Chapter 1

AC Circuits

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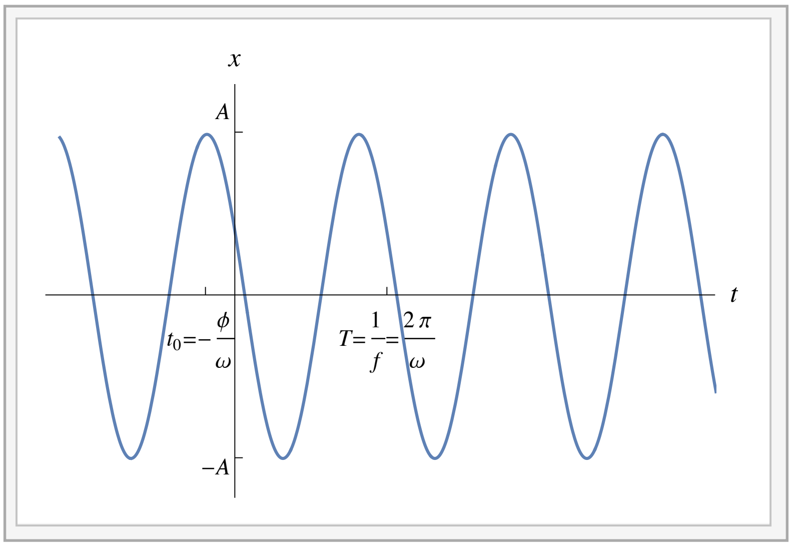
1.1 - Sinusoidal Voltage and Current

The general form of a sinusoidal function of voltage is:

$$V(t) = V_{\max} \cos(\omega\, t + \phi)$$

$V_{\max}$ is the *peak voltage*, ω is the *angular frequency* and ϕ is the *phase angle*.



Interactive Figure

Frequency, Angular Frequency and Period

The rate of oscillation of a sinusoidal function is described by the *angular frequency* ω. The *period* T is the time for each cycle and the *frequency* f is the cycles per time and thus is $f = 1/T$. When time changes by one period the argument of the trig function, $\omega\, t + \phi$, shifts by 2π, which implies that $\omega\, T = 2\pi$.

$$\omega = \frac{2\pi}{T} = 2\pi f$$

Phase Angle

Changing the phase shifts the graph along the time axis. (A positive ϕ shifts the graph in the negative time direction.) Note that a choice of ϕ can shift the function from sin to cos,

$$\cos\left(\omega\, t \mp \frac{\pi}{2}\right) = \pm \sin \omega\, t.$$

The phase angle ϕ is related to t_0, which is some time when the function achieves its peak value.

$$V(t_0) = V_{\max} \quad \Longrightarrow \quad \phi = -\omega\, t_0$$

rms and Peak Quantities

In calculus, the average of a function over some interval is the integral of the function over the interval divided by the width of the interval:

$$\bar{f} = \frac{1}{(b-a)} \int_a^b f(x)\, dx.$$

When dealing with periodic functions, like sinusoidal functions, we define its average as the average over one period (or over some integer number of periods) of that function. It is clear, either by evaluating the integral or by viewing the graph, that the average of any sinusoidal function is zero.

To see how much a function that averages to zero deviates from zero we use the root-mean-square, rms, quantity. This is defined, as its name implies, as the square root (root) of the average (mean) of the square of the function.

$$V_{\text{rms}} = \sqrt{\overline{V^2}} = \sqrt{V_{\max}^2 \,\overline{\cos^2}}$$

We will show in the next subsection that the average value of $\overline{\cos^2}$ is 1/2. It follows that:

$$\overline{\cos^2} = \frac{1}{2} \implies V_{\text{rms}} = \sqrt{V_{\max}^2 \,\overline{\cos^2}} = \frac{1}{\sqrt{2}} V_{\max}$$

The above expression applies to any sinusoidal function, so in AC it applies as well to the current.

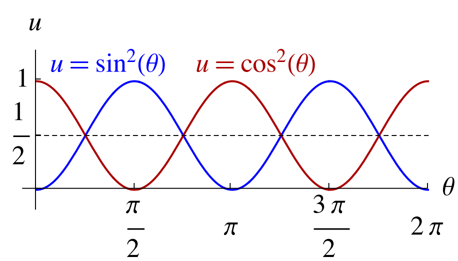
$$V_{\text{rms}} = \frac{1}{\sqrt{2}} V_{\max} \text{ and } I_{\text{rms}} = \frac{1}{\sqrt{2}} I_{\max}$$

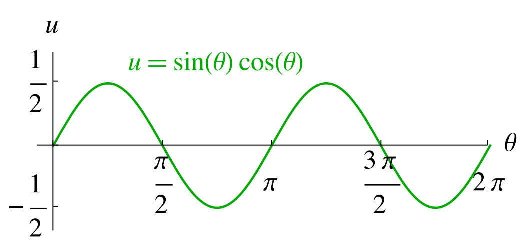
Averages of Products of Sine and Cosine

Above we used the fact that the average value of the square of cosine over one period is 1/2. The average value of $\sin^2$ is also 1/2. Moreover, the average of sin times cos is zero.

$$\overline{\sin^2} = \frac{1}{2} = \overline{\cos^2} \text{ and } \overline{\sin\cos} = 0.$$

The argument of the trig function, whether θ or $\omega t + \phi$, is excluded because they do not affect the average over one period. These average values can be seen by looking at the graphs





or by using the trig identities

$$\left.\begin{matrix}\cos^2\theta \\ \sin^2\theta\end{matrix}\right\} = \frac{1}{2}(1 \pm \cos 2\theta) \text{ and } \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta.$$

The second result, $\overline{\sin\cos} = 0$, will be useful in the discussion of power in AC circuits.

A Trig Identity

Using the identity $\cos(\theta + \phi) = \cos\theta\,\cos\phi - \sin\theta\,\sin\phi$ with $\theta = \omega t$ we can write

$$\begin{aligned} V(t) &= V_{\max}\cos(\omega t + \phi) \\ &= V_{\max}\,(\cos\phi\cos\omega t - \sin\phi\sin\omega t). \end{aligned}$$

This applies to any sinusoidal function, not just voltage.

Standard US Outlet

A standard US outlet has

$$V_{\text{rms}} = 120\text{ V} \quad \text{and} \quad f = 60\text{ Hz},$$

It follows that $V_{\max} = \sqrt{2}\, V_{\text{rms}} = 169.7\text{ V}$ and $\omega = 2\pi f = 377\, s^{-1}$. Note that it is the obligation of power companies to make the frequency average precisely to 60 Hz to guarantee that AC electric clocks run at the proper rate.

I.2 - Relating Voltage and Current

Impedance and Phase

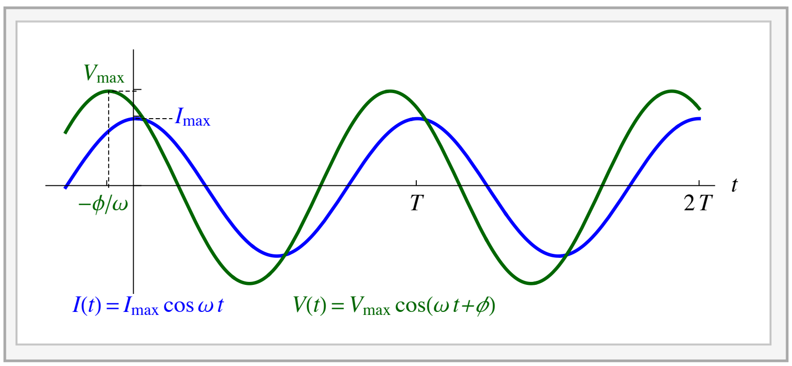
In DC circuits we can relate the voltage and current through a resistor by just one number, the resistance. In AC it takes two numbers, the impedance Z and the phase ϕ. The peak values, and similarly the rms quantities, are related by the impedance.

$$Z = \frac{V_{\max}}{I_{\max}} = \frac{V_{\text{rms}}}{I_{\text{rms}}}$$

Sinusoidal functions also have a phase angle. The phase angle of some function by itself is unimportant; it just refers to a time shift. What we define as the phase angle ϕ is the relative phase between the voltage and current. The voltage is ahead of the current by the phase angle.

$$I(t) = I_{\max} \cos \omega t \quad \text{and} \quad V(t) = I_{\max} Z \cos(\omega t + \phi)$$

Increasing the phase angle shifts the voltage vs. time graph to the left (ahead in time.)



Interactive Figure

Resistors, Capacitors and Inductors

Using the definitions above of impedance and phase we can describe the effect of the resistor, capacitor and inductor in AC circuits.

For a resistor the voltage to current relationship is Ohm's law, $V = I R$. For a capacitor it is given by $Q = C V$, where $I = \frac{d}{dt} Q$. Inductors have the relation $V = L \frac{d}{dt} I$. Using these basic voltage and current relations we can find the impedance and phase of these three components taken by themselves, which will describe their effect in an AC circuit.

	Voltage and Current	$I(t) = I_{\max} \cos \omega t$ $V(t) = I_{\max} Z \cos(\omega t + \phi)$	Impedance Z	Phase ϕ
Just R	$V = I R$	$V(t) = I_{\max} R \cos \omega t$	R	0
Just C	$\frac{d}{dt} Q = I$ $V = \frac{Q}{C}$	$Q(t) = I_{\max} \left(\frac{1}{\omega} \sin \omega t\right)$ $V(t) = I_{\max} \left(\frac{1}{\omega C}\right) \sin \omega t$	$X_C = \frac{1}{\omega C}$	$-\frac{\pi}{2}$

Just L	$V = L \frac{d}{dt} I$	$\frac{d}{dt} I(t) = I_{max}\,(-\omega \sin \omega\, t)$ $V(t) = I_{max}\,(\omega\, L)\,(-\sin \omega\, t)$	$X_L = \omega\, L$	$\frac{\pi}{2}$

In this table we have used $\cos\left(\omega\, t \mp \frac{\pi}{2}\right) = \pm \sin \omega\, t$.

I.3 - Power in AC

The power dissipated in a circuit at some instant is the voltage times the current at that instant.

$$\mathcal{P}(t) = V(t)\, I(t)$$

Using

$$\begin{aligned} V(t) &= V_{max} \cos(\omega\, t + \phi) \\ &= V_{max}\;\; (\cos \phi \cos \omega\, t - \sin \phi \sin \omega\, t) \end{aligned}$$

and $I(t) = I_{max} \cos \omega\, t$ we can derive an expression for the power

$$\mathcal{P}(t) = V_{max}\, I_{max} \left(\cos \phi\;\; \cos^2 \omega\, t - \sin \phi\;\; \sin \omega\, t\;\; \cos \omega\, t\right)$$

Averaging the above expression gives

$$\overline{\mathcal{P}} = V_{max}\, I_{max} \left(\cos \phi\;\; \overline{\cos^2} - \sin \phi\;\; \overline{\sin \cos}\right)$$

$$\overline{\mathcal{P}} = V_{rms}\, I_{rms} \cos \phi\ .$$

where we have used

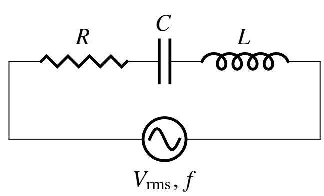
$$\overline{\cos^2} = \frac{1}{2}, \quad \overline{\sin \cos} = 0 \text{ and}$$

$$\frac{1}{2} V_{max}\, I_{max} = V_{rms}\, I_{rms}.$$

In DC we have $\mathcal{P} = I\, V$ and $V = I\, R$. For a purely resistive circuit $\phi = 0$, so we get $\overline{\mathcal{P}} = V_{rms}\, I_{rms}$. Since $V_{rms} = I_{rms}\, R$ as well, we can treat the circuit as if it were DC by using rms values of the voltage and current and by using the average power.

For an inductor and capacitor the phase angle is $\phi = \pm \frac{\pi}{2}$. It follows that $\cos \phi = 0$ and thus $\overline{\mathcal{P}} = 0$. This is important. All energy lost in an AC circuit is lost in the resistors. Inductors and capacitors can store energy and release that energy later in each AC cycle; this has the effect of changing the phase of the circuit.

I.4 - The Series RCL Circuit



The only AC circuit we will discuss in detail is the series RCL circuit. This is a circuit consisting of a resistor, capacitor and inductor connected in series across an AC voltage source with frequency f. In a series the currents through all components are the same and the voltages add.

$$I(t) = I_R(t) = I_C(t) = I_L(t)$$

$$V(t) = V_R(t) + V_C(t) + V_L(t)$$

Beginning with the current of

$$I(t) = I_{max} \cos \omega\, t$$

We get that voltage across all three components is

$$
\begin{aligned}
V(t) &= I_{max}\, Z \cos(\omega\, t + \phi) \\
&= I_{max}\, Z\ (\cos\phi \cos\omega\, t - \sin\phi \sin\omega\, t)
\end{aligned}
$$

And the voltages across all three are found in the table

$$
\begin{aligned}
V_R(t) &= I_{max}\, R \cos\omega\, t \\
V_C(t) &= I_{max}\, X_C \sin\omega\, t \\
V_L(t) &= -I_{max}\, X_L \sin\omega\, t
\end{aligned}
$$

Inserting the above expressions into $V(t) = V_R(t) + V_C(t) + V_L(t)$ gives

$$
\begin{aligned}
&I_{max}\, [(Z\cos\phi)\cos\omega\, t - (Z\sin\phi)\sin\omega\, t] \\
&= I_{max}\, [R\cos\omega\, t - (X_L - X_C)\sin\omega\, t]
\end{aligned}
$$

For this to be equal the multipliers of $\cos\omega\, t$ and $\sin\omega\, t$ must separately be equal, giving

$$Z\cos\phi = R \quad \text{and} \quad Z\sin\phi = X_L - X_C.$$

Solving for the impedance and phase

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad \text{and} \quad \tan\phi = \frac{X_L - X_C}{R}.$$

We can derive an expression for the power dissipated in a series RCL circuit.

$$\overline{\mathcal{P}} = V_{rms}\, I_{rms} \cos\phi\ = I_{rms}^2\, Z\cos\phi$$

which gives

$$\overline{\mathcal{P}} = I_{rms}^2\, R.$$

This has a simple interpretation. All of the power dissipated in an AC circuit is lost in the resistors; here there is only one resistor and the total current passes through that resistor.

I.5 - Resonance

The reactances vary with frequency. The capacitive reactance decreases with frequency and the inductive reactance increases.

	Low f	High f
$X_C = \frac{1}{2\pi f C}$	large	small
$X_L = 2\pi f L$	small	large
$X_L - X_C$	negative	positive

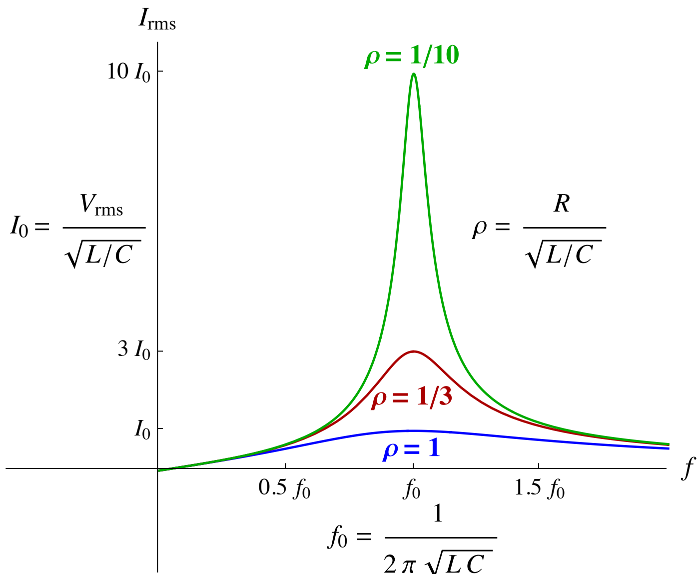
It is clear that there is some frequency where $X_L = X_C$. This is called the resonant frequency f_{res}. At resonance the condition $X_L = X_C$ implies that the frequency is

$$f_{res} = \frac{1}{2\pi}\,\omega_{res} = \frac{1}{2\pi\sqrt{L\,C}}.$$

At frequencies below the resonant frequency a circuit is said to be capacitive and above resonance it is inductive.

	Capacitive	Resonant	Inductive
frequency	$f < f_{res}$	$f = f_{res}$	$f > f_{res}$
X_L and X_C	$X_L < X_C$	$X_L = X_C$	$X_L > X_C$
$\phi = \tan^{-1}\frac{X_L - X_C}{R}$	negative	0	positive
$Z = \sqrt{R^2 + (X_L - X_C)^2}$	$Z > R$	$Z = Z_{min} = R$	$Z > R$

If we fix V_{rms} and vary the frequency, the current I_{rms} will peak at the resonant frequency. At resonance the impedance is the resistance, so it follows that the smaller the resistance the more dramatic the peak in the current. The graph shows the behavior of current versus frequency for different resistances.



I.6 - Aside - Complex Impedance and Phasors

Complex numbers provide an elegant way to deal with phase. They provide an alternative to the awkward trigonometic expressions of the preceding series circuit discussion but, most importantly, they provide a straight-forward method to calculate the impedance of arbitrarily complicated combinations of resistors, capacitors and inductors.

The Complex Plane

A complex number is something of the form $z = x + i\,y$, where $i = \sqrt{-1}$. The real part of z is x and the imaginary part is y.

$$\text{Re}(z) = x \text{ and } \text{Im}(z) = y$$

The starting point for representing phase with complex numbers is the expression, known as Euler's formula

$$e^{i\,\theta} = \cos\theta + i\sin\theta\,.$$

To prove this consider the Taylor expansions of the exponential, cosine and sine functions,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \text{ and } \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots,$$

and use $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$.

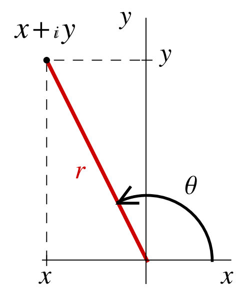
$$e^{i\,\theta} = 1 + i\,\theta + \frac{(i\,\theta)^2}{2!} + \frac{(i\,\theta)^3}{3!} + \frac{(i\,\theta)^4}{4!} + \frac{(i\,\theta)^5}{5!} + \ldots = 1 + i\,\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \ldots$$
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right) = \cos\theta + i\sin\theta$$

It follows from Euler's formula that

$$i = e^{i\,\pi/2} \text{ and } -i = e^{-i\,\pi/2}.$$

There is a simple connection to polar coordinates. The absolute value (or modulus) of z is the r of polar coordinates $r = |z| = |x + i\,y| = \sqrt{x^2 + y^2}$ and the angle θ ($\tan\theta = y/x$) is called the argument of z, $\theta = \text{Arg}(z)$.

$$z = x + i\,y = r\cos\theta + i\,r\sin\theta = r\,e^{i\,\theta}$$



Multiplication of complex numbers takes a simple form using this polar representation

$$z_1\, z_2 = r_1\, e^{i\,\theta_1}\, r_2\, e^{i\,\theta_2} = r_1\, r_2\, e^{i\,(\theta_1+\theta_2)}$$

Note that the trig identities for the sum of angles, used in the preceding discussion for series circuits, follows trivially from this.

$$\begin{aligned}\cos(\theta+\phi) + i\sin(\theta+\phi) &= e^{i\,(\theta+\phi)} = e^{i\,\theta}\, e^{i\,\phi} \\ &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i\,(\cos\theta\sin\phi + \sin\theta\cos\phi)\end{aligned}$$

Complex Current, Voltage and Impedance

We may look at sinusoidal function as the real part of a complex function.

$$I(t) = I_{\max}\cos\omega\, t = \mathrm{Re}\left(I_{\max}\, e^{i\,\omega\, t}\right)$$

$$\begin{aligned}V(t) &= V_{\max}\cos(\omega\, t+\phi) = \mathrm{Re}\left(V_{\max}\, e^{i\,(\omega\, t+\phi)}\right) \\ &= \mathrm{Re}\left(I_{\max}\, Z\, e^{i\,(\omega\, t+\phi)}\right) = \mathrm{Re}\left(Z\, e^{i\,\phi}\, I_{\max}\, e^{i\,\omega\, t}\right)\end{aligned}$$

We then extend both current and voltage to complex functions, $\tilde{I}(t)$ and $\tilde{V}(t)$, where it is implied that the physically significant part is the real part of these functions. Because we are studying linear circuits the imaginary part of the functions doesn't affect the real part.

$$\tilde{I}(t) = I_{\max}\, e^{i\,\omega\, t} \text{ and } \tilde{V}(t) = V_{\max}\, e^{i\,(\omega\, t+\phi)} = Z\, e^{i\,\phi}\, I_{\max}\, e^{i\,\omega\, t} = Z\, e^{i\,\phi}\, \tilde{I}(t)$$

$$\text{where } I(t) = \mathrm{Re}\left(\tilde{I}(t)\right) \text{ and } V(t) = \mathrm{Re}\left(\tilde{V}(t)\right)$$

The voltage to current relationship is then given by the complex impedance $\tilde{Z} = Z\, e^{i\,\phi}$.

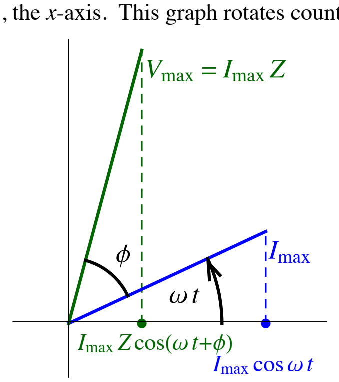
$$\tilde{V}(t) = \tilde{Z}\, \tilde{I}(t)$$

Now AC circuits can be treated just like DC circuits, except the voltage and current are now complex functions and the complex impedance $\tilde{Z} = Z\, e^{i\,\phi}$ behaves like a resistance with a complex value. The impedance and phase angle are the absolute value and argument of the complex impedance.

$$Z = |\tilde{Z}| \text{ and } \phi = \mathrm{Arg}(\tilde{Z})$$

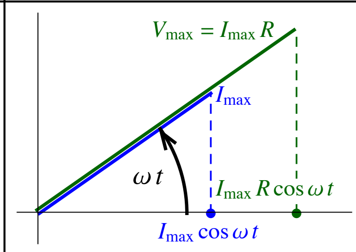
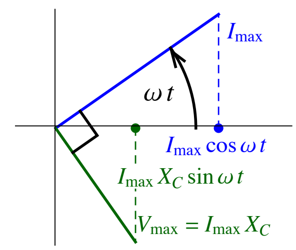
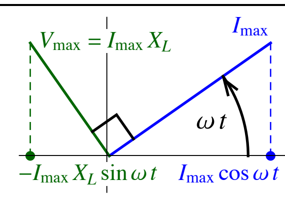
Phasor Diagrams

A phasor diagram represents the voltage and current in the complex plane. The real part of the complex value is the physically interesting part, so this represents the projection on the the Reals, the x-axis. This graph rotates counterclockwise with angular velocity ω.



Reactances and Complex Impedance

We can now find simple expressions for the complex impedance of resistors, capacitors and inductors.

	Impedance Z	Phase ϕ	Complex Impedance $\tilde{Z}=Z\,e^{i\,\phi}$	Phasor Diagram
Just R	R	0	R	
Just C	$X_C = \frac{1}{\omega\,C}$	$-\frac{\pi}{2}$	$\frac{1}{i\,\omega\,C}$	
Just L	$X_L = \omega\,L$	$\frac{\pi}{2}$	$i\,\omega\,L$	

In the above table we have used $e^{i\,\pi/2} = i$ and $e^{-i\,\pi/2} = -i = 1/i$.

Combinations of Circuit Elements

For a series circuit the voltages add and there is one current that flows through everything. A parallel circuit has one voltage and currents that add. It follows that the complex impedances $\tilde{Z} = Z\,e^{i\,\phi}$ combine in AC just like resistors in DC.

$$\tilde{Z}_{eq} = \tilde{Z}_1 + \tilde{Z}_2 + \ldots \quad \text{(for series)}$$

$$\tilde{Z}_{eq} = \left(\frac{1}{\tilde{Z}_1} + \frac{1}{\tilde{Z}_2} + \ldots\right)^{-1} \quad \text{(for parallel)}$$

$$\tilde{Z}'_{ij} = \frac{\tilde{Z}_i\,\tilde{Z}_j}{\tilde{Z}_{\shortparallel}} \text{ where } \tilde{Z}_{\shortparallel} = \left(\frac{1}{\tilde{Z}_1} + \frac{1}{\tilde{Z}_2} + \frac{1}{\tilde{Z}_3} + \ldots\right)^{-1} \quad \text{(node reduction)}$$

The Series RCL Circuit with Phasors

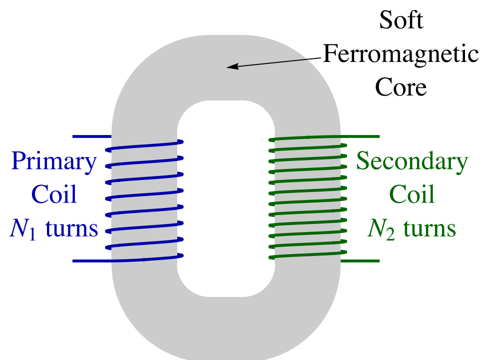
For a series RCL circuit we find the equivalent complex impedance by adding the three complex impedances.

$$Z\,e^{i\,\phi} = \tilde{Z} = R + \frac{1}{i\,\omega\,C} + i\,\omega\,L = R + i\left(\omega\,L - \frac{1}{\omega\,C}\right) = R + i\,(X_L - X_C)$$

The impedance and phase angle follow.

$$Z = |\tilde{Z}| = \sqrt{R^2 + (X_L - X_C)^2} \quad \text{and} \quad \phi = \text{Arg}(\tilde{Z}) \implies \tan\phi = \frac{X_L - X_C}{R}$$

I.7 - The Transformer



A transformer is a case of mutual inductance where, ideally, all of the flux from one coil passes through the other. Take the primary coil to have N_1 turns and the secondary to have N_2. This can be achieved to a good approximation by wrapping both coils around the same soft ferromagnetic core. The core amplifies the field due to the current and directs the field lines around the loop of the core

The fluxes are assumed to be equal.

$$\Phi = \Phi_1 = \Phi_2$$

Then by Faraday's law

$$V_1 = -N_1 \frac{d\Phi}{dt} \text{ and } V_2 = -N_2 \frac{d\Phi}{dt}$$

we get a proportionality between the voltage and number of turns.

$$\frac{V_2}{V_1} = \frac{N_2}{N_1}$$

Take the voltages to be the rms voltages then we see that the effect of the transformer is to vary the voltage. A step-up transformer increases the voltage $V_2 > V_1$ and a step-down transformer decreases the voltage $V_2 < V_1$. If we consider the power in the circuit then the instantaneous power is $\mathcal{P} = V I$, so it follows that there is an inverse proportionality between the voltage and current.

$$\frac{V_2}{V_1} = \frac{N_2}{N_1} = \frac{I_1}{I_2}$$

Transformers are very common. Many transformers have standard household voltage as its input and DC output at a different voltage. After the transformer a *rectifier circuit* is used to convert to DC. This is discussed next.

I.8 - Diodes and Rectifier Circuits

An ideal diode is a nonlinear circuit element; this means that the voltage to current relationship is not linear. It allows current to flow freely in one direction and blocks the current in the other direction. A mechanical analog is a check-valve for fluids; this allows a fluid to flow in one direction but blocks the flow in the other.

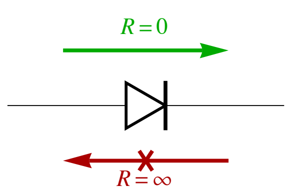


Figure - An ideal diode behaves as if its resistance is zero for current flowing in one direction and infinite for current in the other direction.

By allowing current to flow in just one direction diodes give a way of converting AC to DC. A rectifier is something that converts AC to DC. A simplest rectifier can be made with just one diode. The obvious problem with this is half of the electrical signal is lost.

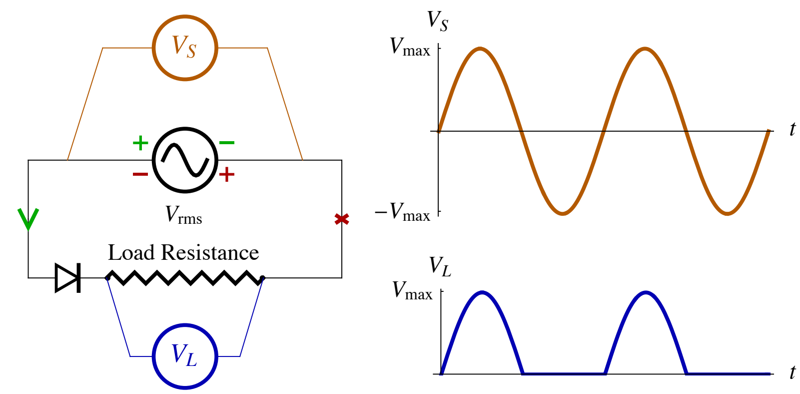


Figure - Half-wave rectifier: The voltage of the AC source is V_S and the voltage across the load is V_L. V_L truncates the negative parts of V_S.

A somewhat more complicated circuit can be used to recover the other half of the signal.

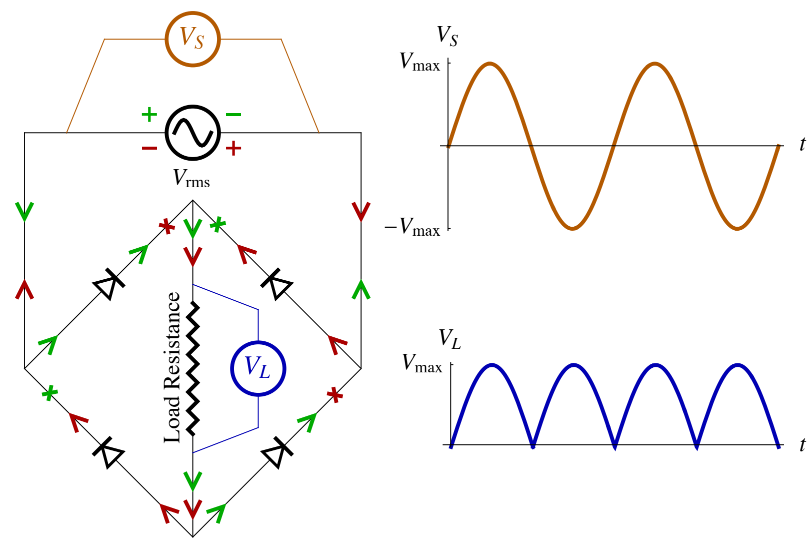


Figure - Full-wave rectifier: The voltage of the AC source is V_S and the voltage across the load is V_L, the absolute value of V_S.