Chapter J

Electromagnetic Radiation

Blinn College - Physics 2426 - Terry Honan

As an immediate consequence of Maxwell's equations in a vacuum we get a wave equation. Solving for the speed of the wave we will get

$$v = c = \frac{1}{\sqrt{\mu_0\,\varepsilon_0}}.$$

To derive this using the integral form of Maxwell's equations is awkward.

We discuss electromagnetic radiation, including radio waves, microwave, infrared radiation, visible light, ultraviolet radiation, x-rays and gamma rays. This chapter serves as a bridge between electromagnetism and optics.

J.1 - The One Dimensional Wave Equation

We will take some generic wave variable to be u which is some disturbance that varies as a function of x, the position, and t, time. For example, consider waves on a stretched string. The position along the string is labelled by x, t is time and u is the distance of a point on the string from its equilibrium position. The one dimensional wave equation is

$$\frac{\partial^2}{\partial x^2}u = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}u,$$

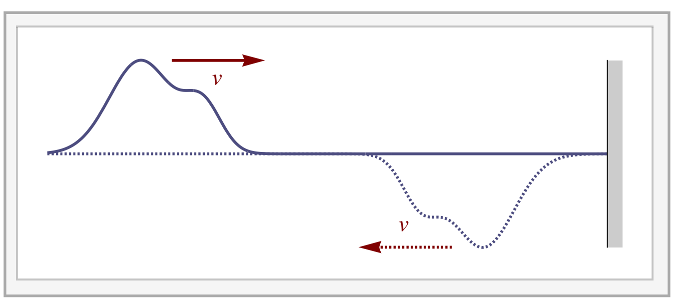
where v is the wave speed. This is a second order partial differential equation for $u(x,\,t)$.

The General Solution

The general solution of an ordinary differential equation (ODE), where we solve for functions of one variable, involves arbitrary constants. The general solution of a partial differential equation (PDE), where we are solving for functions of several variables, will involve arbitrary functions. The general solution is

$$u(x,\,t) = f(x - v\,t) + g(x + v\,t),$$

where f and g are arbitrary functions. If we take the graph $u = f(x)$ and shift it in the positive x-direction by a we get $u = f(x - a)$. This allows us to interpret our solution. The $u = f(x - v\,t)$ corresponds to a pulse of shape $u = f(x)$ moving in the positive direction with speed v and $u = g(x + v\,t)$ describes pulses of shape $u = g(x)$ moving in the negative x-direction with the same speed.



Interactive Figure

Sinusoidal Waves

We often consider waves where the shape of the pulse f (or g) are sinusoidal.

$$f(x) = A\cos k\,x$$

A is called the amplitude. k is called the wave number; this is related to the wavelength λ, which is the spatial period of the function. Since the period of cosine is 2π and the period of f is λ we get $k\lambda = 2\pi$ or

$$k = \frac{2\pi}{\lambda}.$$

If we take this function f and move it in the positive or negative direction we get $f(x \mp v\,t) = A\cos[k\,(x \mp v\,t)]$ or

$$u(x,\ t) = A\cos(k\,x \mp \omega\,t),$$

where the angular frequency ω and wave number are related to the wave speed by $k\,v = \omega$. Since the angular frequency is related to the frequency by $\omega = 2\pi f$ the wave speed can also be written in terms of the frequency and wavelength.

$$v = \frac{\omega}{k} = f\,\lambda$$

A less mathematical derivation is to observe that the wave moves one wavelength in one period, $v = \lambda/T$, and then use the fact that the period and frequency are related by $f = 1/T$.

J.2 - Plane Waves in a Vacuum

Maxwell's Equations in a Vacuum

In a vacuum there is no matter so it follows that there is no charge or currents.

$$Q = 0 \text{ and } I = 0$$

Removing the source terms from Maxwell's equations we get

$$\oint \vec{E} \cdot d\vec{A} = 0$$

$$\oint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{B} \cdot d\vec{r} = \mu_0\,\varepsilon_0 \frac{d}{dt}\Phi_e$$

$$\oint \vec{E} \cdot d\vec{r} = -\frac{d}{dt}\Phi_m$$

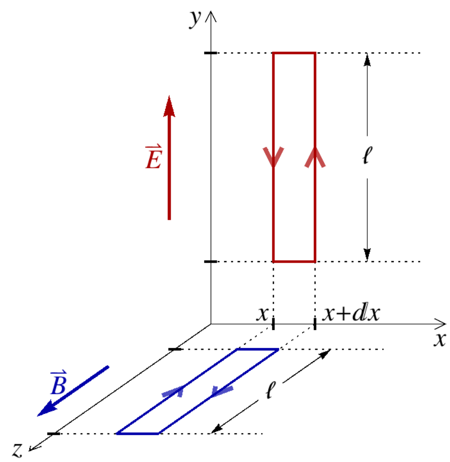
Plane Wave Assumption

A plane wave is a one dimensional wave in three dimensional space. If the wave propagates in the x-direction then we assume that nothing depends on the variables y and z, making the fields constant on a plane. This makes the problem one dimensional.

We will make an additional assumption, which will be justified when the solution is verified, that the electric and magnetic fields are mutually perpendicular and both are perpendicular to the direction of propagation. Take the electric field to be in the y-direction, the magnetic field to be in the z-direction and the x-direction to be the direction of wave propagation. The fields then take the form:

$$\vec{E} = \hat{y}\,E(x,\ t)$$
$$\vec{B} = \hat{z}\,B(x,\ t).$$

Maxwell's Equations with the Plane Wave Assumption



Applying the plane wave assumption to Maxwell's equations in a vacuum will give us a pair of coupled first order equations. Combining these equations will give a one dimensional wave equation.

First apply Faraday's law to a contour in the xy-plane. Take the contour to be of length dx in the x-direction and length ℓ in the y-direction. Now evaluate the line integral of the electric field around this contour. The segments in the x and $-x$ directions contribute 0 since the segment is perpendicular to the field. The segment in the positive y-direction at $x + dx$ contributes a $E(x + dx, t)\,\ell$ term and the segment in the negative y-direction at x gives a $-E(x, t)\,\ell$ term.

$$\oint \vec{E}\cdot d\vec{r} = E(x + dx, t)\,\ell - E(x, t)\,\ell = \frac{\partial E}{\partial x}\ell\, dx$$

The right hand side of Faraday's law involves the derivative of the magnetic flux. The magnetic field is in the z-direction which is the same as the direction of the positive normal to the surface. The magnetic flux becomes $\Phi_m = B\,\ell\,dx$ and the time derivative becomes a partial time derivative. Faraday's law becomes:

$$\oint \vec{E}\cdot d\vec{r} = -\frac{d}{dt}\Phi_m \implies \frac{\partial E}{\partial x}\ell\, dx = -\frac{\partial B}{\partial t}\ell\, dx \implies \frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}.$$

We may similarly apply Ampere's law in the xz-plane to a contour of length dx in the x-direction and length ℓ in the z-direction.

$$\oint \vec{B}\cdot d\vec{r} = B(x + dx, t)\,\ell - B(x, t)\,\ell = \frac{\partial B}{\partial x}\ell\, dx$$

The electric flux becomes $\Phi_e = -E\,\ell\,dx$ where we have the negative sign because the positive normal is in the negative y-direction with is opposite the direction of the electric field. Ampere's law then gives:

$$\oint \vec{B}\cdot d\vec{r} = \mu_0\,\varepsilon_0\,\frac{d}{dt}\Phi_e \implies \frac{\partial B}{\partial x} = -\mu_0\,\varepsilon_0\,\frac{\partial E}{\partial t}.$$

The Wave Equation and the Speed of Light

Maxwell's equations with the plane wave assumption gave us a pair of coupled first order partial differential equations.

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} \quad \text{and} \quad \frac{\partial B}{\partial x} = -\mu_0\,\varepsilon_0\,\frac{\partial E}{\partial t}$$

If we take $\frac{\partial}{\partial t}$ of the second equation and use the equality of mixed partial derivatives, $\frac{\partial}{\partial t}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}\frac{\partial}{\partial t}$, we get

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x}B = -\mu_0\,\varepsilon_0\,\frac{\partial^2}{\partial t^2}E \implies \frac{\partial}{\partial x}\frac{\partial}{\partial t}B = -\mu_0\,\varepsilon_0\,\frac{\partial^2}{\partial t^2}E.$$

Using the first equation and cancelling the sign gives a one dimensional wave equation.

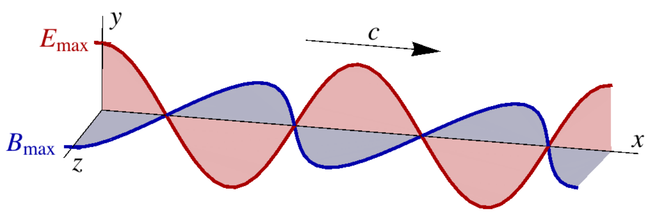
$$\frac{\partial^2}{\partial x^2}E = \mu_0\,\varepsilon_0\,\frac{\partial^2}{\partial t^2}E$$

It is clear that the above equation is of the form of the 1D wave equation $\frac{\partial^2}{\partial x^2}u = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}u$, if we associate the electric field E with the disturbance u and the wave speed v with

$$v = c = \frac{1}{\sqrt{\mu_0\,\varepsilon_0}}.$$

The constant c is the speed of light in a vacuum; it is a most fundamental constant in physics. When Maxwell first calculated this value for the speed of electromagnetic radiation, which was written in terms of the two electromagnetic constants ε_0 and μ_0, and compared it with the previously measured value of the speed of light it was clear that visible light was an example of electromagnetic radiation. This was later verified as true.

Sinusoidal Plane Waves



Interactive Figure

If we choose a sinusoidal form of the electric field, which is in the y-direction, we get

$$E(x, t) = E_{max} \cos(k\,x - \omega\,t).$$

Using one of our first order expressions $\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$ we can find the magnetic field by differentiating with respect to x and then integrating over t.

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} \implies \frac{\partial B}{\partial t} = k\,E_{max} \sin(k\,x - \omega\,t) \implies B = \frac{k}{\omega} E_{max} \cos(k\,x - \omega\,t).$$

It is clear the magnetic field, which recall is in the z-direction, is in phase with the electric field and its peak value is given by

$$B_{max} = \frac{k}{\omega} E_{max}.$$

Since the wave speed is c we can write

$$c = \frac{\omega}{k} = f\,\lambda$$

and this relates the electric and magnetic fields:

$$c = \frac{E_{max}}{B_{max}} = \frac{E}{B}.$$

Aside - Alternate Derivation of Wave Equation

Using the differential form of Maxwell's equations, discussed as an aside in Chapter G, we can derive a wave equation much more efficiently. Maxwell's equations in a vacuum ($\rho = 0$ and $\vec{J} = \vec{0}$) in differential form become:

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \text{(Gauss's Law)}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{(Gauss's Law for Magnetism)}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \, \varepsilon_0 \, \frac{\partial}{\partial t} \, \vec{E} \quad \text{(Ampere−Maxwell Law)}$$

$$-\vec{\nabla} \times \vec{E} = \frac{\partial}{\partial t} \, \vec{B} \quad \text{(Faraday's Law)}$$

We want to derive the three dimensional wave equation. Since the one-dimensional wave equation for $u(x, t)$ is $\frac{\partial^2}{\partial x^2} u = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} u$, it follows that its natural three-dimensional generalization for $u(x, y, z, t)$ is $\nabla^2 u = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} u$, where $\nabla^2 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u$.

Begin by taking the curl of Faraday's Law.

$$-\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla} \times \frac{\partial}{\partial t} \vec{B}$$

A vector calculus identity, that holds for any vector field $\vec{F}$, is $\nabla^2 \vec{F} = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{F}\right) - \vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right)$. Also, the equality of mixed partial derivatives $\left(\frac{\partial}{\partial x} \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \frac{\partial}{\partial x}\right)$ implies that $\vec{\nabla} \times \frac{\partial}{\partial t} \vec{B} = \frac{\partial}{\partial t} \vec{\nabla} \times \vec{B}$. Applying these results gives

$$\nabla^2 \vec{E} - \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) = \frac{\partial}{\partial t} \vec{\nabla} \times \vec{B}.$$

Using Gauss's Law and the Ampere-Maxwell Law we get an equation of the form of the three dimensional wave equation

$$\nabla^2 \vec{E} = \mu_0 \, \varepsilon_0 \, \frac{\partial^2}{\partial t^2} \, \vec{E}.$$

From this we can read off the speed.

$$v = c = \frac{1}{\sqrt{\mu_0 \, \varepsilon_0}}$$

J.3 - Energy and Electromagnetic Radiation

Energy Density

The energy density u in an electromagnetic field can be written as a sum over electric and magnetic contributions

$$u = u_e + u_m = \frac{1}{2} \varepsilon_0 \, E^2 + \frac{1}{2 \, \mu_0} B^2$$

We can find the average energy density $\overline{u}$ using the fact that the fields are sinusoidal and that the average of $\cos^2$ is $\frac{1}{2}$.

$$\overline{u} = \frac{1}{2} \varepsilon_0 \, \frac{1}{2} \, E_{\max}^2 + \frac{1}{2 \, \mu_0} \, \frac{1}{2} \, B_{\max}^2$$

We can verify that the electric and magnetic contributions are equal, $\overline{u}_e = \overline{u}_m$.

$$\frac{E_{\max}}{B_{\max}} = c = \frac{1}{\sqrt{\mu_0 \, \varepsilon_0}} \implies \overline{u}_e = \overline{u}_m$$

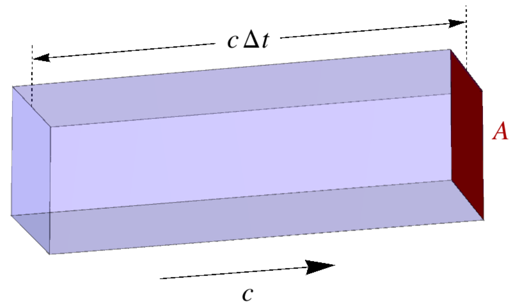
It follows that

$$\overline{u} = \frac{\varepsilon_0}{2} E_{\max}^2 = \frac{B_{\max}^2}{2 \, \mu_0} = \frac{E_{\max} \, B_{\max}}{2 \, \mu_0 \, c} = \frac{E_{\max}^2}{2 \, \mu_0 \, c^2}$$

Intensity

Intensity I is a measure of the power (energy/time) per area. If A is the area of a surface normal to the radiation, Δt is a time, U is the energy passing through the surface and $\mathcal{P}$ is the power $(U / \Delta t)$ through the surface, then these are related by

$$I = \frac{\mathcal{P}}{A} = \frac{U}{A \, \Delta t}.$$



Consider all the radiation in the right cylinder (with any shape cross-section, even rectangular) and length $c\Delta t$. Since all the radiation, and thus all the energy, is flowing at speed c, it follows that all the energy in the cylinder passes A in Δt. The volume of the cylinder is $Ac\Delta t$, so

$$U = \overline{u} A\, c\, \Delta t \text{ and } I = \frac{\mathcal{P}}{A} = \frac{U}{A\,\Delta t} = \overline{u}\, c$$

This gives several equivalent expressions for the intensity

$$I = \overline{u}\, c = \frac{E_{max}^2}{2\,\mu_0\, c} = \frac{E_{max}\, B_{max}}{2\,\mu_0}$$

Poynting Vector

The Poynting vector $\vec{S}$ is defined by

$$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0}.$$

Application of the right hand rule shows that if the electric field is in the y-direction and the magnetic field is in the z-direction, then the Poynting vector is in the x-direction. Since the fields are perpendicular, the magnitude of the cross product is just the product of the fields. Taking the average of the magnitude we get

$$\overline{S} = \frac{\overline{E\,B}}{\mu_0} = \frac{E_{max}\, B_{max}}{2\,\mu_0}.$$

This expression follows from averaging $\cos^2$ and getting 1/2. It is clear that the average value of the pointing vector is the intensity.

$$\overline{S} = I$$

The Poynting vector's direction is the direction of flow of electromagnetic waves and its magnitude is the intensity. This is quite generally the case, meaning that it is true independent of our plane wave assumptions. If $d\vec{A}$ is some infinitesimal surface and $d\mathcal{P}$ is the power (energy/time) passing through the surface then

$$d\mathcal{P} = \vec{S}\cdot d\vec{A}.$$

J.4 - Momentum and Pressure

Momentum Carried by Radiation

Electromagnetic radiation carries energy. Since there is moving energy there is also momentum. To see this consider Einstein's famous formula $E = m\,c^2$. This was shown to be generally true in 1905, but in the case of electromagnetism an analogous expression had been derived previously. We are using U for energy, so let us write this as $U = m\,c^2$. Since momentum is $p = m\,v$ and the radiation is moving at c we can write $p = m\,c$. Combining these expressions we get

$$p = \frac{U}{c}.$$

This is the momentum carried by electromagnetic radiation.

Some comments should be made on the mass referred to above. The mass in $E = m\,c^2$ is known as the *relativistic mass*. This is to be distinguished from the *rest mass*. The tabulated values of the masses of particles are their rest masses. In relativity, a particle with a rest mass can never be accelerated to the speed of light, but it can reach a speed arbitrarily close to that of light. A particle of light is known as a photon; this is called a massless particle meaning that it has no rest mass. Massless particles must always move at c.

Momentum from Radiation Normally Incident on a Surface

If the radiation is normally incident on a surface we can derive simple expressions for the momentum given to the surface. First consider the case of a surface that is a perfect absorber. All of the momentum of the radiation is given to the surface, giving

$$p = \frac{U}{c} \text{ (perfect absorber).}$$

If the surface is a perfect reflector then the change in the momentum of the radiation is twice the value of the incident radiation. Recall that momentum is a vector and here we are subtracting two vectors in the opposite direction. Since momentum must be conserved, the change in the momentum of the radiation is equal (in magnitude) to the momentum given to the surface.

$$p = 2\,\frac{U}{c} \text{ (perfect reflector)}$$

We can interpolate between these two expressions. If κ is the fraction of energy reflected then the momentum gained by the surface is

$$p = (1 + \kappa)\,\frac{U}{c} \text{ (}\kappa\text{ is the fraction reflected).}$$

Pressure and Force on a Surface from Normally Incident Radiation

Newton's second law $\vec{F}_{\text{net}} = \frac{d}{dt}\vec{p}$ relates force to momentum. The force can be related to the momentum in the case of normally incident radiation by the expression

$$F = \frac{p}{\Delta t}.$$

If A is the area of the surface and P is the pressure (using capital 'p' for pressure and lower case for momentum) then pressure is defined as force per area.

$$P = \frac{F}{A}.$$

We can then write the pressure P in terms of the momentum p.

$$P = \frac{p}{A\,\Delta t}$$

Using the definition of intensity

$$I = \frac{\mathcal{P}}{A} = \frac{U}{A\,\Delta t}.$$

We can turn the momentum expressions, which involve the energy U into expressions for the pressure involving the intensity I by dividing both sides of the momentum expressions by $A\Delta$t.

$$P = \frac{I}{c} \text{ (perfect absorber)}$$

$$P = 2\,\frac{I}{c} \text{ (perfect reflector)}$$

$$P = (1 + \kappa)\,\frac{I}{c} \text{ (}\kappa\text{ is the fraction reflected)}$$

J.5 - The Electromagnetic Spectrum

Electromagnetic radiation can be described by its wavelength and frequency. Since frequency and wavelength are related by

$$f\,\lambda = c$$

for radiation in a vacuum, it follows that all electromagnetic radiation can be written along a line of increasing frequency and decreasing wavelength.

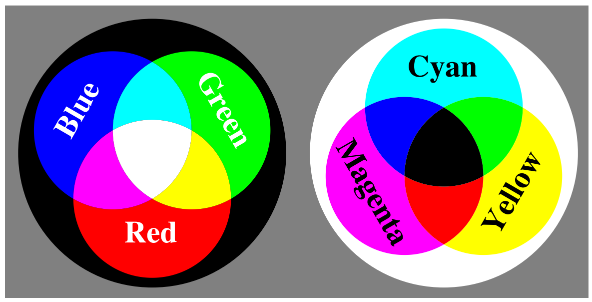
The Full Spectrum

Long wavelength waves are radio waves. Wavelengths shorter than between 1 and 1/10 meter are usually labeled microwaves. Smaller than about 1 mm start the infrared (IR) waves. This leads to the visible spectrum which is a narrow band of wavelengths between 400 and 700 nm. Below 400 nm to about 1 to 10 nm are the ultraviolet (UV) waves. Wavelengths smaller than that are called x-rays and the small wavelength limit are called gamma rays.

The Visible Spectrum and Primary Colors

The visible spectrum consists of the narrow band of wavelengths between 400 and 700 nm. In order of decreasing wavelength (increasing frequency) this is ROYGBIV: Red, Orange, Yellow, Green, Blue, Indigo and Violet. There is approximately a factor of two of wavelengths (and frequencies) we can view; this is crucial to our perception of light. Our brain glues the ends of the visible spectrum into a circle; violet appears as a reddish blue. This makes it possible to represent colors as combinations of three primary colors.

There are two notions of primary colors: additive mixing and subtractive mixing. Additive mixing is used with computer monitors and television screens. We begin with black and add colors. The additive primary colors are Red, Green and Blue. Combining all three we can get white as shown. Subtractive mixing is where we begin with white and remove colors. This is used when mixing paints or for color printers. Here the primary colors are Cyan (a blue-green color), Magenta (a reddish violet) and Yellow. Removing all three subtractive colors gives black.



Additive Mixing Subtractive Mixing