 = 2 \gamma \omega - \beta \omega.$$

The second expression gives

$$\gamma = \frac{\beta}{2}$$

and plugging this into the first expression gives

$$\omega = \sqrt{\omega_0^2 - \frac{\beta^2}{4}}\,.$$



Interactive Figure - Charge as a Function of Time for an LCR Circuit