Chapter L

Interference, Diffraction and Polarization

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L.1 - Simple Interference

A sinusoidal wave traveling in one dimension has the form:

$$A \cos(k\,x \mp \omega\,t)$$

where in the case of electromagnetic radiation the amplitude A is the peak electric field, $A = E_{max}$. At some position x the disturbance varies with time by the general form

$$A \cos(\omega\,t + \phi).$$

This form generally describes waves that vary sinusoidally with time at some position even in two or three dimensions.

Now consider combining waves (at some position) from two sources with the same amplitude but with a different relative phase angle ϕ. To do this we just add the two functions of time

$$A_0 \cos \omega\,t + A_0 \cos(\omega\,t + \phi).$$

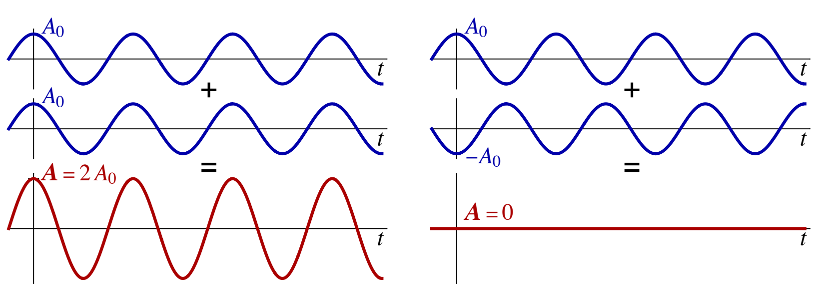
If the two waves are in phase $\phi = 0$ we get *constructive interference* and if they are totally out of phase $\phi = \pi$ we get *destructive interference*. In the constructive case

$$A_0 \cos \omega\,t + A_0 \cos \omega\,t = 2\,A_0 \cos \omega\,t$$

and the resulting amplitude of the combined wave is $A = 2\,A_0$. In the destructive case, since $\cos(\omega\,t + \pi) = -\cos \omega\,t$, we get

$$A_0 \cos \omega\,t + A_0 \cos(\omega\,t + \pi) = 0;$$

the resulting combined amplitude is $A = 0$.



Constructive Interference Destructive Interference

The more general result is a bit more complicated.

$$A_0 \cos \omega\,t + A_0 \cos(\omega\,t + \phi) = A \cos\left(\omega\,t + \frac{\phi}{2}\right) \text{ where } A = 2\,A_0 \cos \frac{\phi}{2}.$$

Before proving this result it should be mentioned that the special cases of constructive and destructive interference follow trivially: $\phi = 0$ gives $A = 2\,A_0$ and $\phi = \pi$ gives $A = 0$. To verify the result, first start with the identity

$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta.$$

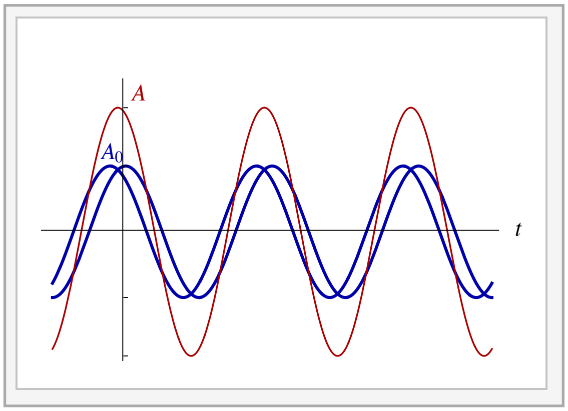
It follows that

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos\alpha \cos\beta.$$

Taking $\alpha = \omega\,t + \frac{\phi}{2}$ and $\beta = \frac{\phi}{2}$ gives

$$\cos(\omega\,t) + \cos(\omega\,t + \phi) = 2 \cos \frac{\phi}{2} \cos\left(\omega\,t + \frac{\phi}{2}\right)$$

which implies the desired result.



Interactive Figure - General Interference

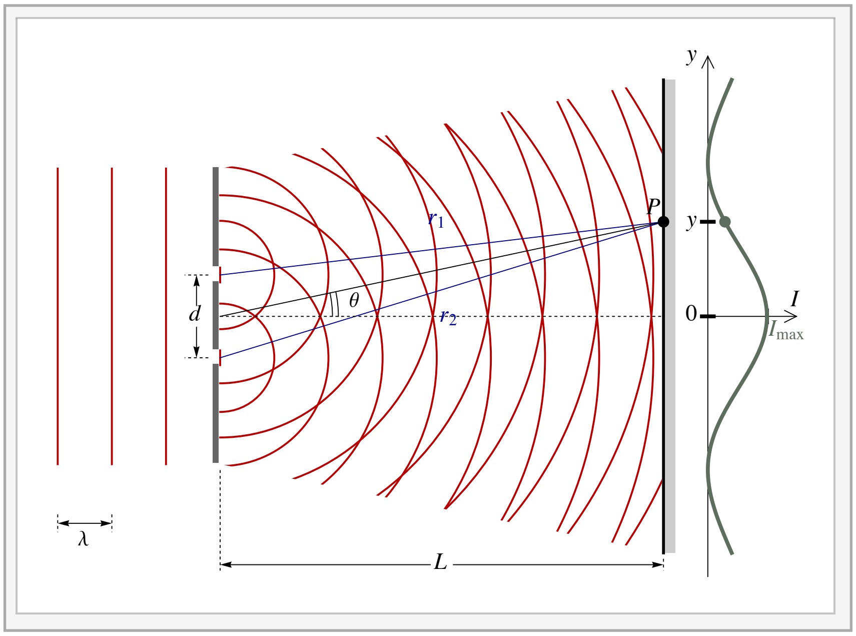
The intensity is proportional to the square of the amplitude, which is peak electric field.

$$A = E_{\max} \quad \text{and} \quad I = \frac{E_{\max}^2}{2\,\mu_0\,c} \quad \Longrightarrow \quad I \propto A^2$$

If two flashlights, each with intensity I_0 shine at the same point the resulting intensity is $2\,I_0$; this is incoherent mixing, meaning that the phase of light of one source is not related to the phase of the other. The key to interference is coherence; the phases of the two sources are related. The following table summarizes the amplitudes and intensities for the special cases of constructive and destructive interference, for the general case and for incoherent mixing. In all cases the amplitude and intensity of the uncombined waves are A_0 and I_0.

	Amplitude	**Intensity**
Constructive Interference $\phi = 0$	$2\,A_0$	$4\,I_0$
Destructive Interference $\phi = \pi$	0	0
General Case (Any ϕ)	$2\,A_0 \cos\frac{\phi}{2}$	$4\,I_0 \cos^2\frac{\phi}{2}$
Incoherent Mixing (two flashlights)	–	$2\,I_0$

L.2 - Young's Double-slit Experiment



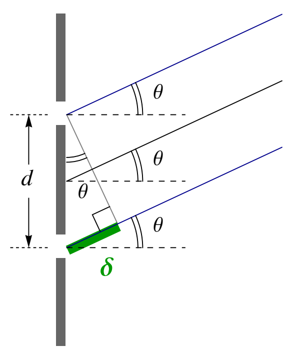
Interactive Figure

Consider monochromatic light of wavelength λ normally incident on a pair of vertical slits each of width a and separated by d, the distance between the centers of the slits. We will make the simplifying assumption of narrow slits, or $a \ll d$. We will take L to be the distance from the slits to some distant screen; by distant we mean that L is much larger than both d and λ. A point P on the screen can be labeled by θ, the angle of deflection or by y, the distance along the screen, where $y = 0$ and $\theta = 0$ describe undeflected rays. θ and y are related by

$$\tan\theta = \frac{y}{L}.$$

The distance from one slit to P is r_1 and r_2 is the distance from the other slit to P. Each r is much larger than d but their difference δ is on the same order. Using trig and the approximation that $d \ll r$ we get $\delta = d \sin\theta$

$$\delta = r_2 - r_1 = d \sin\theta.$$



Constructive and Destructive Interference

Huygen's principle states that every point on a wave front behaves as a separate point source. When a plane wave hits a pair of narrow slits then each slit represents a different source. The key point is that each source is in phase with the other; this coherence is the key to interference. If the two distances r_1 and r_2 are equal (so $\delta = 0$) then the light rays hitting P are in phase and there is constructive interference. Moreover, if the path difference δ is an integer number of wavelengths there is constructive interference. If the path difference is half a wavelength then the rays are out of phase and there is destructive interference.

Using that the path difference is $d \sin\theta$ and taking m to be any integer we can write the conditions for constructive and destructive interference.

$$d \sin\theta = m\lambda \qquad \text{(constructive interference)}$$

$$d \sin\theta = \left(m + \frac{1}{2}\right)\lambda \qquad \text{(destructive interference)}$$

where $m = 0, \pm 1, \pm 2, \ldots$ and $m + \frac{1}{2} = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}.....$

Intensity Formula

We can generalize the preceding results. We can relate the path difference δ to the phase difference ϕ. Every wavelength path difference corresponds to a 2π phase difference, so we can write

$$\frac{\delta}{\lambda} = \frac{\phi}{2\pi} \implies \phi = \frac{2\pi}{\lambda}\delta = \frac{2\pi}{\lambda} d \sin\theta$$

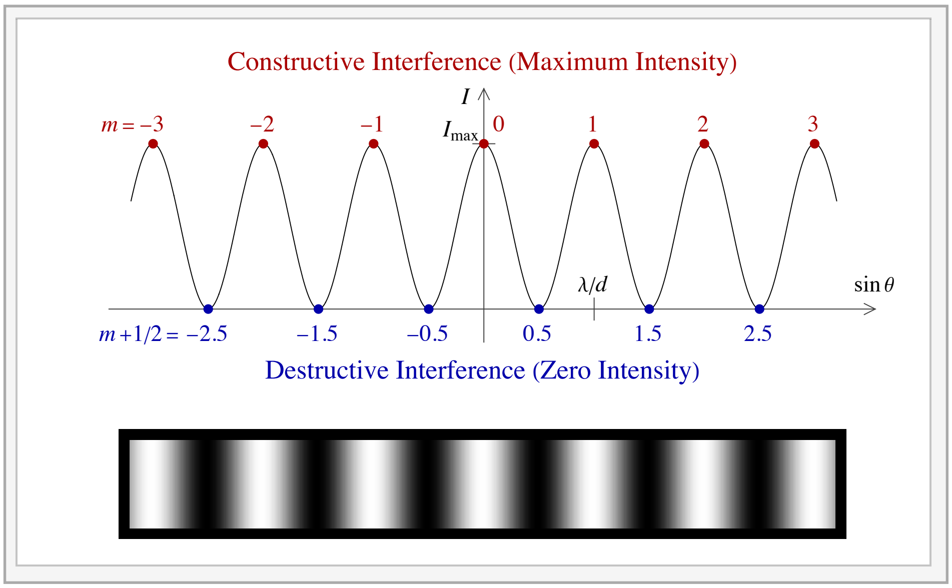
since the intensity for the sum of two waves of equal amplitude out of phase by ϕ satisfies $I \propto \cos^2 \frac{\phi}{2}$ we get

$$I = I_{max} \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right),$$

where I_{max} is defined as the peak intensity, which is the value at $\theta = 0$.

Constructive and destructive interference are simple special cases of this intensity formula. The peak intensity is when $\cos^2$ is one and that is when $\frac{\pi d \sin\theta}{\lambda} = m\pi$; this implies the constructive interference formula. Destructive interference is when $I = 0$ and that is when $\frac{\pi d \sin\theta}{\lambda} = \left(m + \frac{1}{2}\right)\pi$, implying the destructive condition.

Graph of Intensity



Interactive Figure - Intensity Pattern for Double-slit

Small Angle Formula

It is a common case that the angle θ is small. This is equivalent to y being small compared to L.

$$\theta \text{ is small} \iff y \ll L.$$

Since $\tan\theta = y/L$ and for small angles $\sin\theta \simeq \tan\theta$ we can replace $\sin\theta$ with y/L,

$$\sin\theta \simeq \frac{y}{L} \text{ for small } \theta.$$

Example L.1 - Young's Double-slit Experiment

Suppose light shines through two narrow vertical slits separated by 0.0165 mm and onto a screen 3.5 m away.

(a) If light from a He-Ne laser shines through these slits, then at what distance from the center of the central maximum are the first and second dark fringes and the first and second bright fringes? Laser light is monochromatic, meaning that is has only one color or wavelength and the most common red laser light is from a He-Ne laser which has the wavelength 632.8 nm. Use the small-angle approximation.

Solution

First, list what we are given. The slit separation is d and the distance to the screen is L.

$$d = 0.0165 \times 10^{-3}\ \text{m}\ ,\quad L = 3.5\ \text{m}\ \text{ and }\ \lambda = 632.8 \times 10^{-9}\ \text{m}$$

The center of the central bright fringe is at $\theta = 0$ or $y = 0$, so the distance from the center of the central bright fringe is just y. The dark fringes are at $d \sin\theta = (m + 1/2)\ \lambda$ and the first two dark fringes correspond to $m + 1/2$ having the values $1/2$ and $3/2$. We then find the distance from the center y from the angle using $\tan\theta = y/L$. Here we will use the small-angle approximation $\sin\theta \simeq \tan\theta = y/L$

$$d \sin\theta = d\,\frac{y}{L} = \left(m + \frac{1}{2}\right)\lambda \implies y = \left(m + \frac{1}{2}\right)\frac{\lambda L}{d}$$

$$1^{\text{st}}\ \text{dark fringe}:\ y = \frac{1}{2}\,\frac{\lambda L}{d} = 0.067\ \text{m}\quad\text{and}\quad 2^{\text{nd}}\ \text{dark fringe}:\ y = \frac{3}{2}\,\frac{\lambda L}{d} = 0.201\ \text{m}$$

The bright fringes are at $d \sin\theta = m\ \lambda$ and the first two correspond to $m = 1$ and $m = 2$. Repeating the previous procedure gives.

$$d \sin\theta = d\,\frac{y}{L} = m\ \lambda \implies y = m\,\frac{\lambda L}{d}$$

$$1^{\text{st}}\ \text{bright fringe}:\ y = 1\,\frac{\lambda L}{d} = 0.134\ \text{m}\quad\text{and}\quad 2^{\text{nd}}\ \text{bright fringe}:\ y = 2\,\frac{\lambda L}{d} = 0.268\ \text{m}$$

(b) With the setup for part (a), what is the smallest distance from the center of the central bright fringe to where the intensity is half the maximum intensity. Again, use the small-angle approximation.

Solution

The intensity formula with the small-angle approximation becomes:

$$I = I_{\text{max}} \cos^2\!\left(\frac{\pi d \sin\theta}{\lambda}\right) = I_{\text{max}} \cos^2\!\left(\frac{\pi d\, y}{\lambda L}\right)$$

In addition to the given information from part (a), we also know that $I/I_{\text{max}} = 1/2$. We can then solve for y.

$$y = \frac{\lambda L}{\pi d} \cos^{-1} \sqrt{\frac{1}{2}} = 0.0336\ \text{m}$$

Note that when evaluating the arccosine you must have your calculator in the radians mode.

(c) If white light shines through this double-slit then what is the angular separation $\Delta\theta$ between the first bright fringes of red light with $\lambda = 700$ nm and of violet light with $\lambda = 400$ nm.

Solution

The d value is the same. Since we are only concerned with angles here we do not need L.

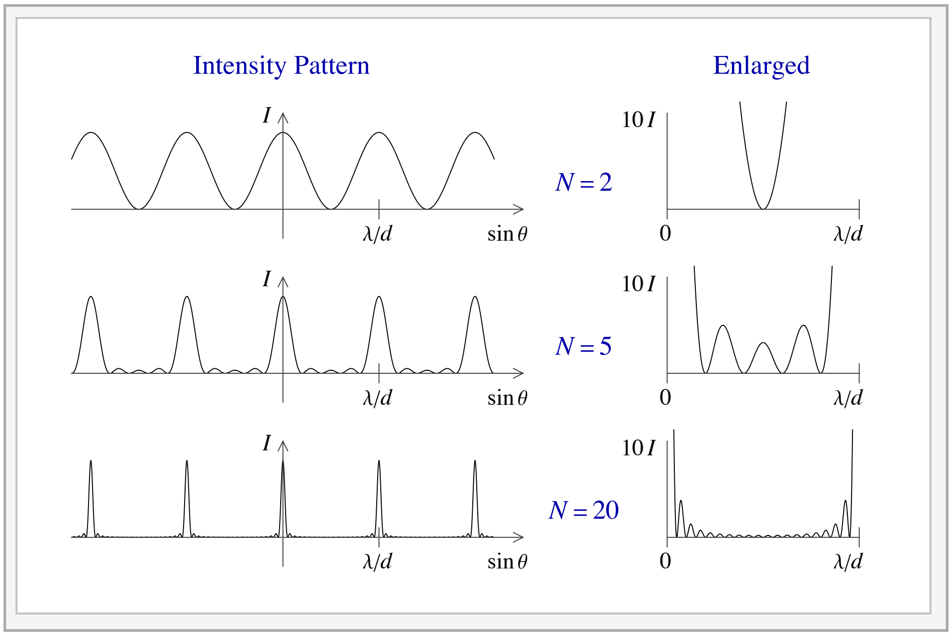
$$\lambda_{\text{red}} = 700 \times 10^{-9}\ \text{m}\ \text{ and }\ \lambda_{\text{violet}} = 400 \times 10^{-9}\ \text{m}$$

The first bright fringe is at $m = 1$ so $d \sin\theta = m\ \lambda$ gives $\theta = \sin^{-1}(\lambda/d)$.

$$\theta_{\text{red}} = \sin^{-1}\!\left(\frac{\lambda_{\text{red}}}{d}\right) = 2.43\,^\circ\ \text{ and }\ \theta_{\text{violet}} = \sin^{-1}\!\left(\frac{\lambda_{\text{violet}}}{d}\right) = 1.39\,^\circ \implies \theta_{\text{red}} - \theta_{\text{violet}} = 1.04\,^\circ$$

L.3 - Many Slits and Diffraction Gratings

Now consider the case of N narrow slits. The following graphs show the intensity graphs. The graphs on the right is an enlargement of one period of the graph where the intensity is multiplied by 10. Note that the condition for constructive interference is the same but as the number of slits increases the intensity goes to zero at all positions in between.



Plot: Parameter {−2, 0} in {{−2, 0}, −2.2, 2.2} is a raw expression and cannot be used as a variable.

Interactive Figure - Interference pattern for *N* slits

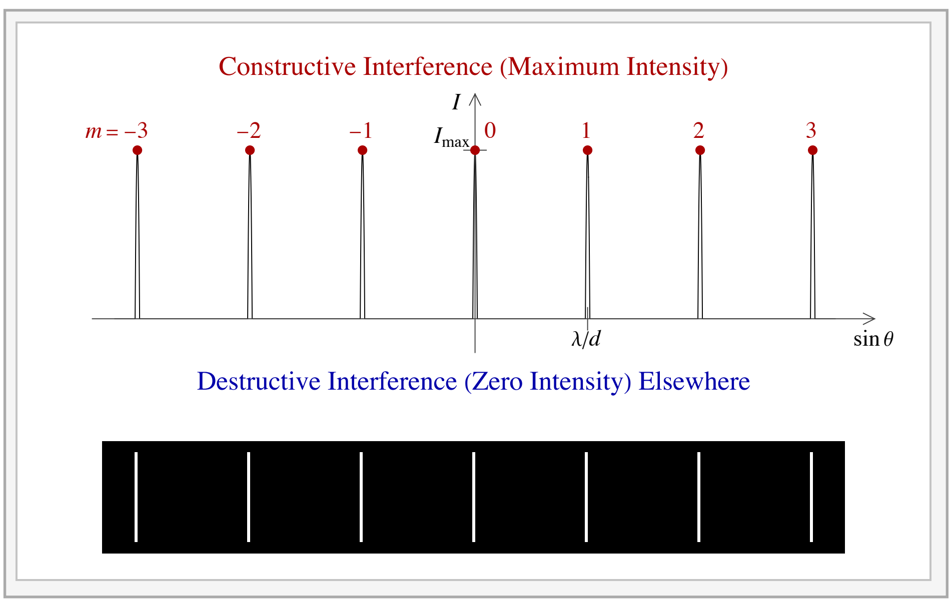
As an aside, it should be mentioned that the intensity formula for the above graphs is

$$I = I_{\max}\left[\frac{\sin(N\,k\,d\sin\theta)}{N\sin(k\,d\sin\theta)}\right]^2 \quad \text{where} \quad k = \frac{2\pi}{\lambda}.$$

A diffraction grating is the limit as N becomes large. The interference conditions for a diffraction grating become

$$d\sin\theta = m\lambda \;\; \text{for constructive interference}$$

and destructive interference elsewhere.



Interactive Figure - Intensity Pattern for DIffraction Grating

When a laser, which has monochromatic light, is shot through a diffraction with vertical slits, you observe a series of horizontally-spaced dots representing the constructive interference. The difference from the double-slit is that it is dark everywhere else.

Note that the position of the m^{th} maximum varies with wavelength. If white light passes through a diffraction grating then the central fringe ($m = 0$) is the same for all wavelengths and thus is white, but the higher order fringes will break light into its spectrum. Diffraction gratings are better for spectroscopy (resolving light into its constituent wavelengths) than prisms. With a prism, red is the color that refracts the least. This is reversed with a diffraction grating; since red has the longest wavelength, it diffracts the most.

Example L.2 - The Visible Spectrum and the *m* = 1 Fringe

Show that for any slit spacing d, the visible spectrum (from 400 nm to 700 nm) for the $m = 1$ fringe does not overlap with the spectrum for the $m = 2$ fringe. Show also that far any d the second-order spectrum always overlaps the third-order spectrum.

Solution

The wavelengths satisfy $400\text{ nm} \le \lambda \le 700\text{ nm}$. With this we can find an expression for the range of possible angles for the m^{th} fringe.

$$d \sin\theta = m\lambda \implies \sin\theta = \frac{m\lambda}{d} \implies \frac{m\,400\text{ nm}}{d} \le \sin\theta \le \frac{m\,700\text{ nm}}{d}$$

The largest value of $\sin\theta$ for the $m = 1$ fringe is $1\times 700\text{ nm}/d$ and the smallest value of $\sin\theta$ for $m = 2$ fringe is $2\times 400\text{ nm}/d$. Because $700 < 2\times 400$ we can see that for any d there is no overlap.

For the $m = 2$ fringe, the largest value of $\sin\theta$ is $2\times 700\text{ nm}/d$ and the smallest value of $\sin\theta$ for $m = 3$ fringe is $3\times 400\text{ nm}/d$. Because $2\times 700 > 3\times 400$ we can see that for any d there is overlap.

Example L.3 - Diffraction Grating

Light from a He-Ne laser (wavelength 632.8 nm) shines through a diffraction grating with 230 lines/mm.

(a) If the light passes onto a screen 5.8-m away from the grating then what is the distance from the central bright fringe to the first, second and third order fringes on the screen?

Solution

First we need to find the slit spacing from the lines per length; since d is the distance between lines we take the reciprocal.

$$d = \frac{1}{230\text{ lines/mm}} = \frac{1\text{ mm}}{230} = \frac{10^{-3}\text{ m}}{230} = 4.348\times 10^{-6}\text{ m}$$

We are also given the wavelength and L.

$$\lambda = 632.8\text{ nm} \quad\text{and}\quad L = 5.8\text{ m}$$

The angle for the m^{th} fringe is found from $d\sin\theta = m\lambda$. From θ we can then find the distance from the center on the screen y using $\tan\theta = y/L$.

$$\text{for } m = 1,\ \theta = \sin^{-1}\left(1\,\frac{\lambda}{d}\right) = 8.37° \implies y = L\tan\theta = 0.853\text{ m}$$

$$\text{for } m = 2,\ \theta = \sin^{-1}\left(2\,\frac{\lambda}{d}\right) = 16.9° \implies y = L\tan\theta = 1.76\text{ m}$$

$$\text{for } m = 3,\ \theta = \sin^{-1}\left(3\,\frac{\lambda}{d}\right) = 25.9° \implies y = L\tan\theta = 2.81\text{ m}$$

(b) From Part (a) of this problem we can see that as the order m increases, the angle increases. There is only a finite number of fringes since the angle must be less than 90°. Using this laser and grating, what is the highest order bright fringe and how many fringes are present in total?

Solution

Since we have the same laser and grating, d and λ are the same as in Part (a). Because the angle cannot be larger than 90° the sine function must be less than 1.

$$d\sin\theta = m\lambda \implies m\frac{\lambda}{d} = \sin\theta < 1 \implies m < \frac{d}{\lambda} = 6.87$$

m is an integer, so it follows that the largest such m is

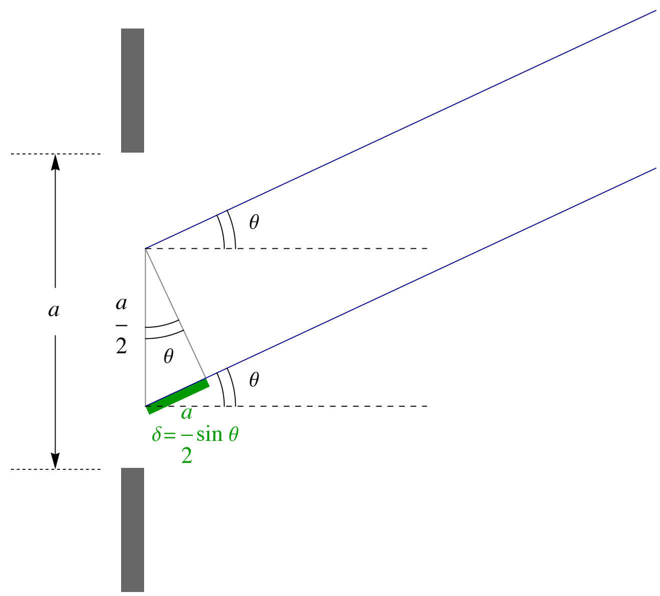
$$m_{\text{max}} = 6$$

This is the highest order fringe. For the total number of bright fringes, remember that there are fringes on either side of the center and one in the center.

$$\text{total number of fringes} = 2\,m_{\max} + 1 = 13$$

L.4 - Single-slit Diffraction

When light passes through a single narrow slit of width a one observes a diffraction pattern. Here each point in the slit is a separate source. It is simple to find the condition for destructive interference.



Consider two rays from points separated by $a/2$ in the slit. The difference of the path lengths δ is given by

$$\delta = r_2 - r_1 = \frac{a}{2}\sin\theta$$

If this distance is half a wavelength.

$$\delta = \frac{a}{2}\sin\theta = \frac{\lambda}{2}.$$

then those two rays destructively interfere. But if this condition is met then the ray leaving *any* point in the slit will be exactly canceled by the ray from $a/2$ away and this is the condition for destructive interference for all the rays.

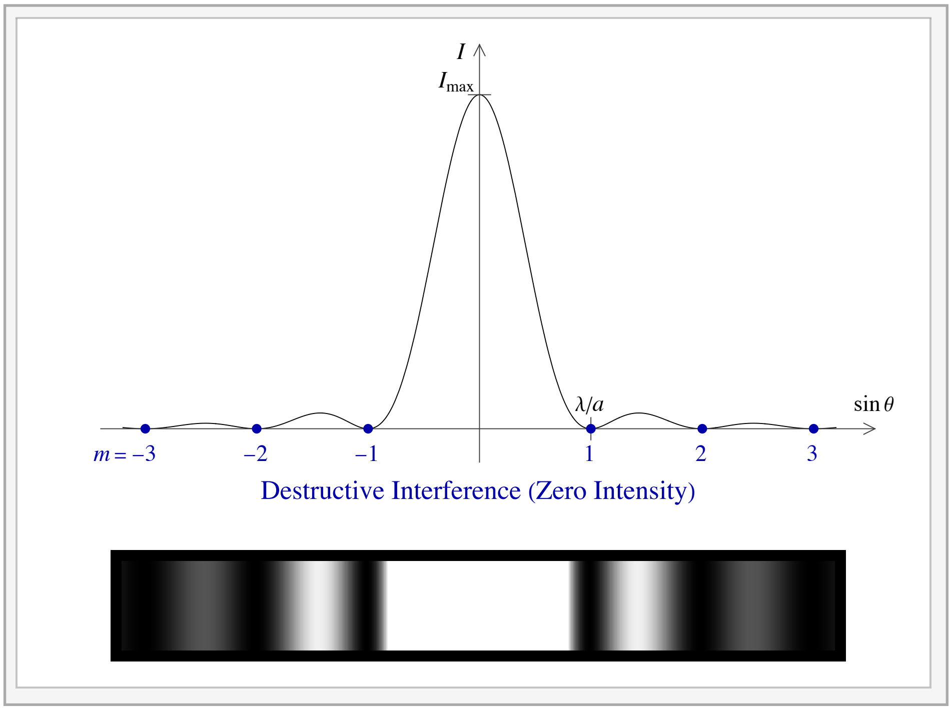
This argument fails if we break the slit into thirds, because there would always be one ray that is not canceled. However if we break the slit into any even number of pieces $2\,m$, then light from a point $\frac{a}{2\,m}$ away would destructively interfere and all the rays could be canceled

$$\delta = \frac{a}{2\,m}\sin\theta = \frac{\lambda}{2}.$$

The condition for destructive interference is then

$$a\sin\theta = m\,\lambda, \quad \text{where } m = \pm 1,\ \pm 2,\ \pm 3,\ \ldots \text{(destructive interference)}$$

Below is the graph of the intensity pattern for a single slit. Note that the $m = 0$ position is not destructive interference; it is the center of the central bright fringe.



Interactive Figure - Intensity Pattern for Single Slit of width *a*

Without proof, we give the intensity formula for the above graph is

$$I = I_{max}\left[\frac{\sin[(\pi\, a/\lambda)\sin\theta]}{(\pi\, a/\lambda)\sin\theta}\right]^2.$$

Example L.4 - SIngle-slit Diffraction

Light from a He-Ne laser (wavelength 632.8 nm) shines through a narrow vertical slit with a width of 0.075mm and onto a screen 3.5 m away.

(a) What is the width of the central maximum on the screen?

Solution

We are given a, L and λ.

$$a = 0.075\times 10^{-3}\ \text{m}\,,\quad L = 3.5\ \text{m}\ \text{and}\ \lambda = 632.8\ \text{nm}$$

The central maximum is from $m = -1$ to $m = 1$ but it will just be twice the distance from the center to the $m = 1$ dark fringe. If this distance is labeled y_1 and the angle θ_1, then we are looking for $2\,y_1$

$$a\sin\theta = m\,\lambda \implies a\sin\theta_1 = 1\,\lambda \implies \theta_1 = \sin^{-1}\left(\frac{\lambda}{a}\right) = 0.483\,° \implies y_1 = L\tan\theta_1 = 0.0295\ \text{m}$$

$$\implies 2\,y_1 = 0.0581\ \text{m}$$

(b) What is the distance from the center to the fifth dark fringe.

Solution

Since we start counting the dark fringes at $m = 1$, the fifth fringe is must $m = 5$. We want the y value for $m = 5$.

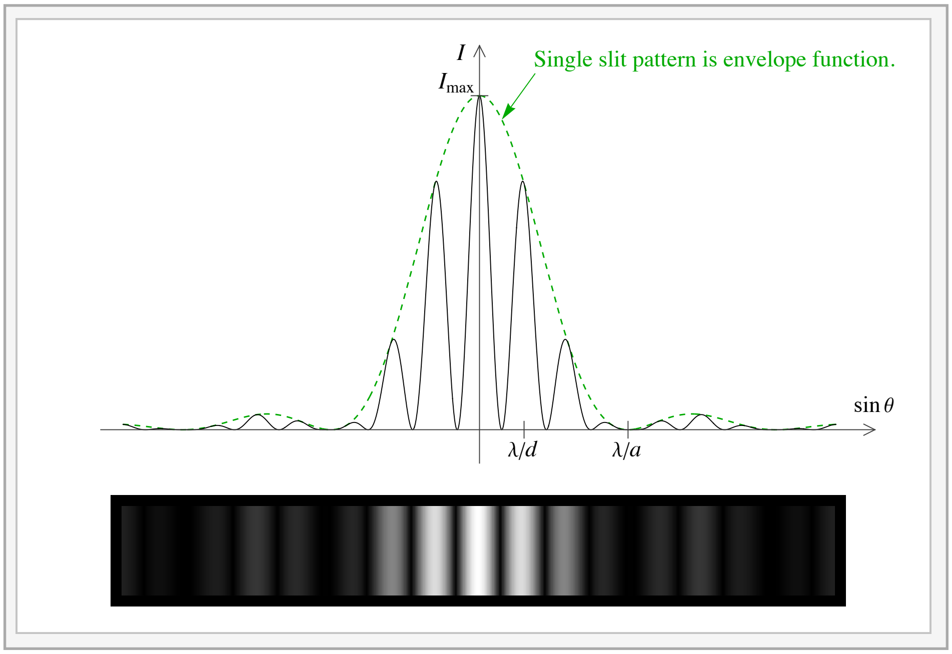
$$a\sin\theta = m\,\lambda \implies a\sin\theta_5 = 5\,\lambda \implies \theta_5 = \sin^{-1}\left(\frac{5\,\lambda}{a}\right) = 2.42\,° \implies y_5 = L\tan\theta_5 = 0.148\ \text{m}$$

The Double-slit with a Finite Width

In the discussion of the double-slit experiment we considered the case of narrow slits. We now consider the case where the slit width a is not negligible compared to d. The effect of this, which we will not prove, is that we get the intensity formula for the double-slit sitting inside the single-slit pattern, which serves as an envelope function. The single-slit intensity formula is: $I = I_{max} \{\sin[(\pi a/\lambda)\sin\theta]/((\pi a/\lambda)\sin\theta)\}^2$ and the double-slit intensity is: $I = I_{max}\cos^2(\pi d \sin\theta/\lambda)$. Combining these two expressions we get the intensity pattern for two slits of width a separated by d.

$$I = I_{max}\left(\frac{\sin[(\pi a/\lambda)\sin\theta]}{(\pi a/\lambda)\sin\theta}\right)^2 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)$$

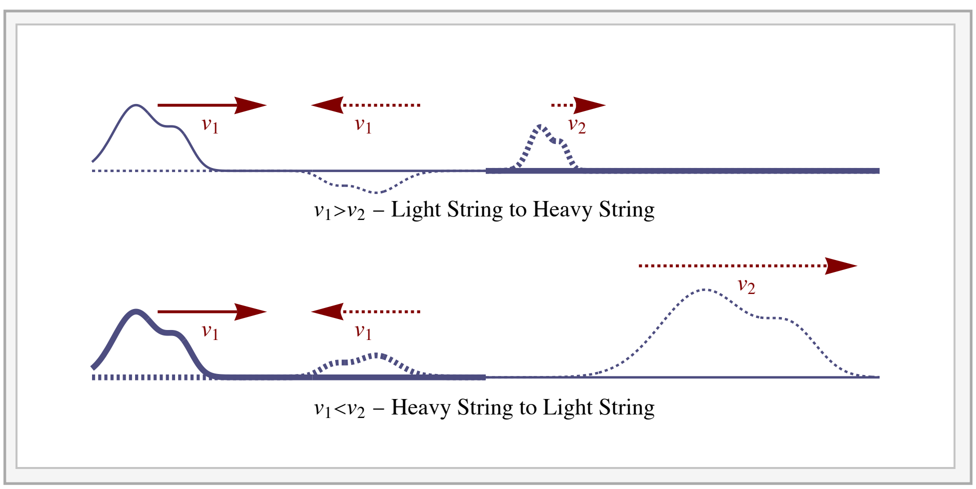
Since d is measured from the centers of the slits, it must be true that $a < d$.



Interactive Figure - Intensity Pattern for Two Slits of width *a* separated by *d*

L.5 - Thin Film Interference and Phase Change under Reflection

Phase Change under Reflection



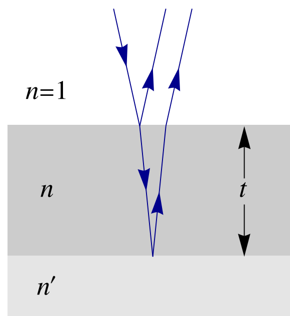
Interactive Figure

When any wave moves from one medium to another there is a reflected wave and a refracted (or transmitted) wave. If the wave moves from a medium with a higher speed to one with a lower speed ($v_1 > v_2$) there is a 180° phase shift in the reflected wave. If it moves from lower to higher speed ($v_1 < v_2$) there is no phase shift. In the case of light a smaller speed corresponds to a larger index of refraction. The behavior of light under reflection can then be summarized.

$n_1 < n_2$	180 ° phase shift
$n_1 > n_2$	no phase shift

Thin Film Interference

Consider light normally incident on a thin film with index n of thickness t sitting on a medium with index n'. At the top interface there is always a phase shift of 180°. At the bottom surface there may or may not be a phase shift; if $n < n'$ there is a 180° phase shift and if $n > n'$ there is no phase shift. The interference is between the two reflected pulses, one of the top interface and one off the bottom.



The path difference is $2\,t$ and the wavelength in the film is λ/n. For the case of $n < n'$ the two phase shifts cancel. If the path difference is an integer number of wavelengths there is constructive interference and a half integer number of wavelengths corresponds to destructive interference. The other case of $n > n'$ adds a relative 180° phase shift between the two reflected rays and has the effect of swapping the conditions of constructive and destructive interference. The table summarizes this.

	Constructive Interference	Destructive Interference
$n < n'$	$2\,t = m\,\frac{\lambda}{n}$	$2\,t = \left(m + \frac{1}{2}\right)\frac{\lambda}{n}$
$n > n'$	$2\,t = \left(m + \frac{1}{2}\right)\frac{\lambda}{n}$	$2\,t = m\,\frac{\lambda}{n}$

Example L.5 - Anti-glare Coating

An anti-glare coating on a lens minimizes reflective light which then maximizes the transmitted light. A plastic lens for eyeglasses has an index of 1.60. What is the thinnest layer of an anti-glare coating with index of 1.28 that will minimize reflection for light in the middle the visible spectrum at 550 nm.

Solution

Here we have destructive interference and $n < n'$, since $n = 1.28$ is the index of the film and $n' = 1.60$ is the index of the plastic.

$$n < n' \text{ and destructive interference} \implies 2\,t = \left(m + \frac{1}{2}\right)\frac{\lambda}{n}$$

The minimum thickness is when $m + \frac{1}{2} = \frac{1}{2}$

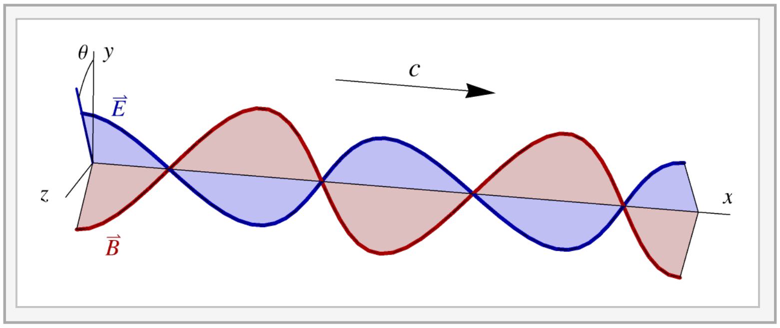
$$\lambda = 550\text{ nm} \implies t_{\text{min}} = \frac{1}{4}\,\lambda_{\text{film}} = \frac{1}{4}\,\frac{\lambda}{n} = 107\text{ nm}$$

L.6 - Polarization

In discussing plane waves in Chapter J we found a solution of the form:

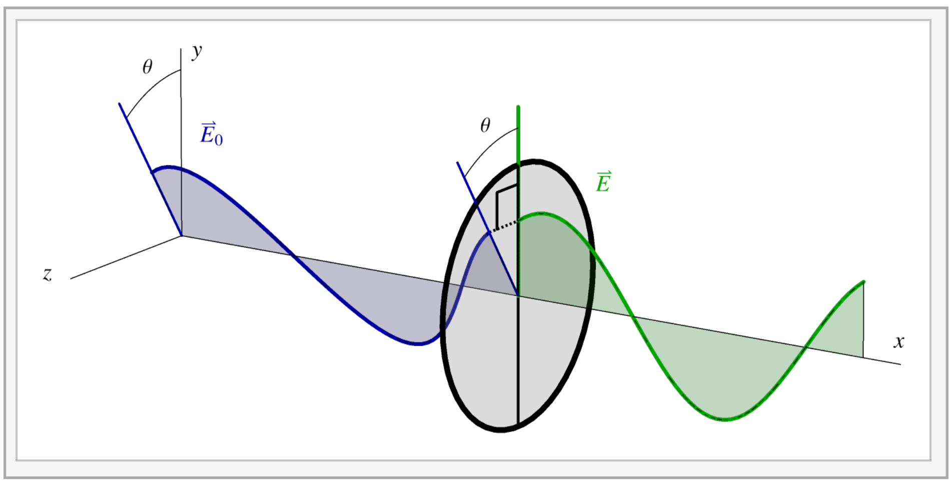
$$\vec{E} = \hat{y}\, E_{\max} \cos(k\,x - \omega\,t)$$
$$\vec{B} = \hat{z}\, B_{\max} \cos(k\,x - \omega\,t).$$

Here we have waves propagating in the x - direction with electric and magnetic fields in the y and z - directions, respectively. Generally for plane wave solutions the electric and magnetic fields are mutually perpendicular and both are perpendicular to the direction of propagation. There is a plane of possible directions perpendicular to the direction of propagation, representing different polarizations. We take the direction of the electric field to be the direction of polarization of the wave; this is the y - direction in the solution shown above. For a general polarization we rotate the solution above for the electric field by an angle θ in the yz-plane as shown below.



Polarized Light through a Filter

A polarizing filter lets through only the component of the electric fields along the axis of polarization. Suppose polarized light is incident on a polarizing filter where the angle between the polarizing axis of the filter and the polarization of the light is θ.



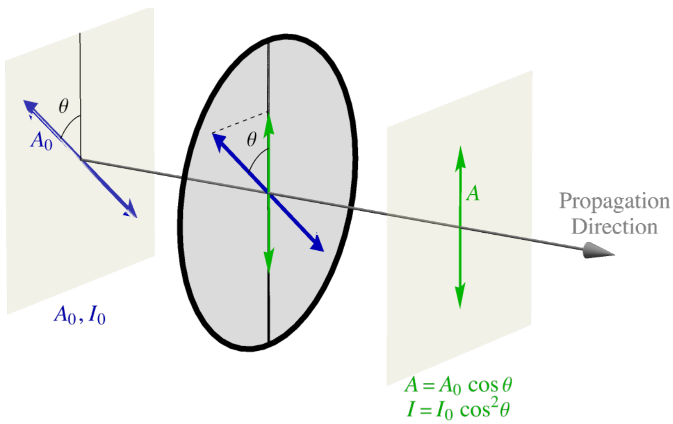
If $A_0 = E_{\max}$ is the amplitude of polarized light incident on a filter then

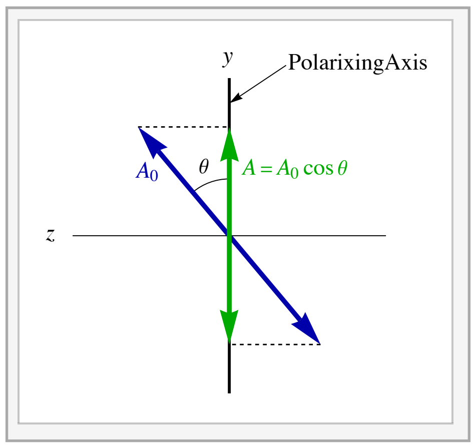
$$A = A_0 \cos\theta$$

is the amplitude of light leaving the filter. Since the intensity is proportional to the square of the amplitude $I \propto A^2$, we get

$$I = I_0 \cos^2\theta$$

relating the intensities of the light before (I_0) and after (I) the filter. This relation is known as Malus's law.



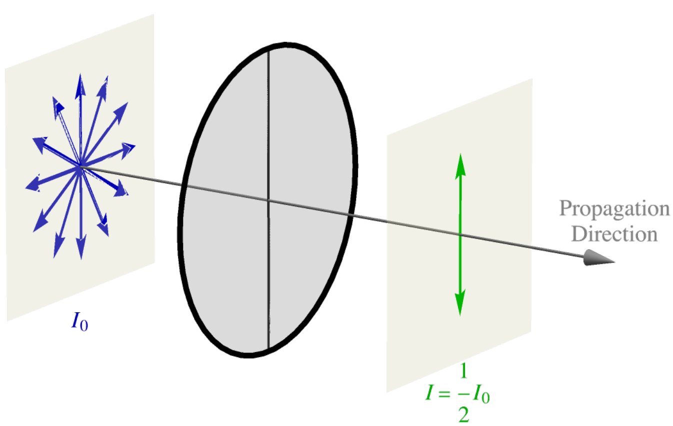


Unpolarized Light through a Filter

Viewing normal ambient light through a polarizing filter usually shows no effect when the filter is rotated. This is because the light is unpolarized, meaning that it is a random mixture of all polarizations. Since the average value of $\cos^2$ is 1/2 we get.

$$I = \frac{1}{2} I_0$$

relating the intensities before and after the filter.

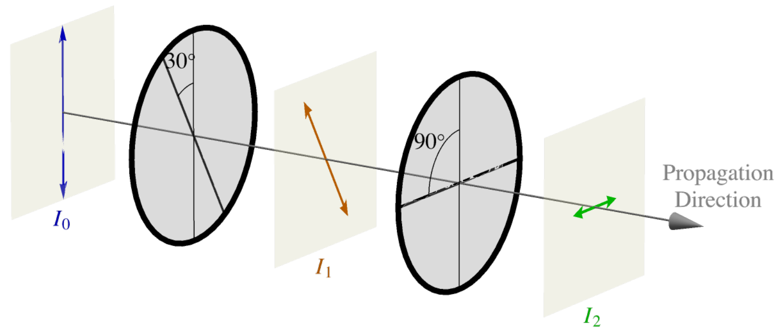


Example L.6 - Light through Two Polarizing Filters

Light with an intensity of 2000 $\mathrm{W/m^2}$ passes through two polarizing filters, the first has an axis at an angle of 30° from vertical and the second has a horizontal (90° from vertical) axis.

(a) If the initial light is vertically polarized, then what is the intensity of the light between the filters and after the filters.

Solution



Take $I_0 = 2000\ \mathrm{W/m^2}$ to be the intensity of the light before the filters.

$$I = I_0 \cos^2\theta$$

Label the intensity between to be I_1 and after to be I_2. The angle between the polarization of the incoming light and the first filter is $\theta_1 = 30°$, so we get

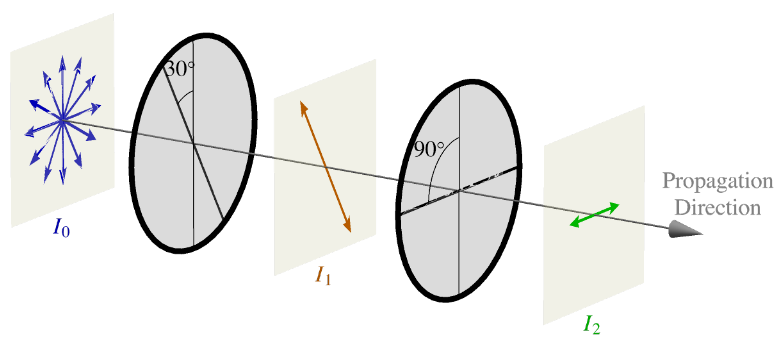
$$I_1 = I_0 \cos^2\theta_1 = 1500\ \mathrm{W/m^2}$$

The light between the filters is polarized at θ_1. Take the polarizing angle of the second filter to be $\theta_2 = 90°$. It follows that the angle between the polarization angle of the light hitting the second filter and the axis of the second filter is $\theta_2 - \theta_1 = 60°$. We can then find the intensity after the second filter.

$$I_2 = I_1 \cos^2(\theta_2 - \theta_1) = 375\ \mathrm{W/m^2}$$

(b) If the initial light is unpolarized, then what is the intensity of the light between the filters and after the filters.

Solution



Now we have the same I_0 before the filters but when unpolarized light passes through a filter it leave with half the intensity and polarized along the axis of the filter.

$$I = \frac{1}{2} I_0$$

So now between the filters we have

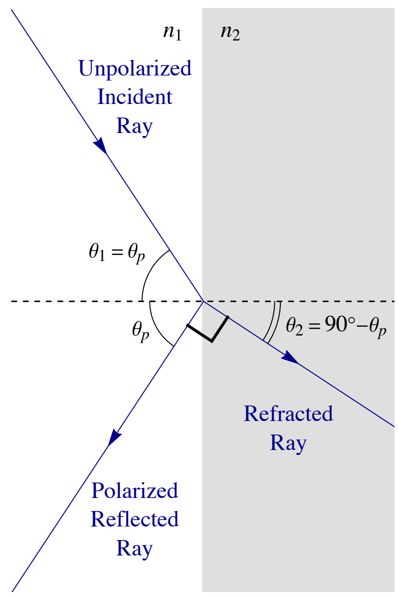
$$I_1 = \frac{1}{2} I_0 = 1000\ \mathrm{W/m^2}$$

The light between the filters is polarized at θ_1. Take the polarizing angle of the second filter to be $\theta_2 = 90°$. It follows that the angle between the polarization angle of the light hitting the second filter and the axis of the second filter is $\theta_2 - \theta_1 = 60°$. We can then find the intensity after the second filter.

$$I_2 = I_1 \cos^2(\theta_2 - \theta_1) = 250\ \mathrm{W/m^2}$$

Polarization by Reflection

Reflected light tends to be polarized. When light reflects of a surface at some angle (not normally incident) there is one possible polarization that is parallel to the surface. The reflected light tends to be polarized in this parallel direction. For instance, light reflecting off a horizontal surface tends to be horizontally polarized. Polarizing sunglasses have filters with vertical axes to remove this reflected light or glare.



It turns out that when the reflected light ray is perpendicular to the refracted ray, the reflected ray is *totally polarized*. A simple application of Snell's law gives that this occurs when the incident angle is $\theta_1 = \theta_p$ where θ_p, the polarizing angle or Brewster's angle, is

$$\tan\theta_p = \frac{n_2}{n_1}.$$

Example L.7 - Polarization of Reflected Light

At a time near sunset when sunlight is horizontal, sunlight reflects off the side window of a bus.

(a) How is the light polarized, more horizontally or more vertically?

Solution

Since the surface is vertical the reflected light will tend to be vertically polarized. Note that standard polarizing sunglasses will select for this light.

(b) Taking the index of refraction for the window glass to be 1.52 , at what incident angle will the reflected light be totally polarized?

Solution

$$n_1 = 1,\ n_2 = 1.52 \text{ and } \tan\theta_p = \frac{n_2}{n_1} \implies \theta_p = \tan^{-1}\left(\frac{n_2}{n_1}\right) = 56.7\,°$$

L.7 - Aside on Quantization

The Photon Hypothesis

Newton published a major work on optics late in his life, well after his work on mechanics and calculus. Newton had a particle theory of light and was able to explain all that we discussed in the previous chapter on geometric optics. For instance, if the speed of a particle of light changes when moving between different media, then that change in speed would cause a bending, explaining Snell’s law of refraction. The shortcomings of a particle theory of light were already becoming clear by the time of Newton’s work. The particle theory was fully unable to explain the interference and diffraction phenomena discussed in this chapter; by early in the nineteenth century, the 1800s, the wave nature of light was fully established.

In 1905, the same year as his first papers on Special Relativity, Einstein published a paper on the photoelectric effect; in that, he introduced the concept of a photon, a particle of light. The photoelectric effect is the physics behind a photocell; when light hits a metal surface the energy carried by the radiation can stimulate an electron to be emitted from the surface. What was observed was baffling. When light with a frequency below some threshold, regardless of the light’s intensity, no electrons were emitted. Above that frequency threshold electrons would be emitted even at low intensities, but the number of electrons would increase with intensity. Einstein’s hypothesis was that light is absorbed and emitted in discrete units, quanta of light which we now call photons, and the energy E of the photon is proportional to its frequency f.

$$E = hf$$

The constant h is known as Planck's constant; it was introduced by Planck five years earlier to give a description of the pattern of radiation produced by a hot glowing body. The photoelectric effect explanation was simple. It requires a minimum amount of energy E_0 to kick an electron off the surface of a metal and if the photon had less energy than E_0 then no electrons would be emitted; the minimum (threshold) frequency then is $f_0 = E_0/h$.

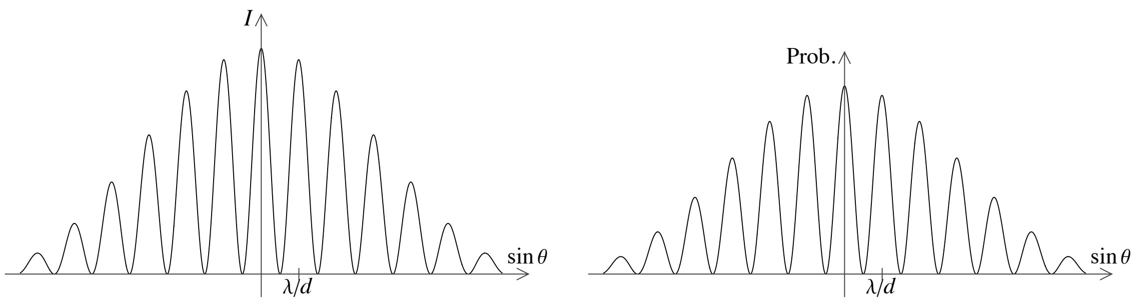
The way we see it now is that light is both a wave and a particle. It travels as a wave but is always absorbed and emitted one photon at a time. That light is both a wave and a particle has far-reaching implications. Einstein never fully accepted these implications and by the time he was middle-aged had largely become an obsolete figure in physics.

Intensity and Probability

Quantum physics is inherently probabilistic. When a beam of light shines through a pair of slits we see the Young's double-slit intensity pattern. If we decrease the intensity of the light beam drastically, where there light passes through the slits essentially one photon at a time, then we observe the same pattern for the probability of a photon landing in particular place. The key concept is that the probability is proportional to the intensity.

$$\text{Probability} \propto \text{Intensity}$$

The total probability always must be one, so the intensity pattern is rescaled (or normalized) to make the its integral one.



On the left: Intensity for a beam of light passing through two slits with a small but finite width.
On the right: The probability of an individual photon landing in a particular place with this setup.

For another example, consider a beam of vertically polarized light passing through a polarizing filter with a polarizing axis at a 45° angle from vertical. The intensity of the light after the filter I is related to the intensity before the filter I_0 by

$$I = I_0 \cos^2 \theta = I_0 \cos^2 45\,° = \frac{1}{2} I_0$$

Since the probability is proportional to the intensity, the probability of a photon passing through the filter is one-half.

$$\text{Probability} = \frac{1}{2}$$

This is a quantum coin-flip; just as a coin has a 50-50 chance of landing on heads, the photon has a 50-50 chance of passing through the filter.

Quantum physics is inherently probabilistic. We can no longer predict the outcomes of experiments; all we can do is predict the probabilities of the outcomes. By performing an experiment many times, for example by using a beam of many photons, we can test our predictions of the probabilities. Classical physics as described by Newton's three laws is deterministic. That means that if we know the initial conditions of some experiment exactly, then Newton's second law gives the differential equation that exactly predicts the outcome of that experiment. With quantum physics, this determinism was lost from fundamental physics.

de Broglie Matter Waves

In section J.4, we saw that the momentum carried by radiation is related to the energy of that radiation by $p = U/c$, where U is the energy, p is the momentum and c is the speed of light. There we used U instead of E to avoid confusing it with the electric field; here we will use E. It follows then for a photon, we have

$$p = E/c = hf/c = h/\lambda$$

where $\lambda = c/f$ is the wavelength. So particles make matter waves. If a beam of electrons is shot through two slits, as in the optical double-slit experiment, then we observe the same interference pattern with the electron probabilities.

Einstein, with his photon hypothesis, showed that a wave can have particle properties. In 1924, de Broglie proposed the reverse; particles like electrons have a wave nature. The wavelength of a particle with momentum p is given by

$$\lambda = h/p$$

An electron microscope is based on this wave property of electrons. With visual wavelengths of light one cannot make out details that are smaller than the wavelength of the light, and for visible light that is, as we have seen, on the order of hundreds of nanometers or less than a micron = μm. Electrons can, with sufficient momentum, have much smaller wavelengths and thus can used to view objects much smaller.

de Broglie's matter wave hypothesis was the key insight that led to modern quantum mechanics. If there are matter waves then what wave equation do they satisfy. A year later Schrodinger built on de Broglie's wave hypothesis and found the wave equation that matter waves satisfied. That was the birth of modern quantum mechanics. Between de Broglie and Schrodinger, Heisenberg published an alternative version of quantum mechanics that proved to be equivalent to the Schrodinger approach.