Chapter B - Problems

Blinn College - Physics 2426 - Terry Honan

Problem B.1

A disk with a 12 cm radius is rotated through all possible orientations in a uniform electric field. If it is found that the maximum electric flux is 300 $\mathrm{N \cdot m^2/C}$ then what is the magnitude of the electric field?

Solution to B.1

$$\Phi = \vec{E} \cdot \vec{A} = E\,A \cos\theta \implies \Phi_{max} = E\,A$$
$$\implies E = \frac{\Phi_{max}}{A} = \frac{\Phi_{max}}{\pi\, r^2} = \frac{300}{\pi\; 0.12^2} = 6630\,\frac{\mathrm{N}}{\mathrm{C}}$$

Problem B.2

A flat surface of area A sits in an electric field of $\vec{E} = \langle a, b, 0\rangle$. (This field is in the xy-plane.) What is the flux through the surface if it sits
(a) in the yz-plane?
(b) in the xz-plane?
(c) in the xy-plane?

Solution to B.2

$\Phi = \int \vec{E} \cdot d\vec{A} = \vec{E} \cdot \vec{A}$, where the first expression is the general definition of flux. This becomes the second form is the case of a uniform field and a flat surface. For a flat surface we write $\vec{A} = A\,\hat{n}$, where A is the area of the surface and $\hat{n}$ is the unit normal vector to the surface. If $\hat{n}$ is a unit normal to a surface then so is $-\hat{n}$, giving a sign ambiguity in the flux.

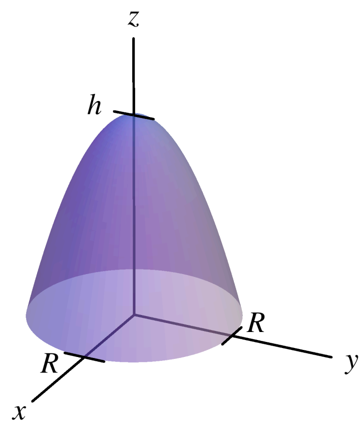
Note that the dot product of a vector with a unit vector gives the component of the vector in the direction of the unit vector.

(a) $\hat{n} = \pm\hat{x} \implies \Phi = \pm E_x\,A = \pm a\,A$

(b) $\hat{n} = \pm\hat{y} \implies \Phi = \pm E_y\,A = \pm b\,A$

(c) $\hat{n} = \pm\hat{z} \implies \Phi = \pm E_z\,A = 0$

Problem B.3



Consider the truncated paraboloid shown. What is the flux through the paraboloid due to the electric fields:

(a) $\vec{E} = \langle a, 0, 0 \rangle$?

(b) $\vec{E} = \langle 0, b, 0 \rangle$?

(c) $\vec{E} = \langle 0, 0, c \rangle$?

(d) $\vec{E} = \langle a, b, c \rangle$?

Solution to B.3

(a) and (b) This is a field in the x-direction (or y-dirrection). Any field line will enter the paraboloid on one side and leaves on the other. This gives zero net contribution.

$$\Phi = 0$$

(c) This is a field in the z-direction. Flux is a measure of the number of field lines passing through a surface. In this problem every line that passes through the parabaloid passes through the flat end at its base. The disk at the base is perpendicular to the field. The area vector for the base is $\vec{A} = A\,\hat{n} = \pi\,R^2\,\hat{z}$.

$$\Phi = \vec{E} \cdot \vec{A} = E\,A = c\,\pi\,R^2$$

(d) As in the previous case the flux through the disk at the base is the same as flux through the paraboloid. The only lines that pass through the paraboloid that don't pass through the base enter on one side and leave on the other and thus contribes zero. If the field is an an arbitrary direction then it is only the z-component of the field that matters.

$$\Phi = \vec{E} \cdot \vec{A} = \vec{E} \cdot \hat{z}\,\pi\,R^2 = E_z\,\pi\,R^2 = c\,\pi\,R^2$$

Problem B.4

A point charge Q sits above an infinite horizontal plane. What is the flux of the charge through the plane?

Solution to B.4

Half of the field lines of a point charge will pass below a horizontal plane containint the charge and half will pass above it. If a charge sits above a horizontal plane then any half the field lines will eventually pass through the plane. Thus

$$\Phi_{\text{plane}} = \frac{1}{2}\,\Phi_{\text{total}} = \frac{Q}{2\,\varepsilon_0}.$$

We can neglect the contribution of any horizontal field lines because they will not contribute to the flux. On a sphere the equator doesn't contribute to its area.

For an infinite plane this is independent of the distance of the charge from the plane.

Problem B.5

A point charge Q sits at the center of a cube with sides of length ℓ. What is the electric flux through one face of the cube?

Solution to B.5

The total flux through all six faces of the cube is, by Gauss's law, given by: $\Phi_{\text{total}} = \frac{Q}{\varepsilon_0}$. Since the charge is at the center of the cube we know, by symmetry, that each face will have the same flux. Thus $\Phi_{\text{total}} = 6\,\Phi_{\text{face}}$ and

$\Phi_{\text{face}} = \frac{Q}{6\,\varepsilon_0}$. Note that this answer is independent of the length of the sides.

Problem B.6

(a) An insulating sphere with a 12 cm radius has a uniform charge of 216 μC. What is the charge inside spherical Gaussian surfaces of radius 4 cm, 6 cm and 15 cm?

(b) A conducting sphere with a 12 cm radius has a net charge of 216 μC. What is the charge inside spherical Gaussian surfaces of radius 4 cm, 6 cm and 15 cm?

Solution to B.6

For a sphere of radius $R = 12$ cm and uniform charge $Q = 216\,\mu$C, we can find the charge inside a spherical surface of radius r by considering the two cases:

(a) For $r < R$ the fraction of the charge is the fraction of the volume.

$$r < R \implies Q_{\text{inside}} = Q\,\frac{V_{\text{inside}}}{V_{\text{total}}} = Q\,\frac{\frac{4}{3}\pi\,r^3}{\frac{4}{3}\pi\,R^3} = Q\left(\frac{r}{R}\right)^3$$

This applies to the 4 cm and 6 cm cases:

$$r = 4\text{ cm} \implies Q_{\text{inside}} = 216\left(\frac{4}{12}\right)^3 = 8\,\mu\text{C}$$

$$r = 6\text{ cm} \implies Q_{\text{inside}} = 216\left(\frac{6}{12}\right)^3 = 27\,\mu\text{C}$$

For $r > R$ all af the charge is inside the Gaussian surface.

$$r = 15\,\text{cm} \implies Q_{\text{inside}} = Q = 216\,\mu\text{C}$$

(b) For any conductor in electrostatics, since the electric field is zero inside it all of the charge is at its surface.

$$r < R \;\; (r = 4\,\text{cm and } r = 6\,\text{cm}) \implies Q_{\text{inside}} = 0$$

As before, for $r > R$ all af the charge is inside the Gaussian surface.

$$r = 15\,\text{cm} \implies Q_{\text{inside}} = Q = 216\,\mu\text{C}$$

Problem B.7

A 12 m long line with a uniform charge of 3 μC. Consider a tube of radius 8 cm and of length 2 cm; adding 8 cm radius disks to each end makes it a closed surface. Take this to be our Gaussian surface. Take the line to be the central axis of the tube and the center of the line to be the center of the tube.
(a) What is the total charge enclosed by the Gaussian surface?
(b) Using the assumptions that 2 cm and 8 cm are both much less than 12 m, find the electric field at the surface of the tube.

Solution to B.7

(a) The charge per length of the line of charge is $\lambda = 3\times10^{-6}/12 = 2.5\times10^{-7}\,C/m$. The charge per length times the length of the Gaussian surface gives Q_{inside}.

$$Q_{\text{inside}} = \lambda\times0.02\,\text{m} = 5\times10^{-9}\,\text{C}$$

(b) Making the assumptions that the length of the line charge is much longer than the other lengths causes the field to point radially away from the line. At every point on the tube the field is perpendicular to the tube. We get

$$\int_{\text{tube}} \vec{E}\cdot d\vec{A} = E\times A_{\text{tube}} = E\times2\,\pi\,r\,L.$$

The flux through the entire Gaussian surface is the same as the flux through the tube. The field is parallel to the face of the disks at the ends and thus the flux through them is zero.

$$\oint \vec{E}\cdot d\vec{A} = E\times2\,\pi\,r\,L$$

Putting this together using Gauss's law gives the electric field magnitude.

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{\text{inside}}}{\varepsilon_0} \implies E\times2\,\pi\,0.08\times0.02 = \frac{5\times10^{-9}}{8.85\times10^{-12}} \implies E = 56\,200\,\frac{\text{N}}{\text{C}}$$

The direction of the field points radially away from the line. Call $\hat{r}$ the unit vector in this direction. The field then becomes:

$$\vec{E} = 56\,200\,\frac{\text{N}}{\text{C}}\,\hat{r}.$$

Problem B.8

Consider the general case of spherical symmetry where there is a charge distribution given by $\rho(r)$, where r is the radial distance from some origin.
(a) Find a general expression for the electric field as a function of postion in terms of an integral over the charge density.
(b) For the case of $\rho(r) = a/r$ find the field as a function of position.

Solution to B.8

(a) In any case of spherical symmetry we have:

$$E \times 4\pi r^2 = \oint \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_0} Q_{\text{inside}} \implies \vec{E} = k_e \frac{Q_{\text{inside}}}{r^2} \hat{r}$$

Here the charge distribution is not uniform so we must integrate to get $Q_{\text{inside}}(r)$. The variable of integration is a dummy variable; it can be anything except r, since we already have an r. Call the variable of integration r'; its limits are: $0 \le r' \le r$.

The integral is over spherical shells of radius r' and thickness dr'. The infinesimal volume of each shell is:

$$d\,\text{Volume} = 4\pi r'^2\, dr'.$$

(Think of this as the area $4\pi r'^2$ multiplied by its thickness dr'.)

$$Q_{\text{inside}}(r) = \int_0^r \rho(r')\, 4\pi r'^2\, dr'$$

This gives the general expression:

$$\vec{E} = k_e \frac{Q_{\text{inside}}}{r^2} \hat{r} = \vec{E} = \hat{r} \frac{k_e}{r^2} \int_0^r \rho(r')\, 4\pi r'^2\, dr'.$$

(b) Using $\rho(r) = a/r$ gives a simple expression for $Q_{\text{inside}}(r)$.

$$Q_{\text{inside}}(r) = \int_0^r \frac{a}{r'} 4\pi r'^2\, dr' = 4\pi a \int_0^r r'\, dr' = 2\pi a r^2$$

This gives the electric field as:

$$\vec{E} = k_e \frac{Q_{\text{inside}}}{r^2} \hat{r} = 2\pi k_e a \hat{r} = \frac{a}{2\varepsilon_0} \hat{r}.$$

Note that this charge distribution has the odd property that the magnitude of the electric field is constant.

Problem B.9

What is the magnitude of the electric field a perpendicular distance of 6 cm from the surface of a large uniformly charged plane with a charge per area of $9\ \mu\text{C}/m^2$?

Solution to B.9

We derived in the lecture notes that $\vec{E} = \frac{\sigma}{2\varepsilon_0} \hat{n}$ where $\hat{n}$ is the unit normal pointing away from the surface. The field magnitude is just $E = \frac{\sigma}{2\varepsilon_0}$. The field is uniform, so it is independent of the distance from the sheet and the distance of 6 cm is irrelevant.

$$E = \frac{\sigma}{2\varepsilon_0} = \frac{9\times10^{-6}}{2 \cdot 8.85\times10^{-12}} = 5.08\times10^5\ \frac{\text{N}}{\text{C}}$$

Problem B.10

Consider a disk in the xy-plane of radius R with a uniform charge Q. Take z to be the perpendicular distance above the center of the disk.

(a) What is the electric field just above (a small z value $z \ll R$) this surface?

(b) What is the field at a large perpendicular distance z ($z \gg R$) above the disk?

Solution to B.10

(a) For $z \ll R$ we can treat the disk as infinite with a surface charge density σ given by $\sigma = \frac{Q}{\pi R^2}$. Since for a uniform infinite plane $\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$ where $\hat{n}$ is the unit normal pointing away from the surface. Here $\hat{n} = \hat{z}$, so

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n} = \frac{Q}{2\pi\varepsilon_0 R^2}\hat{z}$$

(b) For large z ($z \gg R$) we can treat the distribution as a point charge.

$$\vec{E} = k_e \frac{Q}{r^2}\hat{r} = k_e \frac{Q}{z^2}\hat{z}$$

Problem B.11

A long straight conducting rod of radius R is give a charge per length of λ.

(a) What is the electric field as a function of r, the perpendicular distance from the center? Give answers for both possible cases: $r < R$ and $r > R$.

(b) If $R = 5$ cm and $\lambda = 30$ nC/m, then what is the electric field magnitude at $r = 3$ cm, 10 cm and 100 cm? What is the surface charge density (charge per area) on the conductor?

Solution to B.11

In any case of cylindrical symmetry we have: $\vec{E} = \frac{Q_{\text{inside}}/L}{2\pi\varepsilon_0 r}\hat{r}$. This was derived in the lecture notes.

(a) Since $r < R$ is inside the conductor, we have $\vec{E} = \vec{0}$ for $r < R$.

For $r > R$ all the charge is inside a Gaussian surface so $Q_{\text{inside}}/L = \lambda$. It follows that $\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$ for $r > R$.

(b) At 3 cm: since $r < R$, $E = 0$.

At 10cm and 100cm: since both outside the rod so $E = \frac{\lambda}{2\pi\varepsilon_0 r} = \frac{30\times10^{-9}}{2\pi\cdot 8.85\times10^{-12}\, r}$.

$$r = 0.1\text{ m} \implies E = 5400\,\frac{\text{N}}{\text{C}} \quad\text{and}\quad r = 1\text{ m} \implies E = 540\,\frac{\text{N}}{\text{C}}$$

The total charge inside a segment of length L is $Q = \lambda L$. The surface area of the conductor for that length is $A = 2\pi R L$. We can now find σ, the charge per area.

$$\sigma = \frac{Q}{A} = \frac{\lambda}{2\pi R}$$

Problem B.12

The electric field at the surface of a conducting sphere is 300 N/C.

(a) What is the surface charge density (charge per area) on the conductor?

(b) If the total charge on the conductor is 2 nC, then what is the radius of the sphere?

Solution to B.12

(a) The surface sharge density σ is related to the field by: $E = \sigma/\varepsilon_0$. This gives

$$\sigma = \varepsilon_0 E = 8.85\times 10^{-12}\, 300 = 2.66\times 10^{-9}\ \frac{\text{C}}{m^2}.$$

(b) Since σ is the charge per area and $Q = 2\times 10^{-9}$ C

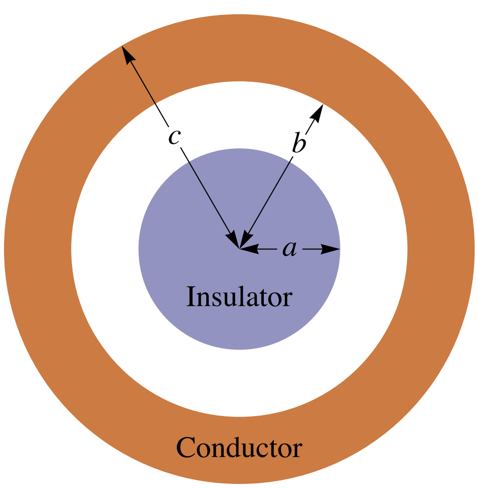
$$\sigma = \frac{Q}{A} = \frac{Q}{4\pi r^2} \implies r = \sqrt{\frac{Q}{4\pi\sigma}} = 0.245\text{ m}.$$

Problem B.13

A solid insulating sphere of radius a has a uniform charge Q. This sits inside of a hollow conducting sphere with and inside radius b and outside radius c. The conductor is given a net charge of q. (All spherical surfaces are concentric.)

(a) What is the electric field as a function of position? Give answers for all cases: $r < a$, $a < r < b$, $b < r < c$ and $r > c$.

(b) Specify the distribution of charge by giving the charge densities. If the charge is spread over a volume then give its volume density ρ and if it is over a surface give its surface density σ.



Solution to B.13

(a) For spherical symmetry Gauss's law gives: $\vec{E} = k_e \frac{Q_{\text{inside}}}{r^2} \hat{r}$.

For $r < a$ the uniform charge distribution gives

$$Q_{\text{inside}} = \frac{Q}{\frac{4}{3}\pi a^3} \frac{4}{3}\pi r^3 = Q \frac{r^3}{a^3}.$$

Which gives: $\vec{E} = k_e \frac{Q}{a^3} r \hat{r} \quad (r < a)$.

Between a and b: $Q_{\text{inside}} = Q$ so $\vec{E} = k_e \frac{Q}{r^2} \hat{r} \quad (a < r < b)$.

Inside the conductor, between b and c, the electric field is zero.
$\vec{E} = \vec{0} \quad (b < r < c)$.

Outside the conductor $Q_{\text{inside}} = Q + q \Longrightarrow \vec{E} = k_e \frac{Q+q}{r^2} \hat{r} \quad (r > c)$.

(b) Inside the insulating sphere the charge density is $\rho = \frac{Q}{\frac{4}{3}\pi a^3}$ (for $r < a$).

Charge on a conductor is on its surface. In this case there are two surfaces.
Since the electric field is zero inside a conductor, Gauss's law applied to a gaussian surface enclosing the hole in the conductor (between b and c) gives $Q_{\text{inside}} = 0$. It must be true that the charge on the inside surface cancels the charge Q at the center. The charge on the inside surface (at $r = b$) is $-Q$. Since the total charge on the conductor is q the charge on the outside surface is $Q + q$.

The surface charge density is the charge per area. For a spherical surface: $\sigma = \frac{Q}{4\pi r^2}$.

On the inside surface at b: $\sigma = \frac{-Q}{4\pi b^2}$.

On the outside surface at a: $\sigma = \frac{Q+q}{4\pi a^2}$.