Chapter C - Problems

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Problem C.1

Here we will study the speed v of an electron after it is accelerated from rest across a potential difference of magnitude V.

(a) What is the sign of the change in potential? In other words: Is $\Delta V = +V$ or $\Delta V = -V$?

(b) Using the standard nonrelativistic kinetic energy formula of $K = \frac{1}{2} m v^2$ find the speed.

(c) Find the value for the speed in part (b) when $V = 12\,\text{V}$. The standard nonrelativistic expression for kinetic energy given above only applies to particles with a speed that is significantly less than the speed of light. What fraction of the speed of light is this value?

(d) In the picture tube of a television it is typical to have voltages (potential differences) of $V = 25\,\text{kV}$. Find the speed of an electron after accelerating across this voltage using the nonrelativistic expression. What fraction of the speed of light is this?

(e) The correct relativistic expression for kinetic energy is

$$K = \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right) m c^2.$$

Use this to find the correct speed of an electron after accelerating across $V = 25\,\text{kV}$. What is the percent error in using the nonrelativistic formula?

Solution to C.1

(a) The direction of a force is always toward lower potential energy. A particle released at rest will then move that way.

$$0 > \Delta U = Q \Delta V \text{ and } Q = -e < 0 \implies \Delta V > 0$$

(b) Use the conservation of energy, $\Delta U + \Delta K = 0$. The potential energy change is: $\Delta U = Q \Delta V$. The easiest way to approach such a problem is to avoid signs by taking absolute values. For an electron $|Q| = e$ and take V to be the magnitude of the potential difference: $V = \Delta V$.

$$|\Delta U| = |\Delta K| \implies e V = \frac{1}{2} m v^2 \implies v = \sqrt{\frac{2 e V}{m}}$$

(c) For an electron: $e = 1.60 \times 10^{-19}\,\text{C}$ and $m = 9.11 \times 10^{-31}\,\text{kg}$. Using $V = 12\,\text{V}$ we get

$$v = \sqrt{\frac{2 e V}{m}} = \sqrt{\frac{2 \times 1.60 \times 10^{-19} \times 12}{9.11 \times 10^{-31}}} = 2.05 \times 10^6 \,\frac{\text{m}}{\text{s}}$$

The speed of light is $c = 3.00 \times 10^8\, m/s$. The fraction of the speed of light is tiny.

$$\frac{v}{c} = 0.00684$$

(d) Using $V = 25\,000\,\text{V}$ we get

$$v = \sqrt{\frac{2\,e\,V}{m}} = \sqrt{\frac{2\times1.60\times10^{-19}\times25\,000}{9.11\times10^{-31}}} = 9.37\times10^{7}\ \frac{\text{m}}{\text{s}}$$

The speed of light is $c = 3.00\times10^{8}\ m/s$. The fraction of the speed of light is now a significant value.

$$\frac{v}{c} = 0.312$$

Since this value is so high, the above value for the speed should not be trusted.

(e) Using the relativistic expression will give the correct answer for the speed.

$$K = \left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right) m\,c^2.$$

$$|\Delta U| = |\Delta K| \implies e\,V = \left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right) m\,c^2 \implies v = \text{c}\sqrt{1 - \left(1 + \frac{e\,V}{m\,c^2}\right)^{-2}}$$

Using $V = 25\,000$ V we get

$$v = 9.04\times10^{7}\ \frac{\text{m}}{\text{s}}.$$

Problem C.2

A some distance from a point charge the voltage is -180 V and the electric field magnitude is $800\ V/m$. Find the charge and the distance from the charge?

Solution to C.2

We need to solve for r and Q from E and V.

$$E = k_e\,\frac{|Q|}{r^2} \ \text{ and } \ V = k_e\,\frac{Q}{r}$$

Since $V < 0$ we know that $Q < 0$. To simplify matters we will just use the absolute values of V and Q.

$$|V| = k_e\,\frac{|Q|}{r} \ \text{ and } \ E = k_e\,\frac{|Q|}{r^2} \implies r = \frac{|V|}{E}$$

$$V = k_e\,\frac{Q}{r} \implies |Q| = \frac{r\,|V|}{k_e} = \frac{V^2}{k_e\,E}$$

Using the values $V = -180$ and $E = 800\ V/m$, we can find r and Q.

$$r = \frac{|V|}{E} = 0.225\ \text{m} \ \text{ and } \ |Q| = \frac{V^2}{k_e\,E} = 4.5\times10^{-9} \implies Q = -4.5\ \text{nC}$$

Problem C.3

A 5 μC charge sits at the origin and a -8 μC charge sits at (2 m, -3 m). What is the potential at (0, -2 m)? Compare this to problem A.5, where the electric field at (0, -2 m) was found.

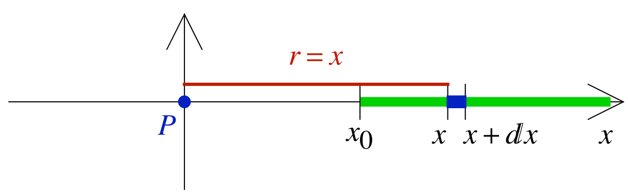
Solution to C.3

$$V = k_e \sum_i \frac{Q_i}{r_i} = k_e \frac{Q_1}{r_1} + k_e \frac{Q_2}{r_2} = 9\times 10^9 \frac{5\times 10^{-6}}{2} + 9\times 10^9 \frac{-8\times 10^{-6}}{\sqrt{2^2+1^2}}$$
$$= 22\,500 - 32\,200. = -9700 \text{ V}$$

Problem C.4

What is the potential at the origin due to a line of charge from x_0 to x_1 along the positive x-axis with a uniform linear charge density (charge/length) of λ. Compare this to problem A.7, where the electric field at the origin was found and where $x_0 \to \infty$.

Solution to C.4



Limits of integration are $x_0 \le x < \infty$. The infinitesimal charge dq has the form: $dq = \lambda\, dx$.

In the electric field problem we had $\vec{r} = -x\,\hat{e}_x$. Here all we need is the magnitude of it: $r = x$.

$$V = k_e \int \frac{dq}{r} = k_e\,\lambda \int_{x_0}^{x_1} \frac{dx}{x} = k_e\,\lambda \ln \frac{x_1}{x_0}$$

Note that in the limit that $x_1 \to \infty$, as in the chapter A problem, the result becomes infinite.

Problem C.5

Consider a flat annulus in the xy-plane with an inside radius a and an outside radius b and with a uniform surface charge density σ. What is the potential at a point z_0 along the positive z-axis?

Solution to C.5

Break up the annulus into concentric rings between r and $r + dr$. The infinitesimal charge in each ring is

$$dq = \sigma\, d\text{Area} = \sigma\, 2\,\pi\, r\, dr.$$

The distance from the ring to the point z_0 is $\sqrt{r^2 + z_0^2}$ and the limits of integration are $a \le r \le b$. The integral becomes:

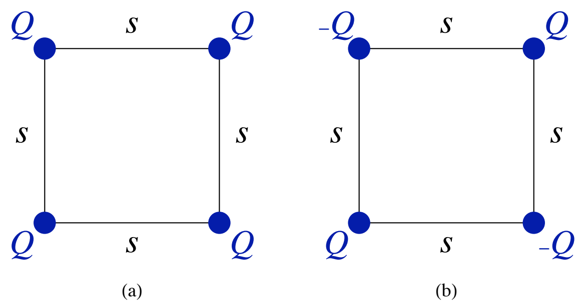
$$V = k_e \int \frac{dq}{\sqrt{z_0^2 + r^2}} = \sigma\, 2\,\pi\, k_e \int_a^b \frac{r\, dr}{\sqrt{z_0^2 + r^2}}$$

This is an integral of the form $\int u^{-1/2}\, du$. Evaluating it gives

$$V = \sigma\, 2\,\pi\, k_e \left(\sqrt{z_0^2 + b^2} - \sqrt{z_0^2 + a^2} \right)$$

Problem C.6

What is the total potential energy of the configuration for each configuration shown?



Solution to C.6

The potential energy for a distribution of point charges is

$$U = k_e \sum_{i<j} \frac{Q_i\, Q_j}{r_{ij}},$$

where the sum is over all pairs of charges.

For a square with side a the diagonals have length $a\sqrt{2}$. To get the total potential energy we must sum over all six pairings of the four charges. These pairing correspond to the 4 sides and the 2 diagonals.

(a) $U = 4 \times k_e \dfrac{Q^2}{a} + 2 \times k_e \dfrac{Q^2}{a\sqrt{2}} = \left(4 + \sqrt{2}\right) k_e \dfrac{Q^2}{a} = 5.414\, k_e \dfrac{Q^2}{a}$

(b) $U = 4 \times k_e \dfrac{Q\,(-Q)}{a} + 2 \times k_e \dfrac{Q^2}{a\sqrt{2}} = \left(-4 + \sqrt{2}\right) k_e \dfrac{Q^2}{a} = -2.586\, k_e \dfrac{Q^2}{a}$

Problem C.7

Two protons are released from rest from a distance of 1 nm. What is their speed when they are a large distance apart? Both will have the same speed. You may assume all speeds are nonrelativistic.

Solution to C.7

The potential energy for two charges separated by a distance r is

$$U = k_e \frac{Q_1 Q_2}{r}.$$

If charges of e start at $r = d$ and move to infinity then $U_i = k_e e^2 / d$ and $U_f = 0$. The total kinetic energy is $K = 2\times \frac{1}{2} m v^2$, since both protons are moving. It is easiest when using the conservation of energy here to just take absolute values.

$$|\Delta K| = |\Delta U| \implies 2\times \frac{1}{2} m v^2 = k_e \frac{e^2}{d} \implies v = \sqrt{\frac{k_e e^2}{m d}}$$

Using $k_e = 9.0\times 10^9 \text{ N m}^2/\text{C}^2$, $e = 1.60\times 10^{-19}$ C, $m = 1.67\times 10^{-27}$ kg and $d = 10^{-9}$ m, we get

$$v = 117\times 00 \frac{\text{m}}{\text{s}}.$$

Problem C.8

Consider a ring of radius R uniformly charged with a charge Q. How much work is required to move a point charge q from a distance z_0 from the center along the central axis to the center.

Solution to C.8

$V(z) = k_e \frac{Q}{\sqrt{z^2+R^2}}$ is the potential as a function of z, the distance from the center along the central axis.

$$\Delta V = V(0) - V(z_0) = k_e Q \left(\frac{1}{R} - \frac{1}{\sqrt{z_0^2+R^2}} \right)$$

The work done by some agent moving a charge is $W = \Delta U$. (Note that the work done by some agent moving something against a field is opposite the work done by the field itself.)

$$W = q\,\Delta V = k_e q Q \left(\frac{1}{R} - \frac{1}{\sqrt{z_0^2+R^2}} \right)$$

Problem C.9

The potential as a function of position is $V(x, y, z) = 6x^2 - 5yz^3 - 8x^3 z$ in SI units.

(a) Find the electric field as a function of position.

(b) What is the value of the field at (3 m, −2 m, 4 m)? What is the magnitude of the field there?

Solution to C.9

(a) The electric field is found from the potential by evaluating derivatives. When evaluating derivatives of functions of more than one variable partial derivatives are used. To take a partial derivative with respect to some variable, just take the ordinary derivative with respect to that variable while treating all the other variables as constants.

$$E_x = -\frac{\partial V}{\partial x} = -\left(12\,x - 0 - 24\,x^2\,z\right) = 24\,x^2\,z - 12\,x$$

$$E_y = -\frac{\partial V}{\partial y} = -\left(0 - 5\,z^3 - 0\right) = 5\,z^3$$

$$E_z = -\frac{\partial V}{\partial z} = -\left(0 - 15\,y\,z^2 - 8\,x^3\right) = 15\,y\,z^2 + 8\,x^3$$

$$\text{or}\quad \vec{E} = \left\langle 24\,x^2\,z - 12\,x,\ 5\,z^3,\ 15\,y\,z^2 + 8\,x^3\right\rangle$$

(b) At the position $(x, y, z) = (3\text{ m}, -2\text{ m}, 4\text{ m})$ we have

$$\vec{E} = \langle 828, 320, -264\rangle\ \frac{\text{V}}{\text{m}}.$$

The magnitude of the field is

$$E = \sqrt{E_x^2 + E_y^2 + E_z^2} = 926\ \frac{\text{V}}{\text{m}}.$$

Problem C.10

Consider a uniform electric field of magnitude $300\ V/m$ in the negative z-direction. A $-20\ \mu$C charge is moved from the point $(3\text{ m},\ -2\text{ m},\ 5\text{ m})$ to the origin.
(a) What is the change in the potential for the charge?
(b) How much work is needed to move the charge?

Solution to C.10

(a) The general expression for the potential from an electric field is the integral $\Delta V = -\int \vec{E} \cdot d\vec{r}$. In this case we have a uniform field and this becomes

$$\Delta V = -\vec{E} \cdot \Delta \vec{r}$$

The field and $\Delta \vec{r}$ have the values

$$\vec{E} = -300\ \frac{\text{V}}{\text{m}}\ \hat{z} = \left\langle 0, 0, -300\ \frac{\text{V}}{\text{m}}\right\rangle \text{ and } \Delta \vec{r} = \langle -3\text{ m},\ 2\text{ m},\ -5\text{ m}\rangle.$$

It follows that

$$\Delta V = -\vec{E} \cdot \Delta \vec{r} = 0 + 0 - (-300)\,(-5) = -1500\text{ V}.$$

(b) The wording of the problem implies that we want the wok done by some agent moving the charge against the field and not the work done by the field itself. For instance it takes positive work to lift something but the work done by gravity is the negative of that.

$$W = \Delta U = Q\,\Delta V = -20 \times 10^{-6}\,(-1500) = 0.030\text{ J}$$

Problem C.11

How many electrons must be removed from a conducting sphere with a 12 cm radius to give it a voltage of 5000 V?

Solution to C.11

The potential of the conductor is constant and thus is the same as the voltage at its surface.

$$V = k_e \frac{Q}{R} \implies Q = \frac{V R}{k_e} = \frac{5000\times 0.12}{9\times 10^9} = 6.67\times 10^{-8}\ \mathrm{C}$$

When electrons are removed something gets a positive charge. The number of electrons is the number of fundamental charges in it.

$$Q = n\, e \implies n = \frac{Q}{e} = \frac{Q}{1.60\times 10^{-19}} = 4.16\times 10^{11}$$

Problem C.12

Two conducting spheres of radius 6 cm and 9 cm are separated by a large distance and connected by a conducting wire. If a total charge of 25 μC is added to the configuration then what charge flows to each conductor? What is the potential of each conductor?

Solution to C.12

When 2 conductors are connected, charge will flow until the potentials become equal. The potential of an isolated conducting sphere of radius R with charge Q is $V = k_e\, Q/R$. If two distant spheres are connected by a wire then the final charge distribution will satisfy:

$$V_1 = V_2 \implies \frac{Q_1}{R_1} = \frac{Q_2}{R_2} \implies \frac{Q_2}{Q_1} = \frac{R_2}{R_1} = \frac{9}{6} = 1.5\ .$$

We also know that

$$Q_1 + Q_2 = Q_{\text{total}} = 25\ \mu\mathrm{C} \implies Q_1 + 1.5\, Q_1 = 25$$
$$\implies Q_1 = 10\ \mu\mathrm{C} \text{ and } Q_2 = 15\ \mu\mathrm{C}\ .$$

The potential of each is the same so we just need one.

$$V_2 = V_1 = k_e \frac{Q_1}{R_1} = 9.0\times 10^9 \frac{10\times 10^{-6}}{.06} = 1.5\times 10^6\ \mathrm{V}$$

Problem C.13

What is the potential as a function of position for a thin spherical shell of radius R with a uniform charge Q?

Solution to C.13

We need to first find the electric field as a function of position using Gauss's law. For any problem with spherical symmetry we have

$$\oint \vec{E}\cdot d\vec{A} = \frac{1}{\varepsilon_0} Q_{\text{inside}} \implies \vec{E} = k_e \frac{Q_{\text{inside}}}{r^2}\,\hat{r} \implies E_r = k_e \frac{Q_{\text{inside}}}{r^2}.$$

The last expression above is the radial component of the field. This is what we will need to find the voltage.

For $r < R$: $Q_{\text{inside}} = 0 \implies E_r = 0$

For $r > R$: $Q_{\text{inside}} = Q \implies E_r = k_e \dfrac{Q}{r^2}$

To find the potential we use $\dfrac{dV}{dr} = -E_r$ and integrate E_r to get $V(r)$. Each integration will give and arbitrary constant. To fix the values of these constants we insist the voltage goes to zero at infinity and that the voltage is a continuous function.

Since our reference potential is at infinity (where $V = 0$) we will begin at infinity and work our way in.

For $r > R$:

$$\frac{dV}{dr} = -E_r = -k_e \frac{Q}{r^2} \implies V(r) = k_e \frac{Q}{r} + C_1$$

We now find the integration constant

$$0 = V(\infty) = 0 + C_1 \implies C_1 = 0.$$

The potential becomes:

$$V(r) = k_e \frac{Q}{r} \text{ for } r \geq R.$$

For $r < R$:

$$\frac{dV}{dr} = -E_r = 0 \implies V(r) = C_2$$

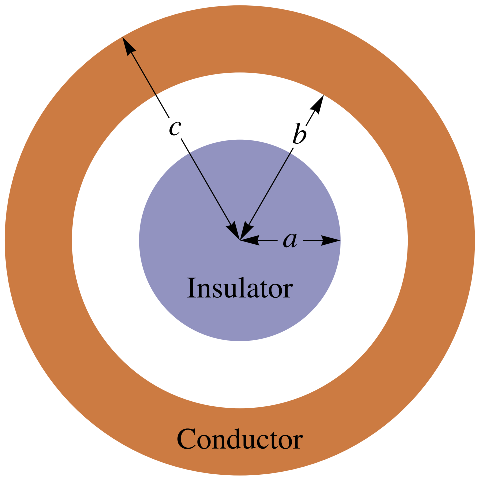
The new constant is found by insisting the potential is continuous. It must have the same value at $r = R$ for the two expressions

$$V(R^-) = V\left(R^+\right) \implies C_2 = k_e \frac{Q}{R}$$

$$V(r) = k_e \frac{Q}{R} \text{ for } r \leq R.$$

Problem C.14

A solid insulating sphere of radius a has a uniform charge Q. This sits inside of a hollow conducting sphere with and inside radius b and outside radius c. The conductor is given a net charge of q. (All spherical surfaces are concentric.) What is the potential as a function of position? Give answers for all cases: $r<a$, $a<r<b$, $b<r<c$ and $r>c$. This is an extension of Problem B.13 from the previous chapter.



Solution to C.14

We found the electric field as a function of position in Problem B.13 . The magnitude of the field is E_r.

$$\vec{E} = k_e \frac{Q}{a^3} r\,\hat{r} \implies E_r = k_e \frac{Q}{a^3} r \text{ (for } r<a)$$

$$\vec{E} = k_e \frac{Q}{r^2} \hat{r} \implies E_r = k_e \frac{Q}{r^2} \text{ (for } a<r<b)$$

$$\vec{E} = \vec{0} \implies E_r = 0 \text{ (for } b<r<c)$$

$$\vec{E} = k_e \frac{Q+q}{r^2} \hat{r} \implies E_r = k_e \frac{Q+q}{r^2} \text{ (for } r>c)$$

To find the voltage we use $\frac{dV}{dr} = -E_r$ and integrate E_r to get $V(r)$. Each integration will give and arbitrary constant. To fix the values of these constants we insist the voltage goes to zero at infinity and that the voltage is a continuous function.

Since our reference potential is at infinity (where $V=0$) we will begin at infinity and work our way in.

For $r>c$:

$$\frac{dV}{dr} = -E_r = -k_e \frac{Q+q}{r^2} \implies V(r) = k_e \frac{Q+q}{r} + C_1$$

We now find the integration constant

$$0 = V(\infty) = 0 + C_1 \implies C_1 = 0.$$

The potential becomes:

$$V(r) = k_e \frac{Q+q}{r} \text{ for } r \ge c.$$

For $b < r < c$:

$$\frac{dV}{dr} = -E_r = 0 \implies V(r) = C_2$$

The new constant is found by insisting the potential is continuous. It must have the same value at $r = c$ for the two expressions

$$V(c^-) = V\left(c^+\right) \implies C_2 = k_e \frac{Q+q}{c}$$

$$V(r) = k_e \frac{Q+q}{c} \text{ for } b < r < c.$$

For $a < r < b$:

$$\frac{dV}{dr} = -E_r = -k_e \frac{Q}{r^2} \implies V(r) = k_e \frac{Q}{r} + C_3$$

We find C_3 by insisting the function is continuous at b.

$$V(b^-) = V\left(b^+\right) \implies k_e \frac{Q}{b} + C_3 = k_e \frac{Q+q}{c} \implies C_3 = k_e \frac{Q+q}{c} - k_e \frac{Q}{b}$$

The voltage becomes:

$$V(r) = k_e \frac{Q}{r} + \left(k_e \frac{Q+q}{c} - k_e \frac{Q}{b}\right) \text{ for } a < r < b.$$

The part of the above expression in brackets is just a constant.

For $r < a$:

$$\frac{dV}{dr} = -E_r = -k_e \frac{Q}{a^3} r \implies V(r) = -\frac{1}{2} k_e \frac{Q}{a^3} r^2 + C_4$$

We find C_4 by insisting the function is continuous at a.

$$V(a^-) = V\left(a^+\right) \implies -\frac{1}{2} k_e \frac{Q}{a^3} a^2 + C_4 = k_e \frac{Q}{a} + k_e \frac{Q+q}{c} - k_e \frac{Q}{b}$$

$$\implies C_4 = k_e \frac{Q+q}{c} - k_e \frac{Q}{b} + \frac{3}{2} k_e \frac{Q}{a}$$

The potential becomes:

$$V(r) = -\frac{1}{2} k_e \frac{Q}{a^3} r^2 + \left(k_e \frac{Q+q}{c} - k_e \frac{Q}{b} + \frac{3}{2} k_e \frac{Q}{a}\right) \text{ for } r < a.$$

The very ugly part of the above expression in brackets is just a constant.